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INDIRECT ASSESSMENT OF THE BIVARIATE SURVIVAL FUNCTION

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Abstract. Estimating the bivariate survival function has been a major goal of many researchers. For that purpose many methods and techniques have been published. However, most of these techniques and methods rely heavily on bivariate failure data. There are situations in which failure time data are difficult to obtain and thus there is a growing need to assess the bivariate survival function for such cases. In this paper we propose two techniques for generating families of bivariate processes for describing several variables that can be used to indirectly assess the bivariate survival function. An estimation procedure is provided and a simulation study is conducted to evaluate the performance of our proposed estimator.

Key words and phrases: Compound Poisson process, Gamma process, inverse Gaussian, Laplace transformation, Poisson process, survival function.

1. Introduction

In recent years substantial research effort has been devoted to developing methodology for estimating multivariate survival function. Applications of multivariate survival analysis arise in various fields. Example in reliability include a device having two integral parts and our desire is to assess the joint survival distribution of the parts.

Suppose $T_1, T_2, \dots$ and T_k are failure times. A multivariate distribution can be specified in terms of the joint survival function,

$$S(t_1, \dots, t_k) = P(T_1 > t_1, \dots, T_k > t_k). \tag{1.1}$$

The main problem connected to $T_1, \dots, T_k$ is estimating the joint survival function in the equation (1.1). The traditional approach is that the survival function is estimated from the experimental data on $T = (T_1, \dots, T_k)$, see Van der Laan (1996), and Tsai and Crowley (1998) and references cited there. Unfortunately, there are situations in which the failure data is not available. For example with high reliability devices and short development times, tests must be conducted with severe time constraints and frequently no failures occur during such test. Thus, for this case it is impossible to assess the joint survival function $S(t_1, \dots, t_k)$ with traditional lifetests that records failure times.

To circumvent such difficulties, $T_1, \dots, T_k$ are defined as functions of several explanatory variables refered to as covariates. For example, thinking of degradation as a covariate, there are situations where degradation measures can be taken over time and where a relationship between failure time and amount of degradation makes it possible to use degradation models and data to make inference about the joint survival function. Usually, degradation models begin with a deterministic description of the degradation process that is often in the form of a differential equation or system of differential equations. Randomness can be introduced using probability distributions to describe variability in initial conditions and model parameters. To illustrate this, let $D_L(t)$ and $D_R(t)$ be the amount of automobile left and right front tires tread wear at time t respectively. Then, the failure times for tires can be the first time when $D_L(t)$ and $D_R(T)$ cross the known thresholds.

To express T_i, $i = 1, \ldots, k$, as functions of covariates as well as to assess the joint survival function, there are 3 separate problems:

(i) Choosing appropriate models for covariates

(ii) Expressing the joint survival function in terms of covariates

(iii) Identifying the parameters of the joint survival function by statistical means.

For notational simplicity, all the statements in this paper are given in the bivariate case. Multivariate generalizations are straight forward and do not require any new concepts. Also, for simplicity of exposition, we use reliability terminology in this article. We emphasize that the proposed methods can be applied to other areas as well.

Let T_i be the failure time of a device i, $i = 1, 2$, and let T_1 and T_2 be defined in terms of covariates. Since, in many practical situations, the covariates that are commonly identified have been observed to display substantial variation, the stochastic process is a reasonable way to model their behaviors. Suppose $\{S_i(t); t \geq 0\}$ be the value of covariate i at time t, $i = 1, 2$ and suppose

$$T_i = \text{Inf}\{t : 0 \leq t < \infty, S_i(t) > \Delta_i\}, \qquad i = 1, 2. \tag{1.2}$$

In other words the device i fails if $S_i(t)$ exceeds a known threshold Δ_i, $i = 1, 2$. From (1.2) the joint survival function of T_1 and T_2 is

$$S(t_1, t_2) = P(T_1 > t_1, T_2 > t_2) = P\left(\sup_{0 \leq u \leq t_i} S_i(u) \leq \Delta_i, \ i = 1, 2\right). \tag{1.3}$$

In this article we propose families of processes $S(t) = (S_1(t), S_2(t))$ that have a state space $[0, \infty) \times [0, \infty)$ and non-decreasing sample paths. Families of univariate processes for modeling $S_1(t)$ was proposed by Dufresne *et al.* (2000) and our work extends that to the bivariate case. At first it would seem that to propose and study properties of the process $S(t) = (S_1(t), S_2(t))$ are little more than mathematically interesting exercises. However, suppose that we are given a two dimensional stochastic vector process $\{S(t) = (S_1(t), S_2(t)); t \geq 0\}$, where $S_i(t)$ is the total damages to device i over the interval $[0, t]$, $i = 1, 2$. It is clear that $S_i(t)$ must have non-decreasing sample paths and its state space must be $[0, \infty)$. Now, let T_i be the failure time of the device i, $i = 1, 2$, then T_i satisfies the equation (1.2), $i = 1, 2$. In other words the device i fails if the total damage to it exceeds a known threshold Δ_i, $i = 1, 2$. Also, the joint survival function of T_1 and T_2 reduces to

$$S(t_1, t_2) = P(S_i(t_i) \leq \Delta_i, \ i = 1, 2). \tag{1.4}$$

The structure of the paper is as follows. In Section 2 we describe two techniques for constructing families of bivariate processes and study their properties. In Section 3, we give several examples. In Section 4, we develop a methodology for assessing $S(t_1, t_2)$ in the equation (1.4). We illustrate the method using a specific stochastic model for $S_1(t)$ and $S_2(t)$.

2. The construction

To motivate our approach and to fix ideas and notations, consider two physical devices. Suppose that as time goes on, damage or stress builds up. Considerations suggest modeling device i damage by a stochastic process $\{S_i(t); t \geq 0\}$ with starting state corresponding to initial level of damage, $i = 1, 2$. Throughout this paper, without loss of generality, we assume that $S_i(0) = 0$, $i = 1, 2$ almost surely. Since both processes describe the damage it is natural to assume that they are non-decreasing in time. In this section we describe two methods for constructing a model for the bivariate process $\{(S_1(t), S_2(t)); t \geq 0\}$.

2.1 *Method* 1

Let $Q(x, y)$ be a non-negative and right continuous non-increasing function of x, y, $x, y > 0$, with the properties that $Q(x, y) \to 0$ as $x, y \to \infty$, $\int_0^\infty Q(0, y)dy < \infty$, $\int_0^\infty Q(y, 0)dy < \infty$, $\int_0^\infty \int_0^\infty Q(x, y)dxdy < \infty$ and $Q(x, y)$ is supermodular, that is, $Q(x_1, y_1) + Q(x, y) \geq Q(x_1, y) + Q(x, y_1)$ whenever $x_1 \geq x$, $y_1 \geq y$. Also, let $K(x)$ and $L(y)$ be two non-negative and right continuous non-increasing functions of x and y respectively with the properties that $K(x) \to 0$ as $x \to \infty$ and $L(y) \to 0$ as $y \to \infty$ and $\int_0^\infty K(x)dx < \infty$ and $\int_0^\infty L(y)dy < \infty$. Suppose two devices are subjected to shocks occurring randomly in time and suppose there are three independent sources of shocks present in the environment. A shock from source 1 damages device 1. A shock from source 2 damages the second device. Finally, a shock from source 3 damages both devices. For each x, let $N_1(t; x)$ denote the number of shocks from source 1 with a damage to device 1 greater than x that occur before time t. We assume that $\{N_1(t; x), t \geq 0\}$ is a Poisson process with parameter $K(x)$. For each y, we let $N_2(t; y)$ to denote the number of shocks from source 2 with a damage to device 2 greater than y that occur before time t. We assume that $\{N_2(t; y); t \geq 0\}$ is a Poisson process with parameter $L(y)$. Finally, for each x, y, let $N_3(t; x, y)$ denote the number of shocks from source 3 with damages to devices 1 and 2 exceeding x and y respectively. We assume that the process $\{N_3(t; x, y); t \geq 0\}$ is a Poisson process with parameter $Q(x, y)$.

From source 1, we assume that the damage distribution is $K^*(x_1; x) = P$ (damage to device 1 be less than x_1 given that the damage exceeded x), where

$$K^*(x_1; x) = \begin{cases} \dfrac{K(x) - K(x_1)}{K(x)}, & x \leq x_1 \\ 0, & x > x_1. \end{cases} \tag{2.1}$$

From source 2, we assume that the damage distribution is

$$L^*(y_1; y) = \begin{cases} \dfrac{L(y) - L(y_1)}{L(y)}, & y \leq y_1 \\ 0, & y > y_1. \end{cases} \tag{2.2}$$

Also, from source three the joint damage distribution is

$$Q^*(x_1, y_1; x, y) = \begin{cases} \dfrac{Q(x, y) - Q(x_1, y) - Q(x, y_1) + Q(x_1, y_1)}{Q(x, y)}, & \\ 0, \quad \text{otherwise}. & \end{cases} \tag{2.3}$$

$$x_1 \geq x, y_1 \geq y$$

Define

$$S_1(t;x,y) = \sum_{i=0}^{N_1(t;x)} Z_i + \sum_{i=0}^{N_3(t;x,y)} X_i, \tag{2.4}$$

and

$$S_2(t;x,y) = \sum_{i=0}^{N_2(t;y)} W_i + \sum_{i=0}^{N_3(t;x,y)} Y_i. \tag{2.5}$$

Here Z_i and X_i are the damages to device 1 from sources 1 and 3 that exceed x respectively. Also, W_i and Y_i are damages to device 2 from sources 2 and 3 that exceed y respectively. It is assumed that Z_i and W_i and (X_i, Y_i) are independent. Also $Z_1, \ldots, Z_n, \ldots$ are independent with the common distribution $K^*(x_1; x)$, $W_1, W_2, \ldots,$ $W_n, \ldots$ are independent with the common distribution $L^*(y_1; y)$ and $(X_1, Y_1), \ldots,$ $(X_n, Y_n), \ldots$ are independent with the common joint distribution function $Q^*(x_1, y_1; x, y)$. The processes $\{S_1(t); t \geq 0\}$ and $\{S_2(t); t \geq 0\}$ are defined as the limit of $\{S_1(t; x, y); t \geq 0\}$ and $\{S_2(t; x, y), t \geq 0\}$ as x and y tend to zero.

It should be noted that if $Q(0,0) < \infty$, then $\sum_{i=0}^{N_3(t;0,0)} X_i$ is a compound Poisson process with Poisson parameter $Q(0,0)$. If $Q(0,0)$ is infinite, then $\sum_{i=0}^{N_3(t;0,0)} X_i$ is no longer a compound Poisson process. Because the expected number of shocks received from source 3 per unit time is infinite. Indeed, in this case, with probability one, the number of shocks in any time interval is infinite. Nevertheless, $\sum_{i=0}^{N_3(t;0,0)} X_i$ is finite, as the majority of the damages are very small in some sense. Similar interpretations can be used for $K(0)$ and $L(0)$.

2.2 *Method* 2

Here, like the method 1 we assume that two devices are subjected to three independent sources of shocks. Shocks from source 1 damages device 1 only, shocks from source 2 damages device 2 only, shocks from source 3 damages both devices. We assume that shocks from the third source makes the damage to both devices and damages to the second device is a function of damages to the first device.

Following some of the notations from the previous method, in this case we define

$$S_1(t,x) = \sum_{i=1}^{N_1(t;x)} Z_i + \sum_{i=1}^{N_4(t;x)} U_i, \tag{2.6}$$

and

$$S_2(t;x,y) = \sum_{i=1}^{N_2(t;y)} W_i + \sum_{i=1}^{N_4(t;x)} g(U_i). \tag{2.7}$$

Here $N_4(t;x)$ is the number of shocks from source 3 with damages to both devices such a way that the damage to device 1 exceeding x. We assume that $\{N_4(t;x); t \geq 0\}$ is a Poisson process with parameter $H(x)$.The function $H(x)$ has similar properties as $K(x)$ and $L(x)$. Also, U has the distribution,

$$H^*(u_1;u) = \begin{cases} \dfrac{H(u) - H(u_1)}{H(u)}, & u \leq u_1 \\ 0, & u > u_1, \end{cases} \tag{2.8}$$

and g is a known non-negative function.

Marshall and Shaked (1979) have proposed a model that devices are subjected to shocks occuring randomly in time as events in a Poisson process. Upon occurrence of i-th shock, the devices suffer non-negative random damages. Damages from successive shocks are assumed to be independent and accumulate additively. See also Li (2000). Our proposed models assume three different sources of shocks and allow "infinitely many small shocks" occurring in any finite time interval, such as the case of fatigue.

Assuming that $xK(x) \to 0$ as $x \to 0$, $yL(y) \to 0$ as $y \to 0$, $y\frac{\partial Q(x,y)}{\partial x} \to 0$ as $y \to 0$ or $y \to \infty$ and $x\frac{\partial Q(x,y)}{\partial y} \to 0$ as $x \to 0$ or $x \to \infty$, under the first method, from the equations (2.4) and (2.5) one can show that

$$\begin{aligned} (2.9) \quad E(S_1(t;x,y)) &= E(N_1(t;x)) \int_0^\infty [1 - K^*(x_1;x)]dx_1 \\ &\quad + E(N_3(t;x,y)) \int_0^\infty (1 - Q^*(x_1,\infty;x,y))dx_1 \\ &= tK(x)x + t\int_x^\infty K(x_1)dx_1 + tQ(x,y)x + t\int_x^\infty Q(x_1,y)dx_1 \\ &= tx(K(x) + Q(x,y)) + t\int_x^\infty (K(x_1) + Q(x_1,y))dx_1 \end{aligned}$$

and as x and y tend to zero

$$(2.10) \qquad E(S_1(t)) = t\int_0^\infty (K(x) + Q(x,0))dx.$$

$$(2.11) \qquad E(S_2(t)) = t\int_0^\infty (L(y) + Q(0,y))dy.$$

Letting both x and $y \to 0$, we obtain the covariance between two processes,

$$(2.12) \qquad \mathrm{Cov}(S_1(t_1), S_2(t_2) = \min(t_1,t_2)\int_0^\infty \int_0^\infty Q(x,y)dxdy,$$

and the joint Laplace transform of $S_1(t_1)$ and $S_2(t_2)$,

$$\begin{aligned} (2.13) \quad M_t(u_1,u_2) &= E(\exp(-u_1S_1(t_1) - u_2S_2(t_2))) \\ &= \exp\left\{t\left(\int_0^\infty (\bar{e}^{u_1x} - 1)k(x)dx + \int_0^\infty (\bar{e}^{u_2y} - 1)l(y)dy\right.\right. \\ &\qquad \left.\left. + \int_0^\infty \int_0^\infty (\bar{e}^{u_1x-u_2y} - 1)q(x,y)dxdy\right)\right\} \end{aligned}$$

if $t_1 = t_2 = t$. If $t_1 < t_2$, then

$$\begin{aligned} (2.14) \quad M_{t_1,t_2}(u_1,u_2) &= \exp\left(t_1\int_0^\infty (\bar{e}^{u_1y} - 1)k(y)dy + t_2\int_0^\infty (\bar{e}^{u_2y} - 1)l(y)dy\right. \\ &\qquad + t_1\int_0^\infty \int_0^\infty (\bar{e}^{u_1x-u_2y} - 1)q(x,y)dxdy \\ &\qquad \left. + (t_2 - t_1)\int_0^\infty (\bar{e}^{u_2y} - 1)dQ(0,y)\right). \end{aligned}$$

If $t_1 > t_2$, then

$$M_{t_1,t_2}(u_1,u_2) = \exp\left(t_1 \int_0^\infty (\bar{e}^{u_1 y} - 1)k(y)dy + t_2 \int_0^\infty (\bar{e}^{u_2 y} - 1)l(y)dy + t_2 \int_0^\infty \int_0^\infty (\bar{e}^{u_1 x - u_2 y} - 1)q(x,y)dxdy + (t_1 - t_2)\int_0^\infty (\bar{e}^{u_1 y} - 1)dQ(y,0)\right). \tag{2.15}$$

Here $k(x), l(y)$ and $q(x,y)$ are the derivatives of $K(x), L(y)$ and $Q(x,y)$ with respect to x, y respectively. Under the second method, from the equations (2.6) and (2.7) we get that

$$E(S_1(t)) = t\int_0^\infty (K(x) + H(x))dx = t\int_0^\infty x(k(x) + h(x))dx, \tag{2.16}$$

$$E(S_2(t)) = t\int_0^\infty x(l(x) + g(x)h(x))dx, \tag{2.17}$$

$$\mathrm{Cov}(S_1(t_1), S_2(t_2)) = \min(t_1, t_2)\int_0^\infty xg(x)h(x)dx, \tag{2.18}$$

where $h(x)$ is the derivative of $H(x)$ with respect to x. Also, the joint Laplace transformation of $S_1(t_1)$ and $S_2(t_2)$,

$$M_t(u_1,u_2) = \exp\left\{t\left(\int_0^\infty (e^{-u_1 x} - 1)k(x)dx + \int_0^\infty (e^{-u_2 x} - 1)l(x)dx + \int_0^\infty (e^{-u_1 x - u_2 g(x)} - 1)h(x)dx\right)\right\}, \tag{2.19}$$

if $t_1 = t_2 = t$. For $t_1 < t_2$,

$$M_{t_1,t_2}(u_1,u_2) = \exp\left\{t_1 \int_0^\infty (\bar{e}^{u_1 x} - 1)k(x)dx + t_2 \int_0^\infty (\bar{e}^{u_2 x} - 1)l(x)dx + t_1 \int_0^\infty (\bar{e}^{u_1 x - u_2 g(x)} - 1)h(x)dx + (t_2 - t_1)\int_0^\infty (\bar{e}^{u_2 g(x)} - 1)h(x)dx\right\}. \tag{2.20}$$

For $t_1 > t_2$,

$$M_{t_1,t_2}(u_1,u_2) = \exp\left\{t_1 \int_0^\infty (\bar{e}^{u_1 x} - 1)k(x)dx + t_2 \int_0^\infty (\bar{e}^{u_2 x} - 1)l(x)dx + t_2 \int_0^\infty (\bar{e}^{u_2 g(x) - u_1 x} - 1)h(x)dx + (t_1 - t_2)\int_0^\infty (\bar{e}^{u_1 x} - 1)h(x)dx\right\}. \tag{2.21}$$

Since our emphasis in this paper is to assess the bivariate survival function we only provide several qualitative properties of our models. More properties of the models are under investigation and will be reported elsewhere.

THEOREM 2.1. *Under the method* 1, *if* $\frac{Q(x,y)}{Q(0,0)} \geq \frac{Q(x,0)Q(0,y)}{(Q(0,0))(Q(0,0))}$, *for all* x *and* y. *Then,*

$$P(T_1 > t_1, T_2 > t_2) \geq P(T_1 > t_1)P(T_2 > t_2) \quad \text{for all} \quad t_1, t_2 \geq 0. \tag{2.22}$$

PROOF. Assume $0 \leq t_1 < t_2$, then

$$\begin{aligned} &P(T_1 > t_1, T_2 > t_2) = P(S_1(t_1) \leq \Delta_1, S_2(t_2) \leq \Delta_2) \\ &\sum_{k_2=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \sum_{k_4=0}^{\infty} P(N_1(t_1;0) = k_1)P(N_2(t_2;0) = k_2) \\ &P(N_3(t_1;0,0) = k_3)P(N_3(t_2 - t_1);0,0) = k_4) \\ &P(Z_1 + \cdots + Z_{k_1} + X_1 + \cdots + X_{k_3} < \Delta_1, W_1 + \cdots + W_{k_2} + Y_1 + \cdots + Y_{k_3} + Y_{k_3+1} \\ &\qquad + \cdots + Y_{k_3+k_4} < a_2) \geq P(T_1 > t_1)P(T_2 > t_2). \end{aligned}$$

This completes the proof. □

Under the second method, one can easily show that if g is non-decreasing function, then $P(T_1 > t_1, T_2 > t_2) \geq P(T_1 > t_2)P(T_2 > t_2)$ for all $t_1, t_2 \geq 0$. The condition (2.22) is known as the positive quadrant dependence (PQD). Li and Xu (2001) discussed a similar problem in which the damages are binary random variables.

THEOREM 2.2. *Under both methods* $P(T_1 > t_1 + u_1, T_2 > t_2 + u_2) \leq P(T_1 > t_1, T_2 > t_2)P(T_1 > u_1, T_2 > u_2)$ *whenever* (t_2, t_2) *and* (u_1, u_2) *are similarly ordered, i.e.,* $(u_i - u_j)(t_i - t_j) \geq 0$, $i, j = 1, 2$.

PROOF. Use similar arguments to proof of Theorem 7.1 of Marshall and Shaked (1979) for the case $X_i \geq x$, $Y_i \geq y$ and then take the limit as $x \to 0$, $y \to 0$. □

3. Examples

In this section we give several examples of bivariate processes constructed using methods 1 and 2. We start with several bivariate processes constructed under the second method.

Example 3.1. Assume that

$$\begin{aligned} k(x) &= \frac{a_1}{x} \exp(-bx), \qquad x > 0 \\ l(x) &= \frac{a_2}{x} \exp(-bx), \qquad x > 0 \\ h(x) &= \frac{a_3}{x} \exp(-bx), \qquad x > 0 \end{aligned}$$

and

$$g(x) = x, \qquad x > 0,$$

where a_i and $b, i = 1, 2, 3$ are positive constants. It is clear, using equations (2.15)–(2.20), that

$$E(S_1(t)) = \left(\frac{a_1 + a_3}{b}\right) t, \tag{3.1}$$

$$E(S_2(t)) = \left(\frac{a_2 + a_3}{b}\right) t, \tag{3.2}$$
$$\operatorname{Cov}(S_1(t_1), S_2(t_2)) = \min(t_1, t_2) a_3 b^2.$$

Also,

$$M_{t_1,t_2}(u_1, u_2) = \begin{cases} \left(\frac{b}{u_1+b}\right)^{a_1 t} \left(\frac{b}{u_2+b}\right)^{a_2 t} \left(\frac{b}{u_1+u_2+b}\right)^{a_3 t}, & t_1 = t_2 = t \\ \left(\frac{b}{u_1+b}\right)^{a_1 t_1} \left(\frac{b}{u_2+b}\right)^{a_2 t_2} & \\ \quad \times \left(\frac{b}{u_1+u_2+b}\right)^{a_3 t_1} \left(\frac{b}{u_2+b}\right)^{a_3 (t_2 - t_1)}, & t_1 < t_2 \\ \left(\frac{b}{u_1+b}\right)^{a_1 t_1} \left(\frac{b}{u_2+b}\right)^{a_2 t_2} & \\ \quad \times \left(\frac{b}{u_1+u_2+b}\right)^{a_3 t_2} \left(\frac{b}{u_1+b}\right)^{a_3 (t_1 - t_2)}, & t_1 > t_2. \end{cases} \tag{3.3}$$

Formula (3.3) shows that $(S_1(t), S_2(t))$ is the bivariate Gamma distribution, see Johnson and Kotz ((1972), p. 219), the equation (3.2). Hence, we refer to this as the bivariate Gamma process.

Example 3.2. Define,

$$k(x) = a_1 x^{\alpha_1 - 1} \exp(-bx), \qquad x > 0$$
$$l(x) = a_2 x^{\alpha_2 - 1} \exp(-bx), \qquad x > 0$$
$$h(x) = a_3 x^{\alpha_3 - 1} \exp(-bx), \qquad x > 0$$

and

$$g(x) = x.$$

If $\alpha_i > 0$, $i = 1, 2, 3$, $K(0) = \frac{a_1}{b^{\alpha_1}} \Gamma(\alpha_1) < \infty$, $L(0) = \frac{a_2}{b^{\alpha_2}} \Gamma(\alpha_2) < \infty$, $H(0) = \frac{a_3}{b^{\alpha_3}} \Gamma(\alpha_3) < \infty$. It can be shown, using equations (2.15)–(2.20), that

$$E(S_1(t)) = t \left(\frac{a_1(\Gamma(\alpha_1 + 1)}{b^{\alpha_1}} + \frac{a_3 \Gamma(\alpha_3 + 1)}{b^{\alpha_3} + 1} \right),$$
$$E(S_2(t)) = t \left(\frac{a_2 \Gamma(\alpha_2 + 1)}{b_2^{\alpha} + 1} + \frac{a_3 \Gamma(\alpha_3 + 1)}{b^{\alpha_3} + 1} \right),$$

and

$$\operatorname{Cov}(S_1(t_1), S_2(t_2)) = \frac{\min(t_1, t_2) \Gamma(\alpha_3 + 2)}{b_3^{\alpha_3 + 2}}.$$

Also,

$$(3.4)\quad M_{t_1,t_2}(u_1,u_2)=\begin{cases}\exp\left\{\dfrac{a_1t\Gamma(\alpha_1)}{(u_1+b)^{\alpha_1}}+\dfrac{a_2t\Gamma(\alpha_2)}{(u_2+b)^{\alpha_2}}+\dfrac{a_3t\Gamma(\alpha_3)}{(u_1+u_2+b)^{\alpha_3}}\right. \\ \quad\left.-\dfrac{a_1t\Gamma(\alpha_1)}{b^{\alpha_1}}-\dfrac{a_2t\Gamma(\alpha_2)}{b^{\alpha_2}}-\dfrac{a_3t\Gamma(\alpha_3)}{b^{\alpha_3}}\right\}, & t_1=t_2=t\\ \exp\left\{\dfrac{a_1t_1\Gamma(\alpha_1)}{(u_1+b)^{\alpha_1}}+\dfrac{a_2t_2\Gamma(\alpha_2)}{(u_2+b)^{\alpha_2}}+\dfrac{a_3t_1\Gamma(\alpha_3)}{(u_2+u_1+b)^{\alpha_3}}\right. \\ \quad+\dfrac{a_3(t_2-t_1)\Gamma(\alpha_3)}{(u_2+b)^{\alpha_3}}-\dfrac{a_1t_1\Gamma(\alpha_1)}{b^{\alpha_1}} \\ \quad\left.-\dfrac{a_2t_2\Gamma(\alpha_2)}{b^{\alpha_2}}-\dfrac{a_3t_2\Gamma(\alpha_3)}{b^{\alpha_3}}\right\}, & t_1<t_2\\ \exp\left\{\dfrac{a_1t_1\Gamma(\alpha_1)}{(u_1+b)^{\alpha_1}}+\dfrac{a_2t_2\Gamma(\alpha_2)}{(u_2+b)^{\alpha_2}}+\dfrac{a_3t_2}{(u_1+u_2+b)^{\alpha_3}}\right. \\ \quad+\dfrac{a_3(t_1-t_2)\Gamma(\alpha_3)}{(u_1+b)^{\alpha_3}}-\dfrac{a_1t_1\Gamma(\alpha_1)}{b^{\alpha_1}} \\ \quad\left.-\dfrac{a_2t_2\Gamma(\alpha_2)}{b^{\alpha_2}}-\dfrac{a_3t_1\Gamma(\alpha_3)}{b^{\alpha_3}}\right\}, & t_1>t_2.\end{cases}$$

Suppose $-1\le\alpha_i<0$, $i=1,2,3$, $a_i=\frac{-1}{\Gamma(\alpha_i)}$ and $b=1$, $\beta_i=-\alpha_i=\frac{1}{2}$. Then from the equation (3.4) one can show that

$$M_{t_1,t_2}(u_1,u_2)=\begin{cases}\exp\left\{\dfrac{-t}{(1+u_1)^{-1/2}}-\dfrac{t}{(1+u_2)^{-1/2}}-\dfrac{t}{(u_1+u_2+1)^{-1/2}}+3t\right\}, \\ \quad t_1=t_2=t\\ \exp\left\{\dfrac{-t_1}{(1+u_1)^{-1/2}}-\dfrac{t_2}{(1+u_2)^{-1/2}}-\dfrac{t_1}{(u_1+u_2+1)^{-1/2}}\right. \\ \quad\left.-\dfrac{t_2-t_1}{(u_2+1)^{-1/2}}+t_1+2t_2\right\}, & t_1<t_2\\ \exp\left\{-\dfrac{t_1}{(1+u_1)^{-1/2}}-\dfrac{t_2}{(1+u_2)^{-1/2}}-\dfrac{t_2}{(1+u_1+u_2)^{-1/2}}\right. \\ \quad\left.-\dfrac{t_1-t_2}{(u_1+1)^{-1/2}}+t_2+2t_1\right\}, & t_1>t_2\end{cases}$$

which is the joint density of the Bivariate inverse Gaussian, see Seshadri ((1993), p. 128). Hence, we refer to this as Bivariate inverse Gaussian process.

Now we give an example of a bivariate process using the first method.

Example 3.3. Let

$$k(x)=\frac{a_1}{x}e^{-bx},\ x>0,\qquad l(x)=\frac{a_2}{x}e^{-bx},\ x>0,$$
$$Q(x,y)=\lambda\exp(-\delta x-\delta y-\delta xy),\qquad \delta,\lambda>0,\qquad x,y>0.$$

The function $Q(x,y)$ is proportional to the Bivariate Gumbe distribution, see Gumble (1960). One can easily verify that $K(0)=L(0)=\infty$ and $Q(0,0)=\lambda<\infty$. Intuitively speaking we assume that the expected number of shocks received from sources 1 and 2 per unit time is infinite. However, the expected number from source 3 is λ which is finite. Now, one can easily show, using equations (2.9)–(2.14), that $E(S_1(t))=t(\frac{a_1}{b}+\frac{\lambda}{\delta})$,

$E(S_2(t)) = t(\frac{a_2}{b} + \frac{\lambda}{\delta})$ and $\mathrm{Cov}(S_1(t_1), S_2(t_2)) = \min(t_1, t_2)\lambda\frac{1}{\delta}\exp(\delta) \times \Psi(\delta)$, where $\Psi(u) = \int_u^\infty e^{-z}\frac{1}{z}dz$. Also,

$$M_{t_1,t_2}(u_1,u_2) = \begin{cases} \left(\frac{b}{u_1+b}\right)^{a_1 t}\left(\frac{b}{u_2+b}\right)^{a_2 t} \\ \quad \times \left(\exp\left(\lambda\frac{u_1u_2t}{\delta}e^{(\delta+u_1)(\delta+u_2)/\delta}\Psi\left(\frac{(\delta+u_1)(\delta+u_2)}{\delta}\right)\right.\right. \\ \quad \left.\left. - \frac{u_1\lambda t}{\delta+u_1} - \frac{u_2\lambda t}{\delta+u_2}\right)\right), & t_1 = t_2 = t \\ \left(\frac{b}{u_1+b}\right)^{a_1 t_1}\left(\frac{b}{u_2+b}\right)^{a_2 t_2} \\ \quad \times \exp\left\{\lambda\frac{u_1u_2t_1}{\delta}e^{(\delta+u_1)(\delta+u_2)/\delta}\Psi\left(\frac{(\delta+u_1)(\delta+u_2)}{\delta}\right)\right. \\ \quad + \lambda u_2(t_2-t_1)/\delta e^{(\delta+u_2)/\delta}\Psi\left(\frac{\delta+u_2}{\delta}\right) \\ \quad \left. - \frac{u_1\lambda t_1}{\delta+u_1} - \frac{u_2\lambda t_2}{\delta+u_1}\right\}, & t_1 < t_2 \\ \left(\frac{b}{u_1+b}\right)^{a_1 t_1}\left(\frac{b}{u_2+b}\right)^{a_2 t_2} \\ \quad \times \exp\left\{\frac{\lambda u_1u_2t_2}{\delta}e^{(\delta+u_1)(\delta+u_2)/\delta}\Psi\left(\frac{(\delta+u_1)(\delta+u_2)}{\delta}\right)\right. \\ \quad + \frac{\lambda u_1(t_1-t_2)}{\delta}e^{(\delta+u_1)/\delta}\Psi\left(\frac{\delta+u_1}{\delta}\right) \\ \quad \left. - \frac{u_1\lambda t_1}{\delta+u_1} - \frac{u_2\lambda t_2}{\delta+u_2}\right\}, & t_1 > t_2 \end{cases}$$

which is combination of two independent gamma processes and the Bivariate compount Poisson process with rate λ and the joint density function $((\delta+\delta y)(\delta+\delta x)-1)\exp(-\delta x - \delta y - \delta xy)$.

4. Estimating the Bivariate survival function

In this section we develop general formulation and describe the methodology. We also illustrate the technique using specific stochastic model. Suppose the expression for the survival function $S(t_1, t_2)$ is known except for several unknown parameters. If we have observations from $S_1(t)$ and $S_2(t)$ on a discrete time grid, say $0 < t_1 < t_2 < \cdots < t_n$, then one can write down the likelihood function and estimate the unknown parameters as well as $S(t_1, t_2)$. Specifically, one can show that the joint transition density of $S_1(t)$ and $S_2(t)$ is

$$\begin{aligned} &p(s_1, s_2, x_1, x_2; t_1, t_2, y_1, y_2) \\ &= \frac{\partial^2}{\partial y_1 \partial y_2} P(S_1(t_1) \le y_1, S_2(t_2) \le y_2 \mid S_1(s_1) = x_1, S_2(s_2) = x_2) \\ &= \frac{\partial^2}{\partial y_1 \partial y_2} P(S_1(t_1 - s_1) \le y_1 - x_1, S_2(t_2 - s_2) \le y_2 - x_2). \end{aligned}$$

Thus, the log-likelihood is

$$\sum_{i=1}^{n} \log p(t_{i-1}, t_{i-1}, S_1(t_{i-1}), S_2(t_{i-1}); t_i, t_i, S_1(t_i), S_2(t_i)). \tag{4.1}$$

Here to avoid complexity, we assume that $S_1(t)$ and $S_2(t)$ are observed at the same time points.

Unfortunately, in our case, it is impossible to observe the processes $S_1(t)$ and $S_2(t)$ after T_1 and T_2 respectively. More precisely, we only have observations from the killed processes

$$S_i^*(t) = \begin{cases} S_i(t), & t \leq T_i \\ \Delta_i, & t > T_i \end{cases}, \quad i = 1,2, \tag{4.2}$$

on a discrete time grid. Now, to write the likelihood function based on observations from $S_1^*(t)$ and $S_2^*(t)$ we need to know the joint transition density of $S_1^*(t)$ and $S_2^*(t)$. The following lemma gives the joint transition density.

LEMMA 4.1. *For $s_i < t_i$, $i = 1,2$ and $x_i < \Delta_i$, $i = 1,2$. The joint transition density of $S_1^*(t)$ and $S_2^*(t)$ is*

$$\begin{aligned} &q(s_1, s_2, x_1, x_2; t_1, t_2, y_1, y_2) \\ &= \frac{\partial^2}{\partial y_1 \partial y_2} P(S_1^*(t_1) \leq y_1, S_2^*(t_2) \leq y_2 \mid S_1^*(s_1) = x_1, S_2^*(s_2) = x_2) \\ &= \begin{cases} p(s_1, s_2, x_1, x_2; t_1, t_2, y_1, y_2), \quad y_1 < \Delta_1, \quad y_2 < \Delta_2 \\ -\dfrac{\partial^2}{\partial y_1 \partial t_2} P(S_1(t_1) \leq y_1, S_2(t_2) < \Delta_2 \mid S_i(s_i) = x_i, i = 1,2), \\ \qquad y_1 < \Delta_1, \quad y_2 = \Delta_2 \\ -\dfrac{\partial^2}{\partial y_2 \partial t_1} P(S_1(t_1) \leq \Delta_1, S_2(t_2) \leq y_2 \mid S_i(s_i) = x_i, i = 1,2), \\ \qquad y_1 = \Delta_1, \quad y_2 < \Delta_2 \\ \dfrac{\partial^2}{\partial t_1 \partial t_2} P(T_1 \leq t_1, T_2 \leq t_2) \quad y_1 = \Delta_1, \quad y_2 = \Delta_2. \end{cases} \end{aligned}$$

PROOF. First assume that $y_i < \Delta_i$, $i = 1,2$. Then,

$$\begin{aligned} &q(s_1, s_2, x_1, x_2; t_1, t_2, y_1, y_2) \\ &= \frac{\partial^2}{\partial y_1 \partial y_2} P(S_1^*(t_1) \leq y_1, S_2^*(t_2) \leq y_2, T_1 > t_1, T_2 > t_2 \mid S_1^*(s_i) = x_i, i = 1,2) \\ &= \frac{\partial^2}{\partial y_1 \partial y_2} P(S_1^*(t_1) \leq y_1, S_2^*(t_2) \leq y_2, \\ &\qquad S_1(t_1) < \Delta_1, S_2(t_2) < \Delta_2 \mid S_i^*(s_i) = x_i, i = 1,2) \\ &= p(s_1, s_2, x_1, x_2; t_1, t_2, y_1, y_2). \end{aligned}$$

Now, assume that $y_2 = \Delta_2$, $y_1 < \Delta_1$, then

$$\begin{aligned} &q(s_1, s_2, x_1, x_2; t_1, t_2, a_1, y_2) \\ &= -\frac{\partial^2}{\partial y_1 \partial t_2} P(S_1(t_1) \leq y_1, T_2 > t_2 \mid S_i(s_i) = x_i, i = 1,2) \\ &= -\frac{\partial^2}{\partial y_1 \partial t_2} P(S_1(t_1) \leq y_1, S_2(t_2) \leq \Delta_2 \mid S_i(s_i) = x_i, i = 1,2). \end{aligned}$$

Similar arguments can be used for other two cases.

To estimate the unknown parameters from discrete observations of $S_1^*(t)$ and $S_2^*(t)$ at time points $0 < t_1 < \cdots < t_n$, the log likelihood function is

$$l_n = \sum_{i=1}^{n} \log q(t_{i-1}, t_{i-1}, S_1^*(t_{i-1}), S_2^*(t_{i-1}); t_i, t_i, S_1^*(t_i), S_2^*(t_i)). \tag{4.3}$$

One can handle the estimation of unknown parameters in the usual way by differentiating the log-likelihood l_n.

Now, we apply our methodology to the model described in Example 3.1. Here we have $\{(S_1(t), S_2(t)); t \geq 0\}$ which is the bivariate gamma process with parameters a_1, a_2, a_3 and b. We assume that a_1, a_2, a_3 are known. Otherwise, suppose we can observe the process $N_1(t), N_2(t)$ and $N_3(t)$ for a time interval of (arbitrarily short) length say h, $h > 0$, the values a_1, a_2 and a_3 can be obtained as a limit,

$$A_i(x) = -\frac{N_i(h; x)}{h \log x}, \qquad i = 1, 2, 3,$$

then $\lim_{x \to 0} A_i(x) = a_i$, $i = 1, 2, 3$. Here $N_3(h; x) = N_3(h; x, x)$.

In this case,

$$\begin{aligned} &p(s, s, x_1, x_2; t, t, y_1, y_2) \\ &= \frac{\partial}{\partial y_1 \partial y_2} P(S_1(t-s) \leq y_1 - x_1, S_2(t-s) \leq y_2 - x_2) \\ &= \left(\frac{(b)^{a_i(t-s)}}{\prod_{i=1}^{3} (a_i(t-s))} \right) \exp(-b(y_1 - x_1) - b(y_2 - x_2)) \\ &\times \int_0^{\min(y_1 - x_1, y_2 - x_2)} x_j^{a_3(t-s)-1} (y_1 - x_1 - x_0)^{a_1(t-s)-1} \\ &\times (y_2 - x_2 - x_0)^{a_2(t-s)-1} \exp(b x_0) dx_0. \end{aligned}$$

Using Lemma 4.1, for $y_i < \Delta_i$, $i = 1, 2$, $q(s, s, x_1, x_2; t, t, y_1, y_2) = p(s, s, x_1, x_2, t, t, y_1, y_2)$. Now, one can easily show that for $y_1 = \Delta_1$ and $y_2 < \Delta_2$, $q(s, s, x_1, x_2; t, t, \Delta_1, y_2) = -\frac{\partial^2}{\partial y_2 \partial t_1} [\int_0^{y_2 - x_2} \int_0^{\Delta_1 - x_1} h(u_1, u_2) du_1 du_2] \mid_{t_1 = t_2 = t}$, for $y_2 = \Delta_2$, $y_1 < \Delta_1$, $q(s, s, x_1, x_2; t, t, y_1, \Delta_2) = \frac{\partial^2}{\partial y_1 \partial t_2} [\int_0^{\Delta_2 - x_2} \int_0^{y_1 - x_1} h(u_1, u_2) du_1 du_2] \mid_{t_1 = t_2 = t}$ and finally for

$$\begin{aligned} &y_1 = \Delta_1, \qquad y_2 = \Delta_2, \\ &q(s, s, x_1, x_2; t, t, \Delta_1, \Delta_2) = \frac{\partial^2}{\partial t_1 \partial t_2} \int_0^{\Delta_1 - x_1} \int_0^{\Delta_2 - x_2} [h(u_1, u_2) du_1 du_2] \mid_{t_1 = t_2 = t}. \end{aligned}$$

Here for $t_1 < t_2$, $h(u_1, u_2) = \frac{\exp(-bu_1 - bu_2)}{\Gamma(a_1(t_1 - s))\Gamma(a_3(t_1 - s))\Gamma(a_2(t_2 - s) + a_3(t_2 - t_1))} \int_0^{\min(u_1, u_2)} \times x_0^{a_3(t_1 - s)} (u_1 - x_0)^{a_1(t_1 - s) - 1} (u_2 - x_0)^{a_2(t_2 - s) + a_3(t_2 - t_1)} \exp(b x_0) dx_0$.

The log-likelihood function is,

$$l_n(b) = \sum_{i=1}^{n} \log q(t_{i-1}, t_{i-1}, S_1(t_{i-1}), S_2(t_{i-1}); t_i, t_i, S_1(t_i), S_2(t_i); b).$$

The maximum likelihood of b can be obtained by solving the non-linear equation

$$\frac{\partial l_n(b)}{\partial b} = 0. \tag{4.4}$$

Since the equation (4.4) cannot be solved analytically for b some numerical method must be used. Solution of the equation (4.4) with an iterative procedure such as Newton Raphson method may be employed here to obtain $\hat{b}$. The initializing value of b in such a scheme may be obtained by pretending the observations $S_i^*(t_j)$, $i = 1, 2$, $j = 1, \ldots, n$ are from the original bivariate process $S_1(t)$ and $S_2(t)$. Now, from the equation (2.3),

$$E(S_1(t)) = \left(\frac{a_1 + a_3}{b}\right) t$$

$$E(S_2(t)) = \left(\frac{a_2 + a_3}{b}\right) t$$

one can use

$$b^* = \frac{1}{n}\left[\sum_{i=1}^{n} \frac{S_1(t_i)}{(a_1 + a_3)t_i} + \sum_{i=1}^{n} \frac{S_2(t_i)}{(a_2 + a_3)t_i}\right]$$

as the initializing value for b.

It should be noted that, in general, when the likelihood function is not well behaved or when even first derivatives of the log-likelihood with respect to unknown parameters are inconvenient to compute, a direct search procedure for finding the maximum likelihood estimators may be the best approach.

As an illustration, we use the data generated from the bivariate Gamma process with $a_1 = a_2 = a_3 = 1$ and different values for b from Example 3.1. The bivariate data were truncated at known values $\Delta_1 = \Delta_2 = \Delta$. The numerical results arising from 20 sets of simulated data are given. The Appendix contains the full descriptions of this simulation. Here, let $\Delta = 20$ and $n = 50$ for all cases. Also, let the data be equally spaced: $t_j - t_{j-1} = \gamma$, $j = 1, \ldots, 50$, $S_i(t_0) = 0$, $i = 1, 2$ and $t_0 = 0$.

The results are shown in Table 1. From Table 1, one can observe that our estimators agree well with actual values. However, sometimes there is a larger difference between the actual value of the parameter and its maximum likelihood estimator. This discrepancy between the estimate and the actual value can be a consequence of γ being large. Table 1 shows that as γ gets smaller the difference between the actual value of the parameter and its maximum likelihood estimator gets larger. This make sense, because larger γ implies the processes $S_i^*(t)$, $i = 1, 2$, and being monitored longer.

Given the estimator $\hat{b}$, the bivariate survival function can be estimated by using the equation (2.4). One can assess the bivariate survival function for which the failure data is not available by monitoring the bivariate process $(S_1^*(t), S_2^*(t))$. □

Table 1. Point estimate of b and the standard error estimate.

	$b = 1$	$b = 2$	$b = \frac{1}{2}$
$\gamma = 1$	$\hat{b} = 1.05$	$\hat{b} = 2.1$	$\hat{b} = .52$
	(0.03)	(0.09)	(0.09)
$\gamma = \frac{1}{2}$	$\hat{b} = 1.12$	$\hat{b} = 2.15$	$\hat{b} = .57$
	(0.21)	(0.26)	(0.16)
$\gamma = \frac{1}{4}$	$\hat{b} = 1.18$	$\hat{b} = 2.29$	$\hat{b} = .60$
	(0.23)	(0.28)	(0.17)

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Appendix

This describes, the way 20 observations were generated from the bivariate Gamma process with $a_1 = a_2 = a_3 = 1$, $\Delta = 20$, $\gamma = 1$, $b = 1$. A similar procedure was used to generate observations for various values of a_1, a_2, a_3, Δ and γ. The IMSL Gamma random number generator was used to generate observations from the Gamma distribution. At a specified time point say $t = 1$, three observations were generated. The first observation y_1, was generated from the random variable $Y_1, Y_1 \sim \text{Gamma}(1,1)$. The second observation y_2, was generated from the random variable $Y_2, Y_2 \sim \text{Gamma}(1,1)$. Finally the third observation y_3 was generated from the random variable $Y_3, Y_3 \sim \text{Gamma}(1,1)$. Here Y_1, Y_2 and Y_3 are assumed to be independent. Using the equation (3.3) and the result from Johnson and Kotz ((1972), p. 219) it is clear that $(Y_1 + Y_2, Y_1 + Y_3)$ is an observation from $(S_1(1), S_2(1))$.

This procedure results in 50 observations from the bivariate Gamma process: $(S_1(j), S_2(j))$, $j = 1, \ldots, 50$. To obtain 50 observations from $(S_1^*(j), S_2^*(j))$, define, for $j = 1, \ldots, 50$,

$$S_i^*(j) = \begin{cases} S_i(j) & \text{if } \; S_i(j) < 20 \\ 20 & \text{if } \; S_i(j) \geq 20 \end{cases}, \qquad i = 1, 2.$$

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