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WHEN DOES THE UNION OF RANDOM SPHERICAL CAPS BECOME CONNECTED?

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Abstract. Drop N random caps all of the same angular radius $\theta = c\sqrt{\frac{1}{N}\log N}$ on a unit sphere. Let U denote the part of the surface covered by these caps. We prove that if $c > \sqrt{2}$, then the probability that U is connected tends to 1 as $N \to \infty$, while if $c < 1$, then the probability that U is connected tends to 0 as $N \to \infty$.

Key words and phrases: Coverage problem, random caps, asymptotic probability.

1. Introduction

Consider N random spherical caps $C_1, C_2, \dots, C_N$ of the same angular radius θ placed on the surface of a unit sphere in 3-space. We suppose that the centers of these caps are independently and uniformly distributed over the surface of the sphere. Then, what is the probability that the union of the caps is connected? Penrose (1997, 1999*b*) considered a similar problem in a unit cube, and in a region of the d-dimensional Euclidean space, $d \geq 2$, rather than on the surface of the sphere. For a related epidemic problem, see Penrose (1997).

Let U denote the part of the sphere covered by the N random caps, that is, $U = C_1 \cup C_2 \cup \cdots \cup C_N$. Let $P(\theta, N)$ denote the probability that U is connected. We prove the following inequality.

THEOREM 1.1. $P(\theta, N) \geq 1 - N(\cos^2\theta)^{N-1} - N(N-1)(\sin^2\theta)(1 - \frac{1}{2}\sin^2\theta)^{N-2}$.

For example, $P(\frac{\pi}{3}, 15) > 0.65$, $P(\frac{\pi}{6}, 60) > 0.61$, $P(\frac{\pi}{10}, 200) > 0.76$.
This inequality is applied to prove the following asymptotic result.

THEOREM 1.2. *Let* $\theta = c\sqrt{\frac{1}{N}\log N}$. *If* $c < 1$ *then* $P(\theta, N) \to 0$, *while if* $c > \sqrt{2}$ *then* $P(\theta, N) \to 1$ *as* $N \to \infty$.

The former part of Theorem 1.2 is proved by showing that if $c < 1$ then there is an isolated cap almost surely. This follows also from a more general result of Penrose (1999*a*), where he gave a strong law for the threshold at which there are no isolated caps on a compact C^2 Riemannian d-manifold.

In the range $1 < c < \sqrt{2}$, we do not know the asymptotic probability. It is known (Maehara (1990)) that in the circle case, the threshold for no isolated arcs and the threshold for connectedness are different. More precisely, when we drop N random arcs of the length $2\pi c/\log N$ on the circumference of the unit circle, then $c = 1$ is the threshold for no isolated arcs, while $c = 2$ is the threshold for connectedness. On the other hand, it is known that the thresholds for connectedness and for no isolated caps are the same in a region of d-dimensional space R^d (Penrose (1999b)), and in the d-dimensional torus, $d \geq 2$ (Penrose (1997)).

Threshold for complete coverage (that is, the union of random caps covers the surface completely) is known (Maehara (1988)) in the case of d-dimensional sphere, $d \geq 1$. For related other problems, see Solomon (1978) Chapter 4, and for coverage processes in Euclidean space, see Hall (1988).

2. Proof of Theorem 1.1

For each $i = 1, 2, \ldots, N$, let v_i denote the center of C_i and let D_i denote the cap of angular radius 2θ concentric with C_i. The area of D_i is

$$2\pi(1 - \cos 2\theta) = 4\pi \sin^2 \theta.$$

A spherical cap is divided into two *half-caps* by a great circle passing through the center of the cap. A cap D_i is called *half-empty* if the interior of a half-cap of D_i contains no v_j, $j \neq i$.

LEMMA 2.1. *If $U = C_1 \cup C_2 \cup \cdots \cup C_N$ is disconnected, then some D_i is half-empty.*

PROOF. Suppose that U is disconnected, and let W_1, W_2 be two connected components of U. Then there is a simple closed curve Γ on the unit sphere such that (i) Γ separates W_1 from W_2, and (ii) no cap C_j intersects the curve Γ. Let p be a point on Γ and let p^* be its antipodal point. If necessary, considering W_2 instead W_1, we may suppose that p^* lies either in the same side of Γ as W_2 or on Γ. Then we can connect p and p^* by a curve on the sphere avoiding W_1.

From now on, let us regard p as the North Pole and p^* as the South Pole. Since every cap contained in W_1 never meets the Poles, the longitude of the center of the cap is uniquely determined, which is called simply the *longitude of the cap*. If the difference of the longitudes of two caps C_i, C_j in W_1 is equal to π, then the shortest arc connecting their centers passes through p or p^*. Hence, $C_i \cap C_j = \emptyset$. That is, if two caps in W_1 intersect, then the difference of their longitudes is less than π, and we can tell which lies east (or west) of the other, unless they have the same longitude.

Now, starting from a cap in W_1, we try to go east and east by moving from a cap to another one that intersects the cap and lying east. Then, sooner or later, either (1) we meet again a cap we already visited, or (2) we reach an *eastern-most* cap and cannot go further. If (1) happens, then the union of the caps we visited forms a *belt* that separates p from p^*. This contradicts that p and p^* can be connected by a curve on the sphere with avoiding W_1. Hence (2) happens. Then the cap of angular radius 2θ concentric with the eastern-most cap is half empty.

LEMMA 2.2. *Let B be a unit disk in the xy-plane centered at the origin, and let $p_1, p_2, \ldots, p_k$ be k (> 0) random points distributed independently and uniformly in B. Then the probability that there is a half plane $ax + by \leq 0$ containing these k points is equal to $2k/2^k$.*

PROOF. Since the probability that some p_i coincides with the origin is zero, and the probability that some two points p_i, p_j coincide is also zero, we may suppose that o

(the origin), $p_1, p_2, \ldots, p_k$ are all different. For each i, let $q_i = p_i/\|p_i\|$. Then $q_1, \ldots, q_k$ are distributed independently and uniformly on the boundary circle ∂B. There is a half plane $ax + by \leq 0$ containing $p_1, \ldots, p_k$ if and only if $q_1, \ldots, q_k$ are contained in some semicircle of ∂B. So, we compute the probability that k points distributed independently and uniformly on the circle ∂B are contained in some semicircle of ∂B. This probability is equal to the probability that the largest spacing among these points is greater than or equal to π. Since the largest spacing has a unique counter-clockwise endpoint, this is the k times the probability that the circular interval of length π beginning from q_1 contains all the other $k-1$ points, and since each of these points has probability $1/2$ of lying in that interval, this gives the result.

By the same reasoning we have the following.

COROLLARY 2.1. *Under the condition that certain k (> 0) vertices lie on D_i and the remaining $N-1-k$ vertices lie outside D_i, the probability that D_i is half-empty is equal to $2k/2^k$.*

PROOF OF THEOREM 1.1. First, we compute the probability that D_i is half-empty. The probability that certain k vertices lie on D_i and the remaining $N-1-k$ vertices lie outside D_i is $(\sin^2\theta)^k(1-\sin^2\theta)^{N-1-k} = (\sin^2\theta)^k(\cos^2\theta)^{N-1-k}$. Hence, by the above corollary,

$$\begin{aligned}
&\Pr(D_i \text{ is half-empty}) \\
&\quad = (\cos^2\theta)^{N-1} + \sum_{k=1}^{N-1}\binom{N-1}{k}(\sin^2\theta)^k(\cos^2\theta)^{N-1-k}\frac{2k}{2^k} \\
&\quad = (\cos^2\theta)^{N-1} + 2\sum_{k=0}^{N-1}\binom{N-1}{k}k\left(\frac{\sin^2\theta}{2}\right)^k(\cos^2\theta)^{N-1-k}.
\end{aligned}$$

Since $\sum_{k=0}^{n}\binom{n}{k}kx^ky^{n-k} = x\cdot\frac{\partial}{\partial x}(x+y)^n = nx(x+y)^{n-1}$, we have

$$\begin{aligned}
\Pr(D_i \text{ is half-empty}) &= (\cos^2\theta)^{N-1} + \sin^2\theta(N-1)\left(\frac{1}{2}\sin^2\theta + \cos^2\theta\right)^{N-2} \\
&= (\cos^2\theta)^{N-1} + (N-1)\sin^2\theta\left(1 - \frac{1}{2}\sin^2\theta\right)^{N-2}.
\end{aligned}$$

Therefore, the probability that some D_i is half-empty is at most

$$N(\cos^2\theta)^{N-1} + N(N-1)\sin^2\theta\left(1-\frac{1}{2}\sin^2\theta\right)^{N-2}.$$

Since 'U is connected' implies that there is no half-empty cap D_i, we have

$$P(\theta, N) \geq 1 - N(\cos^2\theta)^{N-1} - N(N-1)\sin^2\theta\left(1-\frac{1}{2}\sin^2\theta\right)^{N-2}.$$

3. Proof of Theorem 1.2

LEMMA 3.1. *Let $f = f(N)$, $g = g(N)$ be two nonnegative function and suppose $f \to 0$ as $N \to \infty$. Then $(1-f)^g < e^{-f\cdot g}$ holds for sufficiently large N. Furthermore, if $f^2 \cdot g \to 0$, then $(1-f)^g \sim e^{-f\cdot g}$, that is,*

$$(1-f)^g = e^{-f\cdot g}(1+o(1)).$$

PROOF. For $0 < t < 1$, $\log(1-t)$ is written as

$$\log(1-t) = -t - \frac{t^2}{2(1-\lambda t)^2}, \qquad 0 < \lambda < 1.$$

Hence, for sufficiently large N (such as $f < 1$),

$$g \cdot \log(1-f) = -f \cdot g - \frac{f^2 \cdot g}{2(1-\lambda f)^2} < -f \cdot g.$$

Therefore $(1-f)^g < e^{-f\cdot g}$. If $f^2 \cdot g \to 0$, then, since

$$\frac{f^2 \cdot g}{2(1-\lambda f)^2} \to 0 \qquad (N \to \infty),$$

we have $(1-f)^g = e^{-f\cdot g}(1+o(1))$.

PROOF OF THEOREM 1.2. First, suppose $c > \sqrt{2}$. To prove $P(\theta, N) \to 1$ as $N \to \infty$, it is enough to show that $N(\cos^2\theta)^{N-1} + N(N-1)\sin^2\theta(1-\frac{1}{2}\sin^2\theta)^{N-2} \to 0$. Since $\theta = c\sqrt{\frac{1}{N}\log N} \to 0$, we have $\sin^2\theta = \theta^2(1+o(1)) = (1+o(1))\frac{c^2}{N}\log N$. Since $(\theta^2)^2 N = o(1)$, by applying Lemma 3.1, we have

$$\begin{aligned}
&N(1-\sin^2\theta)^{N-1} + N(N-1)\sin^2\theta\left(1 - \frac{1}{2}\sin^2\theta\right)^{N-2} \\
&\quad \sim Ne^{-N\theta^2} + N^2\theta^2 e^{-N(1/2)\theta^2} \sim Ne^{-c^2\log N} + N(c^2\log N)e^{-(c^2/2)\log N} \\
&\quad \sim \frac{1}{N^{c^2-1}} + \frac{c^2\log N}{N^{c^2/2-1}},
\end{aligned}$$

which tends to 0 since $c^2 > 2$.

Now, suppose that $c < 1$. We have to show that $P(\theta, N) \to 0$. Denote by v_i the center of C_i. Then $v_1, \ldots, v_N$ are independently and uniformly distributed on the sphere. Let D_i denote the cap of angular radius 2θ with center v_i as in Section 2. Then the area of D_i is equal to $4\pi\sin^2\theta$. Since $C_i \cap C_j \neq \emptyset \Leftrightarrow v_j \in D_i$,

$$\Pr(C_i \cap C_j \neq \emptyset) = \sin^2\theta \sim \theta^2.$$

Similarly,

$$\Pr(D_i \cap D_j \neq \emptyset) = \sin^2 2\theta \sim 4\theta^2.$$

Let us call a cap C_i is *isolated* if $C_i \cap C_j = \emptyset$ for all $j \neq i$. Then, if some cap C_i is isolated, U is not connected. Let X denote the number of isolated caps in $C_1, \ldots, C_N$.

For each i, $1 \le i \le N$, let X_i denote the random variable such that $X_i = 1$ if C_i is isolated cap, and $X_i = 0$ otherwise. Then $X = X_1 + X_2 + \cdots + X_N$. Since $\theta^4 N = o(1)$, and since C_i is isolated if and only if D_i contains no v_j, $j \ne i$, the expected value of X_i is

$$\begin{aligned} E(X_i) &= \Pr(X_i = 1) \sim (1-\theta^2)^{N-1} \\ &\sim e^{-N\theta^2} \sim e^{-c^2 \log N} = N^{-c^2}. \end{aligned}$$

Thus the expected value of $X = X_1 + \cdots + X_N$ is

$$E(X) \sim NE(X_1) \sim N^{1-c^2}.$$

Next, we consider the expected value $E(X_iX_j)$, $i \ne j$.

$$E(X_iX_j) < \Pr(D_i \cap D_j = \emptyset)(1-2\theta^2)^{N-2} + \Pr(D_i \cap D_j \ne \emptyset)(1-\theta^2)^{N-2}.$$

Since $\Pr(D_i \cap D_j \ne \emptyset) \sim 4\theta^2$, we have

$$\begin{aligned} E(X_iX_j) &< (1-4\theta^2)(1-2\theta^2)^{N-2} + 4\theta^2(1-\theta^2)^{N-2} \\ &\sim (1-4\theta^2)e^{-2N\theta^2} + 4\theta^2 e^{-N\theta^2} \\ &\sim (1-4\theta^2)N^{-2c^2} + (4c^2 \log N)N^{-(1+c^2)} \\ &= N^{-2c^2}(1 - 4\theta^2 + (4c^2 \log N)N^{c^2-1}) \sim N^{-2c^2}. \end{aligned}$$

Hence

$$\begin{aligned} E(X^2) &= \sum_{i,j} E(X_iX_j) = \sum_i E(X_i^2) + \sum_{i \ne j} E(X_iX_j) \\ &= NE(X_1) + N(N-1)E(X_1X_2) \\ &< N \cdot N^{-c^2} + N^2 \cdot N^{-2c^2} = N^{2(1-c^2)}\left(\frac{1}{N^{1-c^2}} + 1\right) \\ &\sim N^{2(1-c^2)} \sim E(X)^2. \end{aligned}$$

Since $E(X^2) \ge E(X)^2$ holds generally, we have $E(X^2) \sim E(X)^2$. Now, applying Chebyshev's inequality,

$$\Pr(X = 0) \le \Pr(|X - E(X)| \ge E(X)) < \frac{E(X^2) - E(X)^2}{E(X)^2} \to 0.$$

Hence $\Pr(X \ge 1) \to 1$ as $N \to \infty$. Since $X \ge 1$ implies that U is disconnected, we have $P(\theta, N) \to 0$ as $N \to \infty$.

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