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LINEAR RELATIVE CANONICAL ANALYSIS OF EUCLIDEAN RANDOM VARIABLES, ASYMPTOTIC STUDY AND SOME APPLICATIONS

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Abstract. We introduce the Linear Relative Canonical Analysis (LRCA) of Euclidean random variables. Then similar properties than for usual linear Canonical Analysis are obtained. Furthermore, we develop an asymptotic study of LRCA and apply the obtained results to tests for lack of relative linear association, dimensionality and invariance.

Key words and phrases: Canonical analysis, relative canonical analysis, asymptotic study, linear relative association, invariance, additional information, (relative) canonical coefficient, partial canonical correlation.

1. Introduction

In some practical situations, one may have to make a statistical analysis on some variables when there is a noise variable. An approach for such situations consists in making the chosen analysis after removing the effect of this later variable; this is done by considering residuals of regression on that variable. This approach has led to known methods; an example can be found in discriminant analysis with a covariate which is known to have the same mean in the related groups (see e.g. Fujikoshi and Khatri (1990), Baccini *et al.* (2001)). For canonical analysis, the same approach gave partial (see Rao (1969)) part and bipartial (see Timm and Carlson (1976)) canonical correlation analysis. In this later work, statistical inferences based on the canonical coefficients related to these analyses were proposed. Although the obvious interest of these methods, it seems that there does not exist an extensive study of their properties as it is the case for usual Linear Canonical Analysis (LCA). Particularly, the aforementioned statistical inference shows the interest of making an asymptotic study of these analyses.

In this paper, we define the Linear Relative Canonical Analysis (LRCA) of Euclidean random variables and show that this analysis is in fact a LCA for suitable random variables and can be seen as a generalization of partial canonical correlation analysis. Then, some properties of LRCA are obtained from those of LCA. Next, we focus on the asymptotic study of LRCA. Although this analysis is a particular LCA, the results of asymptotic studies of LCA (see Arconte (1980), Pousse (1992), Anderson (1999), Fine (2000)) can not be applied because we do not have an i.i.d. sample of the related random variables. Indeed, these variables can not be observed since their definition involves covariance operators which are unknown in practice. So, we develop an asymptotic study for LRCA. The obtained results are used for defining tests for lack of relative linear association, dimensionality for LRCA, and invariance when the corresponding variables are transformed by linear maps (see the end of Section 2 for definition and properties and Section 4.3 for inference procedure).

2. Linear relative canonical analysis

Let $(\Omega, \mathcal{A}, P)$ be a probability space; along the paper we will work with random variables (r.v.) defined on $(\Omega, \mathcal{A}, P)$ and valued into Euclidean spaces (i.e. finite dimensional Hilbertian space). When F is such a space, we will denote by $\langle \cdot, \cdot \rangle_F$ its inner product and by $\|\cdot\|_F$ the associated norm. We will use the usual tensor product $\otimes$ such that, for any vectors u and v belonging to Euclidean spaces F and G respectively, $u \otimes v$ is the linear map: $h \in F \mapsto \langle h, u \rangle_F v \in G$; if $u = v$ we will write $u^{\otimes^2}$ instead of $u \otimes u$. The properties of $\otimes$ (and other tensor products) and the related matrix expressions can be found in Dauxois *et al.* (1994). When X is a random variable valued into an Euclidean space $\mathcal{X}$ and satisfying $\mathbb{E}(\|X\|_{\mathcal{X}}^2) < +\infty$, we will denote by L_X the linear map: $u \in \mathcal{X} \mapsto \langle u, X \rangle_{\mathcal{X}} \in L^2(\Omega, \mathcal{A}, P)$. For all operator T, we will denote by T^* its adjoint. It is easily seen that L_X^* is the map: $Z \in L^2(\Omega, \mathcal{A}, P) \mapsto \mathbb{E}(ZX) \in \mathcal{X}$; thus if X is centered, its covariance operator $V_X := \mathbb{E}(X^{\otimes^2})$ verifies $V_X = L_X^* L_X$.

For $m \in \{1, 2, 3\}$, let us consider a centered r.v. X_m defined on $(\Omega, \mathcal{A}, P)$ and valued into a Euclidean space $\mathcal{X}_m$ with dimension p_m; without loss of generality we assume that $p_1 \leq p_2$. Further, we suppose that $\mathbb{E}(\|X_m\|_{\mathcal{X}_m}^2) < +\infty$ and we define

$$E_m := R(L_{X_m}), \tag{2.1}$$

where $R(T)$ denotes the range of the operator T, and for $k \in \{1, 2\}$

$$E_{k \cdot 3} := (E_k + E_3) \ominus E_3, \tag{2.2}$$

where $\ominus$ denotes the orthogonal difference defined by: $E \ominus F = E \cap F^\perp$ where E and F are Euclidean spaces such that $F \subset E$. Notice that, denoting by Π_E the orthogonal projection operator onto the subspace E and by $E^\perp$ its orthogonal space, one has

$$E_{k \cdot 3} = \Pi_{E_3^\perp}(E_k). \tag{2.3}$$

Now, we define:

DEFINITION 2.1. The linear relative canonical analysis (LRCA) of X_1 and X_2 relative to X_3 is the canonical analysis (CA) of $E_{1 \cdot 3}$ and $E_{2 \cdot 3}$.

Notice that since $E_{1 \cdot 3}$ and $E_{2 \cdot 3}$ are finite-dimensional spaces, the previous CA is a linear CA (LCA) of random vectors (see Dauxois and Pousse (1975), Dauxois and Nkiet (1997*a*)). When these spaces are $\mathbb{R}^p$-type ones, the components of these random vectors are r.v. which span the aforementioned spaces. In the more general framework of Euclidean spaces, these random vectors (say $X_{1 \cdot 3}$ and $X_{2 \cdot 3}$) are such that $E_{1 \cdot 3} = R(L_{X_{1 \cdot 3}})$ and $E_{2 \cdot 3} = R(L_{X_{2 \cdot 3}})$. Now, we will search such random vectors. In order to simplify notation, we will write L_m (resp. V_m) instead of L_{X_m} (resp. V_{X_m}). Let us consider

$$V_{mj} := L_m^* L_j = \mathbb{E}(X_j \otimes X_m) \quad \text{for} \quad (m,j) \in \{1,2,3\}^2 \quad \text{with} \quad m \neq j$$

and put

$$\begin{aligned} L_{k\cdot 3} &:= L_k - L_3 V_3^\dagger V_{3k} \\ X_{k\cdot 3} &:= X_k - V_{k3} V_3^\dagger X_3 \qquad (k = 1,2), \end{aligned} \tag{2.4}$$

where $T^\dagger$ denotes the Moore-Penrose inverse of the operator T. We obtain

Lemma 2.1. *For $k \in \{1,2\}$, one has:*
(i) $E_{k\cdot 3} = R(L_{k\cdot 3})$;
(ii) $L_{k\cdot 3} = L_{X_{k\cdot 3}}$.

PROOF. (i) From equation (2.3), we have $u \in E_{k\cdot 3}$ if, and only if, there exists $x \in E_k$ such that $u = \Pi_{E_3^\perp} x = x - \Pi_{E_3} x$. Thus, from the equality $\Pi_{E_3} = L_3 V_3^\dagger L_3^*$ and equation (2.1), $u \in E_{k\cdot 3}$ is equivalent to the existence of a vector $\alpha \in \mathcal{X}_k$ such that $u = L_k\alpha - L_3 V_3^\dagger L_3^* L_k \alpha$, that is $u \in R(L_{k\cdot 3})$.

(ii) For all $\alpha \in \mathcal{X}_k$, we have: $L_{k\cdot 3}\alpha = \langle \alpha, X_k\rangle_{\mathcal{X}_k} - \langle V_3^\dagger V_{3k}\alpha, X_3\rangle_{\mathcal{X}_3} = \langle \alpha, X_k - V_{k3} V_3^\dagger X_3\rangle_{\mathcal{X}_k} = L_{X_{k\cdot 3}}\alpha$; this proves the lemma. □

Since $E_{k\cdot 3} = R(L_{X_{k\cdot 3}})$, we can state:

PROPOSITION 2.1. *The LRCA of X_1 and X_2 relative to X_3 is the LCA of the random variables $X_{1\cdot 3} = X_1 - V_{13} V_3^\dagger X_3$ and $X_{2\cdot 3} = X_2 - V_{23} V_3^\dagger X_3$.*

Let us consider the operators:

$$\begin{aligned} V_{k\cdot 3} &= L_{k\cdot 3}^* L_{k\cdot 3} = \mathbb{E}(X_{k\cdot 3}^{\otimes 2}), \\ V_{km\cdot 3} &= L_{k\cdot 3}^* L_{m\cdot 3} = \mathbb{E}(X_{m\cdot 3} \otimes X_{k\cdot 3}) \qquad \text{for} \quad m \neq k; \end{aligned}$$

we have:

$$\begin{aligned} V_{k\cdot 3} &= (L_k^* - V_{k3} V_3^\dagger L_3^*)(L_k - L_3 V_3^\dagger V_{3k}) \\ &= V_k - V_{k3} V_3^\dagger V_{3k}, \end{aligned}$$

and for $m \neq k$:

$$\begin{aligned} V_{km\cdot 3} &= (L_k^* - V_{k3} V_3^\dagger L_3^*)(L_m - L_3 V_3^\dagger V_{3m}) \\ &= V_{km} - V_{k3} V_3^\dagger V_{3m}. \end{aligned} \tag{2.5}$$

We deduce from the classical theory of LCA that the LRCA of X_1 and X_2 relative to X_3 is obtained for example from the spectral analysis of the selfadjoint operator

$$T_{1\cdot 3} = (V_{1\cdot 3}^\dagger)^{1/2} V_{12\cdot 3} V_{2\cdot 3}^\dagger V_{21\cdot 3} (V_{1\cdot 3}^\dagger)^{1/2},$$

where $T^{1/2}$ denotes the square root of a nonnegative operator T.

Remark 2.1. When $V_{1\cdot 3}$ and $V_{2\cdot 3}$ are invertible the spectral analysis of $T_{1\cdot 3}$ is equivalent to that of $T'_{1\cdot 3} = V_{1\cdot 3}^{-1}V_{12\cdot 3}V_{2\cdot 3}^{-1}V_{21\cdot 3}$. That is known in the literature as partial canonical correlation analysis (see e.g. Rao (1969), Timm and Carlson (1976)). Then this last analysis appears as a particular case of the general relative canonical analysis of subspaces (see Dauxois and Nkiet (2002), Dauxois *et al.* (2004)), obtained by considering subspaces generated by specific linear functions of the original variables. In order to show up this property we prefer to use the terminology linear relative canonical analysis instead of partial canonical analysis. Notice that the part and bipartial canonical correlation analysis developed by Timm and Carlson (1976) can be reobtained from our framework by considering the CA of E_1 and $E_{2\cdot 3}$, and $E_{1\cdot 3}$ and $E_{2\cdot 4}$ respectively, where $E_{2\cdot 4}$ is constructed as in equation (2.2) with another Euclidean r.v. X_4.

The properties of LRCA are deduced from those of LCA. Hence the LRCA of X_1 and X_2 is characterized by a triple

$$\{(\rho_i)_{0\le i\le r}, (\alpha_{1\cdot 3}^{(i)})_{0\le i\le p_1}, (\alpha_{2\cdot 3}^{(i)})_{0\le i\le p_2}\} \tag{2.6}$$

(where r denotes the rank of $T_{1\cdot 3}$ and p_k is the dimension of $\mathcal{X}_k$, $k = 1,2$) satisfying:

(P1) for each $i \in \{1,\dots,r\}$, ρ_i^2 is the i-th greatest eigenvalue of $T_{1\cdot 3}$ and satisfies $0 < \rho_i \le \rho_{i-1} \le 1$ (with $\rho_0 = 1$);

(P2) the system $(\alpha_{1\cdot 3}^{(i)})_{0\le i\le p_1}$ is an orthonormal basis of $\mathcal{X}_1$ such that each $\alpha_{1\cdot 3}^{(i)}$ is an eigenvector of $T_{1\cdot 3}$ verifying:

- if $i \le r$, then $\alpha_{1\cdot 3}^{(i)}$ is associated with ρ_i^2
- if $i > r$, then $\alpha_{1\cdot 3}^{(i)}$ is associated with 0;

(P3) the system $(\alpha_{2\cdot 3}^{(i)})_{0\le i\le p_1}$ is an orthonormal basis of $\mathcal{X}_2$ such that:

- if $i \le r$, then $\alpha_{2\cdot 3}^{(i)} = \rho_i^{-1}(V_{2\cdot 3}^{\dagger})^{1/2}V_{21\cdot 3}(V_{1\cdot 3}^{\dagger})^{1/2}\alpha_{1\cdot 3}^{(i)}$; this equality is equivalent to $\alpha_{1\cdot 3}^{(i)} = \rho_i^{-1}(V_{1\cdot 3}^{\dagger})^{1/2}V_{12\cdot 3}(V_{2\cdot 3}^{\dagger})^{1/2}\alpha_{2\cdot 3}^{(i)}$ and then $\alpha_{2\cdot 3}^{(i)}$ is an eigenvector of $T_{2\cdot 3} := (V_{2\cdot 3}^{\dagger})^{1/2}V_{21\cdot 3}V_{1\cdot 3}^{\dagger}V_{12\cdot 3}(V_{2\cdot 3}^{\dagger})^{1/2}$ associated with ρ_i^2,
- if $i > r$, then $\alpha_{2\cdot 3}^{(i)}$ is an eigenvector of $T_{2\cdot 3}$ associated with 0.

The ρ_i's are termed the *(relative) canonical coefficients* associated to the LCRA; the *(relative) canonical variates* are the random variables defined for $(k,i) \in \{1,2\} \times \{0,\dots,p_k\}$ by:

$$f_{k\cdot 3}^{(i)} := L_{k\cdot 3}(V_{k\cdot 3}^{\dagger})^{1/2}\alpha_{k\cdot 3}^{(i)} = \langle X_{k\cdot 3}, (V_{k\cdot 3}^{\dagger})^{1/2}\alpha_{k\cdot 3}^{(i)}\rangle_{\mathcal{X}_k}. \tag{2.7}$$

Clearly, one has

$$(V_{k\cdot 3}^{\dagger})^{1/2}X_{k\cdot 3} = \sum_{i=1}^{p_k} f_{k\cdot 3}^{(i)}\alpha_{k\cdot 3}^{(i)} \tag{2.8}$$

and

$$\begin{aligned}
\mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)}) &= \langle L_{1\cdot 3}(V_{1\cdot 3}^{\dagger})^{1/2}\alpha_{1\cdot 3}^{(i)}, L_{2\cdot 3}(V_{2\cdot 3}^{\dagger})^{1/2}\alpha_{2\cdot 3}^{(j)}\rangle \\
&= \langle \alpha_{1\cdot 3}^{(i)}, (V_{1\cdot 3}^{\dagger})^{1/2}L_{1\cdot 3}^{*}L_{2\cdot 3}(V_{2\cdot 3}^{\dagger})^{1/2}\alpha_{2\cdot 3}^{(j)}\rangle_{\mathcal{X}_1} \\
&= \langle \alpha_{1\cdot 3}^{(i)}, (V_{1\cdot 3}^{\dagger})^{1/2}V_{12\cdot 3}(V_{2\cdot 3}^{\dagger})^{1/2}\alpha_{2\cdot 3}^{(j)}\rangle_{\mathcal{X}_1}.
\end{aligned}$$

If $j > r$, $\alpha_{2\cdot 3}^{(j)}$ belongs to the kernel of $T_{2\cdot 3}$, that is also the kernel of $(V_{1\cdot 3}^{\dagger})^{1/2}V_{12\cdot 3}(V_{2\cdot 3}^{\dagger})^{1/2}$; hence we have $\mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)}) = 0$. If $j \leq r$ then the use of (P3) and the orthonormality of the $\alpha_{1\cdot 3}^{(i)}$'s gives $\mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)}) = \rho_j\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Consequently, taking $\rho_i = 0$ for $i > r$, we have for any $(i,j) \in \{0,\dots,p_1\} \times \{0,\dots,p_2\}$

$$\mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)}) = \rho_j\delta_{ij}. \tag{2.9}$$

Moreover, when for $k \in \{1,2\}$, $V_{k\cdot 3}$ is invertible, one has for $(i,j) \in \{1,\dots,p_k\}^2$

$$\begin{aligned}\mathbb{E}(f_{k\cdot 3}^{(i)}f_{k\cdot 3}^{(j)}) &= \langle L_{k\cdot 3}V_{k\cdot 3}^{-1/2}\alpha_{k\cdot 3}^{(i)}, L_{k\cdot 3}V_{k\cdot 3}^{-1/2}\alpha_{k\cdot 3}^{(j)}\rangle \\ &= \langle \alpha_{k\cdot 3}^{(i)}, V_{k\cdot 3}^{-1/2}L_{k\cdot 3}^{*}L_{k\cdot 3}V_{k\cdot 3}^{-1/2}\alpha_{k\cdot 3}^{(j)}\rangle_{\mathcal{X}_k} \\ &= \langle \alpha_{k\cdot 3}^{(i)}, \alpha_{k\cdot 3}^{(j)}\rangle_{\mathcal{X}_1} = \delta_{ij}.\end{aligned} \tag{2.10}$$

In order to simplify the previous expressions, conditions for the invertibility of $V_{k\cdot 3}$ $(k = 1,2)$ may be searched. They can be obtained from the following properties:

LEMMA 2.2. *For $k \in \{1,2\}$:*
(i) $E_k \cap E_3 = L_k(\ker(V_{k\cdot 3}))$;
(ii) $E_k \cap E_3 = \{0\} \Leftrightarrow \ker(V_k) = \ker(V_{k\cdot 3})$.

PROOF. (i) Let u be an element of $E_k \cap E_3$, then $\Pi_{E_k}\Pi_{E_3}u = u$ and there exists a vector α which can be chosen in $R(L_k^*)$ (because $\mathcal{X}_k = \ker L_k \oplus R(L_k^*)$) such that $u = L_k\alpha$. Since $\Pi_{E_m} = L_mV_m^{\dagger}L_m^*$ $(m \in \{1,2,3\})$, we then have $L_kV_k^{\dagger}V_{k3}V_3^{\dagger}V_{3k}\alpha = L_k\alpha$. Premultiplying both sides of this equality by L_k^* gives $\Pi_{R(V_k)}V_{k3}V_3^{\dagger}V_{3k}\alpha = V_k\alpha$; since $R(V_{k3}) = R(L_k^*L_3) \subset R(L_k^*) = R(V_k)$, by equation (2.5) we obtain $V_{k\cdot 3}\alpha = 0$. Hence $u \in L_k(\ker(V_{k\cdot 3}))$ and thus $E_k \cap E_3 \subset L_k(\ker(V_{k\cdot 3}))$. Reciprocally, for all $\alpha \in \ker(V_{k\cdot 3})$, one has $V_k\alpha = V_{k3}V_3^{\dagger}V_{3k}\alpha = L_k^*\Pi_{E_3}L_k\alpha$. Premultiplying both sides of this equality by $L_kV_k^{\dagger}$ and noticing that $L_k\Pi_{R(V_k)} = L_k\Pi_{R(L_k^*)} = L_k$ permit to obtain $u = \Pi_{E_k}\Pi_{E_3}u$, where $u = L_k\alpha$. Thus $u \in E_k \cap E_3$ and this proves that $L_k(\ker(V_{k\cdot 3})) \subset E_k \cap E_3$.

(ii) If $E_k \cap E_3 = \{0\}$ then, from (i), $\alpha \in \ker(V_{k\cdot 3})$ implies $L_k\alpha = 0$, that is $\alpha \in \ker(L_k) = \ker(V_k)$. Reciprocally, if $\ker(V_k) = \ker(V_{k\cdot 3})$ then from (i), we have $E_k \cap E_3 = L_k(\ker(V_k)) = L_k(\ker(L_k)) = \{0\}$. □

From this lemma we deduce that, for $k \in \{1,2\}$, if V_k is invertible and $E_k \cap E_3 = \{0\}$ then $V_{k\cdot 3}$ is invertible. These are sufficient but not necessary conditions for the invertibility of $V_{k\cdot 3}$.

Remark 2.2. When, for $m \in \{1,2,3\}$, a basis is chosen in $\mathcal{X}_m$:

1) the invertibility of V_m is equivalent to the linear (algebraic) independence of the components of X_m related to the basis of $\mathcal{X}_m$ which is considered. One can always reduce to that situation by removing some of these components. In the literature, V_m is often supposed to be invertible and it is admitted that this assumption does not restrict the generality;

2) when the V_m's are invertible, the condition $E_k \cap E_3 = \{0\}$ $(k = 1,2)$ means that the system made up by the components of X_k additioned to that of X_3 is linearly independent. That situation can always be obtained by removing some components in X_k and/or in X_3;

Our approach consists in defining LRCA by using the CA of Euclidean spaces (see Dauxois and Nkiet (1997*a*)); one of the interests of this approach is that it permits to see several classical methods as particular cases of this CA. Two examples are given below.

Example 2.1. *LCA with linear constraints*. This method (see Yanai and Takane (1992)) consists in searching canonical variates of X_1 and X_2 having the form $f_k^{(i)} = \langle \alpha_k^{(i)}, X_k \rangle_{\mathcal{X}_k}$ with the linear constraints $A_k \alpha_k^{(i)} = 0$, where A_k is a linear map from $\mathcal{X}_k$ to another Euclidean space $\mathcal{X}'_k$ $(k = 1, 2)$. Suppose that, for $k \in \{1, 2\}$, V_k is invertible, and put $Y_k := A_k V_k^{-1} X_k$ and $E_{Y_k} := R(L_{Y_k})$. It is clear that $E_{Y_k} \subset E_k$; then $(E_k + E_{Y_k}) \ominus E_{Y_k} = E_k \ominus E_{Y_k}$. For any $\alpha \in \mathcal{X}_k$ and any $\beta \in \mathcal{X}'_k$, one has

$$\mathbb{E}(\langle \alpha, X_k \rangle_{\mathcal{X}_k} \langle \beta, Y_k \rangle_{\mathcal{X}'_k}) = \langle \mathbb{E}(X_k \otimes Y_k)\alpha, \beta \rangle_{\mathcal{X}'_k} = \langle A_k \alpha, \beta \rangle_{\mathcal{X}'_k};$$

consequently, a r.v. $f := \langle \alpha, X_k \rangle_{\mathcal{X}_k}$ belongs to $E_k \ominus E_{Y_k}$ if, and only if, for any $\beta \in \mathcal{X}'_k$, $\langle A_k \alpha, \beta \rangle_{\mathcal{X}'_k} = 0$; that is $A_k \alpha = 0$. Consequently, the LCA of X_1 and X_2 with the previous linear constraints is the CA of $E_1 \ominus E_{Y_1}$ and $E_2 \ominus E_{Y_2}$. Now, it will be reformulated as LCA for suitably transformed variates. For $k \in \{1, 2\}$, from Lemma 2.1 we have $E_k \ominus E_{Y_k} = R(L_{Z_k})$, where $Z_k := X_k - U_k W_k^{\dagger} Y_k$ with $U_k := \mathbb{E}(Y_k \otimes X_k)$ and $W_k := \mathbb{E}(Y_k^{\otimes^2})$. Therefore, the previous CA is the LCA of the random vectors Z_1 and Z_2 which can easily be expressed as transformed variates from X_1 and X_2 respectively. Indeed, from $U_k = \mathbb{E}(X_k^{\otimes^2}) V_k^{-1} A_k^* = A_k^*$ and $W_k = A_k V_k^{-1} \mathbb{E}(X_k^{\otimes^2}) V_k^{-1} A_k^* = A_k V_k^{-1} A_k^*$, we obtain $Z_k = (I_k - C_k) X_k$, where I_k denotes the identity operator of $\mathcal{X}_k$ and $C_k := A_k^* (A_k V_k^{-1} A_k^*)^{\dagger} A_k V_k^{-1}$. Notice that Suzukawa (1997) showed that the previous LCA with linear constraints is the LCA of two variates $\widetilde{X}_1 := Q_1 X_1$ and $\widetilde{X}_2 := Q_2 X_2$, where Q_1 and Q_2 are suitable orthogonal operators. In fact, this result is equivalent to the preceding one. Indeed, the LCA of Z_1 and Z_2 is the research of canonical variates $(\alpha_i, \beta_i)_{1 \le i \le m}$ $(m \in \mathbb{N}^*)$ and canonical coefficients $(\rho_i)_{1 \le i \le m}$ satisfying:

$$\begin{cases} \widetilde{V}_{12} \widetilde{V}_2^{\dagger} \widetilde{V}_{21} \alpha_i = \rho_i^2 \widetilde{V}_1 \alpha_i, & \langle \alpha_i, \widetilde{V}_1 \alpha_i \rangle_{\mathcal{X}_1} = 1, \quad \langle \alpha_i, \widetilde{V}_1 \alpha_j \rangle_{\mathcal{X}_1} = 0 \\ \widetilde{V}_{21} \widetilde{V}_1^{\dagger} \widetilde{V}_{12} \beta_i = \rho_i^2 \widetilde{V}_2 \beta_i, & \langle \beta_i, \widetilde{V}_2 \beta_i \rangle_{\mathcal{X}_2} = 1, \quad \langle \beta_i, \widetilde{V}_2 \beta_j \rangle_{\mathcal{X}_2} = 0 \end{cases} \quad (\text{for} \quad i \neq j),$$

where $\widetilde{V}_{12} = \mathbb{E}(Z_2 \otimes Z_1) = \widetilde{V}_{12}^*$ and $\widetilde{V}_k = \mathbb{E}(Z_k^{\otimes^2})$ $(k \in \{1, 2\})$. Since

$$\widetilde{V}_k = (I_k - C_k) V_k (I_k - C_k)^* = (I_k - C_k) V_k = V_k (I_k - C_k)^*,$$
$$\widetilde{V}_{12} = (I_1 - C_1) V_{12} (I_2 - C_2)^*,$$

and

$$\widetilde{V}_k^{\dagger} = (I_k - C_k)^* V_k^{-1} (I_k - C_k), \qquad (I_k - C_k)^* V_k^{-1} = V_k^{-1} (I_k - C_k),$$

the preceding system is equivalent to

$$\begin{cases} (I_1 - C_1) V_{12} V_2^{-1} (I_2 - C_2) V_{21} \gamma_i = \rho_i^2 V_1 \gamma_i, & \langle \gamma_i, V_1 \gamma_i \rangle_{\mathcal{X}_1} = 1, \quad \langle \gamma_i, V_1 \gamma_j \rangle_{\mathcal{X}_1} = 0 \\ (I_2 - C_2) V_{21} V_1^{-1} (I_1 - C_1) V_{12} \xi_i = \rho_i^2 V_2 \xi_i, & \langle \xi_i, V_2 \xi_i \rangle_{\mathcal{X}_2} = 1, \quad \langle \xi_i, V_2 \xi_j \rangle_{\mathcal{X}_2} = 0 \end{cases}$$
$$(\text{for} \quad i \neq j)$$

with $\gamma_i = (I_1 - C_1)^*\alpha_i$, $\xi_i = (I_2 - C_2)^*\beta_i$. This later system is that is shown in Suzukawa (1997) to define the LCA of $\widetilde{X}_1$ and $\widetilde{X}_2$.

Example 2.2. *Relative discriminant analysis*. Let Y be a discrete r.v. valued into $\{1, \ldots, q\}$ and defining q groups. Then put $X_1 := (\mathbf{1}_{\{Y=1\}}, \ldots, \mathbf{1}_{\{Y=q\}})$; we define the relative discriminant analysis (RDA) of Y and X_2 relative to X_3 as the canonical analysis of E_1 and $E_{2\cdot 3}$. Since $E_{2\cdot 3} = R(L_{X_{2\cdot 3}})$, this RDA is the discriminant analysis of Y and $X_{2\cdot 3}$. This method have been introduced in the literature (see, e.g., Fujikoshi and Khatri (1990), Baccini *et al.* (2001)) for the case where X_3 is a covariate having the same mean in the preceding q groups and admiting an invertible covariance operator.

Invariance of multivariate analyses when the related variables are transformed by linear maps have been considered in some particular forms in the literature. For instance, the problem of additional information, tackled in Fujikoshi (1892) and Suzukawa and Sato (1996) for LCA and in Fujikoshi and Khatri (1990) for covariate discriminant analysis, defined as the research of conditions for which the results of a given analysis are the same whether one considers some variables or subcomponents of them, is clearly a problem of invariance of this analysis after transformations of these variables by projectors (see Remark 2.3). Then, it is of interest to generalize the approach of the previous works by searching for conditions such that the considered analysis is invariant when the variables are transformed by linear maps which may not be projectors. This is an important goal since in multivariate analysis it often occurs that, in order to reduce dimensions, one have to work with linear transformations, and not necessarily projections, of original variables; so it may be convenient that these transformations do not affect the results of the given analysis. For the case of linear canonical analysis (LCA), this generalizing approach have been tackled by Dauxois and Nkiet (1997*a*) who determined conditions for having the aforementioned invariance. We will now extend this problem to the case of LRCA. For $k \in \{1, 2\}$, consider an Euclidean space $\mathcal{X}'_k$ with dimension q_k, a linear map A_k from $\mathcal{X}_k$ to $\mathcal{X}'_k$ and the r.v. $Y_k = A_k X_k$. It is easy to verify that, defining $W_{k3} := E(X_3 \otimes Y_k)$, $Y_{k\cdot 3} := Y_k - W_{k3} V_3^\dagger X_3$ and $W_{k\cdot 3} := E(Y_{k\cdot 3}^{\otimes^2})$, one has $W_{k3} = A_k V_{k3}$, $Y_{k\cdot 3} = A_k X_{k\cdot 3}$; this implies: $L_{Y_{k\cdot 3}} = L_{k\cdot 3} A_k^*$. The LRCA of X_1 and X_2 relative to X_3 is the triple given in equation (2.6), and similarly we consider the triple

$$\{(\gamma_i)_{1\leq i\leq s}, (\beta_{1\cdot 3}^{(i)})_{1\leq i\leq q_1}, (\beta_{2\cdot 3}^{(i)})_{1\leq i\leq q_2}\}$$

which characterizes the LRCA of Y_1 and Y_2 relative to X_3. The canonical variates corresponding to the preceding LRCA are

$$f_{k\cdot 3}^{(i)} = L_{k\cdot 3}(V_{k\cdot 3}^\dagger)^{1/2}\alpha_{k\cdot 3}^{(i)}, \qquad 1 \leq i \leq p_k$$

and

$$g_{k\cdot 3}^{(i)} = L_{Y_{k\cdot 3}}(W_{k\cdot 3}^\dagger)^{1/2}\beta_{k\cdot 3}^{(i)}, \qquad 1 \leq i \leq q_k.$$

DEFINITION 2.2. The LRCA of X_1 and X_2 is invariant for the pair (A_1, A_2) if the following conditions are satisfied:

(i) $r = s$ and $\rho_i = \gamma_i$ $(i = 1, \ldots, r)$;

(ii) for all $(k, i) \in \{1, 2\} \times \{1, \ldots, r\}$, $f_{k\cdot 3}^{(i)} = g_{k\cdot 3}^{(i)}$.

Now, we can seek a necessary and sufficient condition for which this invariance property holds. Notice that, since $Y_{k\cdot 3} = A_k X_{k\cdot 3}$ $(k = 1, 2)$, the invariance introduced in the previous definition means the invariance of the LCA of $X_{1\cdot 3}$ and $X_{2\cdot 3}$ for the pair (A_1, A_2). Then, by applying Proposition 4.2 of Dauxois and Nkiet (1997*a*), we obtain:

PROPOSITION 2.2. *The LRCA of X_1 and X_2 is invariant for the pair (A_1, A_2) if, and only if one has*:

$$V_{12\cdot 3} = V_{1\cdot 3}A_1^*(A_1V_{1\cdot 3}A_1^*)^\dagger A_1V_{12\cdot 3} \quad \text{and} \quad V_{21\cdot 3} = V_{2\cdot 3}A_2^*(A_2V_{2\cdot 3}A_2^*)^\dagger A_2V_{21\cdot 3}.$$

Remark 2.3. The previous notion of invariance for LRCA is related to the problem of additional information in canonical analysis which interested some authors. For example, Siotani (1957) studied the effect of adding variates on the canonical coefficients and Fujikoshi (1982) determined conditions for which LCA remains unchanged when subcomponents of the involved variates are omitted. When, for $k \in \{1, 2\}$, we have the decomposition in direct sum $\mathcal{X}_k = \mathcal{X}_k^{(1)} \oplus \mathcal{X}_k^{(2)}$, an analogous problem of additional information can be formulated for LRCA. Consider π_{k1} (resp. π_{k2}) the projection operator on $\mathcal{X}_k^{(1)}$ (resp. $\mathcal{X}_k^{(2)}$) along $\mathcal{X}_k^{(2)}$ (resp. $\mathcal{X}_k^{(1)}$) and put:

$$X_{kj} = \pi_{kj}X_k \qquad (j = 1, 2).$$

We say that the pair (X_{12}, X_{22}) does not provide additional information on the LRCA of X_1 and X_2 relative to X_3 if this later LRCA is invariant for the pair (π_{11}, π_{21}). A necessary and sufficient condition for this invariance is obtained by applying Proposition 2.2 to the pair (π_{11}, π_{21}); now, by taking $X_3 = 0$ we obtain the condition of Fujikoshi (1982).

3. Asymptotic study of LRCA

The asymptotic theory for classical LCA is well known; the earlier works on this subject focused on the asymptotic joint distribution of the sample canonical correlation coefficients under normality (see Hsu (1941)) or nonnormality (see Muirhead and Waternaux (1980)). Later, asymptotic distributions both for these coefficients and for sample canonical vectors and/or projections were derived under normality and when the population canonical correlation coefficients are distinct (see Anderson (1999) and references inside) or under nonnormality and in case the preceding coefficients have multiplicities (see Arconte (1980), Pousse (1992), Larrère (1994) and Fine (2000)). In fact, asymptotic study for LCA or others multivariate statistical analyses reduces to determining consistency and asymptotic distributions for eigenvalues, eigenvectors and eigenprojections of an operator which is known to be consistent and for which an asymptotic distribution is known. That is not technically difficult nowadays since one can apply results of Dossou-Gbete and Pousse (1991) for a selfadjoint random operator, or those of Eaton and Tyler (1994) when one focuses on singular values of a random matrix which may be not symmetric. Finally, making an asymptotic study for a statistical multivariate analysis mainly consists in studying the consistency and in deriving an asymptotic distribution for the related operator.

In this section, we focus on asymptotics for LRCA. Although it is a particular LCA, the results of asymptotic study for this later analysis cannot been applied to it. Indeed, these results hold when an i.i.d. sample of the related random variables is avalaible; but, as we will see below, we don't have any sample of $(X_{1\cdot 3}, X_{2\cdot 3})$. These r.v. are unobservable ones because their definitions (see equation (2.4)) involve covariance operators which are unknown in practice. Then, consistency and asymptotic distribution for the sample operator related to LRCA are not straightforward and there is an interest to determine them.

We suppose that, for all $(k, m) \in \{1, 2\} \times \{1, 2, 3\}$, we have:

(A1) $\mathbb{E}(\|X_m\|^4_{\mathcal{X}_m}) < +\infty$;

(A2) V_m is invertible;

(A3) $E_k \cap E_3 = \{0\}$;

(A4) $V_{k\cdot 3} = I_k$

where I_k denotes the identity of $\mathcal{X}_k$.

Remark 3.1. From Lemma 2.2 we know that the assumptions (A2) and (A3) imply thas $V_{k\cdot 3}$ is invertible. Then Assumption (A4) does not restrict the generality; indeed one can always reduce to that situation by considering $Y_k = V_{k\cdot 3}^{-1/2} X_k$ instead of X_k $(k = 1, 2)$, and since $V_{k\cdot 3}^{-1/2}$ is invertible the transformation $X_k \mapsto V_{k\cdot 3}^{-1/2} X_k$ yields invariance of the LRCA.

Let $(X_1^{(i)}, X_2^{(i)}, X_3^{(i)})_{1 \le i \le n}$ an i.i.d. sample of the triple $Z = (X_1, X_2, X_3)$; for $(m, k, j) \in \{1, 2, 3\} \times \{1, 2\}^2$, we consider:

$$\overline{X}_m^{(n)} = \frac{1}{n} \sum_{i=1}^{n} X_m^{(i)},$$

$$V_m^{(n)} = \frac{1}{n} \sum_{i=1}^{n} (X_m^{(i)} - \overline{X}_m^{(n)})^{\otimes 2}, \tag{3.1}$$

$$V_{kj}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} (X_j^{(i)} - \overline{X}_j^{(n)}) \otimes (X_k^{(i)} - \overline{X}_k^{(n)}) = V_{jk}^{(n)^*} \quad \text{for} \quad k \neq j. \tag{3.2}$$

Notice that $V_m^{(n)} = \frac{1}{n} \sum_{i=1}^{n} X_m^{(i)^{\otimes 2}} - \overline{X}_m^{(n)^{\otimes 2}}$ and $V_{kj}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} X_j^{(i)} \otimes X_k^{(i)} - \overline{X}_j^{(n)} \otimes \overline{X}_k^{(n)}$; then by the strong law of large numbers $V_m^{(n)}$ and $V_{kj}^{(n)}$ almost surely uniformly converge to V_m and V_{kj} respectively, as $n \to +\infty$. This shows that for large values of n, $V_3^{(n)}$ is invertible (a.s.); thus we can define

$$V_{k\cdot 3}^{(n)} := V_k^{(n)} - V_{k3}^{(n)} V_3^{(n)^{-1}} V_{3k}^{(n)}, \quad \text{and} \tag{3.3}$$

$$V_{12\cdot 3}^{(n)} := V_{12}^{(n)} - V_{13}^{(n)} V_3^{(n)^{-1}} V_{32}^{(n)} = V_{21\cdot 3}^{(n)^*}. \tag{3.4}$$

We also have the almost sure uniform convergence of $V_{k\cdot 3}^{(n)}$ to I_k, as $n \to +\infty$. Then, for large n, $V_{k\cdot 3}^{(n)}$ also is invertible (a.s.) and we put:

$$T_{1\cdot 3}^{(n)} = V_{1\cdot 3}^{(n)^{-1/2}} V_{12\cdot 3}^{(n)} V_{2\cdot 3}^{(n)^{-1}} V_{21\cdot 3}^{(n)} V_{1\cdot 3}^{(n)^{-1/2}}.$$

We take the spectral analysis of $T_{1\cdot 3}^{(n)}$ as an estimator of the LRCA of X_1 and X_2 relative to X_3 and our goal is to study the asymptotic properties of this estimator and its eigenelements.

Let $\lambda' := (\lambda'_j)_{1\le j\le s}$ (resp. $\lambda := (\lambda_i)_{1\le i\le p_1}$) be the strictly decreasing (resp. complete nonincreasing) sequence of the eigenvalues of $T_{1\cdot 3}$, we denote by m_j the multiplicity of λ'_j. It is clear that $\lambda_i = \rho_i^2$ with $\rho_i = 0$ if $i > r$ and that, putting $\nu_j = 1+\sum_{l=0}^{j-1} m_l$ (with $m_0 = 0$), one has $\lambda'_j = \lambda_{\nu_j}$. Then we consider the orthogonal projection operator P_j onto the eigenspace of $T_{1\cdot 3}$ associated to λ'_j, that is $P_j := \sum_{i=\nu_j}^{\nu_j+m_j-1} \alpha_{1\cdot 3}^{(i)\otimes^2}$. Moreover, letting $(\lambda_i^{(n)})_{1\le i\le p_1}$ be the complete sequence of eigenvalues of $T_{1\cdot 3}^{(n)}$, we consider an orthonormal basis $(\alpha_{1\cdot 3,n}^{(i)})_{1\le i\le p_1}$ of associated eigenvectors such that $\alpha_{1\cdot 3,n}^{(i)}$ is associated to $\lambda_i^{(n)}$, and we put $P_j^{(n)} := \sum_{i=\nu_j}^{\nu_j+m_j-1} \alpha_{1\cdot 3,n}^{(i)\otimes^2}$.

3.1 *Almost sure convergence*

As already noticed, the empirical covariance operators defined in equation (3.1) and equation (3.2) almost surely uniformly converge to the corresponding covariance operators. Thus, $V_{1\cdot 3}^{(n)}$ (resp. $V_{2\cdot 3}^{(n)}$; $V_{12\cdot 3}^{(n)}$) almost surely uniformly converges to I_1 (resp. I_2; $V_{12\cdot 3}$). Consequently, $T_{1\cdot 3}^{(n)}$ converges almost surely uniformly to $T_{1\cdot 3}$. A direct application of Proposition 3 of Dossou-Gbete and Pousse (1991) gives the following almost sure convergence properties of the eigenelements of $T_{1\cdot 3}^{(n)}$.

PROPOSITION 3.1. (i) *For any $j \in \{1,\dots,s\}$ and any $i \in \{\nu_j,\dots,\nu_j+m_j-1\}$, $(\lambda_i^{(n)})_{n\in\mathbb{N}^*}$ converges almost surely to λ'_j.*

(ii) *For any $j \in \{1,\dots,s\}$, $(P_j^{(n)})_{n\in\mathbb{N}^*}$ converges almost surely uniformly to P_j.*

(iii) *If $m_j = 1$, then $(\alpha_{1\cdot 3,n}^{(\nu_j)})_{n\in\mathbb{N}^*}$ converges almost surely to $\alpha_{1\cdot 3}^{(\nu_j)}$.*

3.2 *Convergence in distribution*

Here, we will derive the asymptotic distribution of $\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3})$ and, consequently, those of the eigenelements of this operator.

We identify $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ with the direct orthogonal sum $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$, the aforementioned orthogonality being related to the inner product of $\mathcal{X}$ defined by $\langle x,y\rangle_{\mathcal{X}} = \sum_{m=1}^3 \langle x_m, y_m\rangle_{\mathcal{X}_m}$ for all $x := \sum_{m=1}^3 x_m \in \mathcal{X}$ and all $y := \sum_{m=1}^3 y_m \in \mathcal{X}$. Then we can write $Z = \sum_{m=1}^3 X_m$ and put $Z_i = \sum_{m=1}^3 X_m^{(i)}$ $(i = 1,\dots,n)$. In the same way, the space $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3)$ of linear maps from $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ into itself will be identified to the orthogonal sum $M = \bigoplus_{1\le m,j\le 3} \mathcal{L}(\mathcal{X}_m, \mathcal{X}_j)$, where for any pair (F,G) of Euclidean spaces, we denote by $\mathcal{L}(F,G)$ the space of linear maps from F into G (when $F = G$, we will write $\mathcal{L}(F)$). Hence we can write $V = \mathbb{E}(Z^{\otimes^2}) = \sum_{1\le m,j\le 3} V_{jm}$ (with $V_{jj} := V_j$). Let us consider:

$$\overline{Z}^{(n)} = \frac{1}{n}\sum_{i=1}^n Z_i = \sum_{m=1}^3 \overline{X}_m^{(n)},$$

$$V_n = \frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z}^{(n)})^{\otimes^2} = \sum_{1\le j,m\le 3} V_{jm}^{(n)} \qquad (\text{with } V_{jj}^{(n)} := V_j^{(n)})$$

and

$$H_n = \sqrt{n}(V_n - V) = \sum_{1\le j,m\le 3} \sqrt{n}(V_{jm}^{(n)} - V_{jm}). \tag{3.5}$$

For $(r,s) \in \{1,2,3\}^2$, we denote by p_{rs} the orthogonal projection operator from $M = \bigoplus_{1\leq m,j\leq 3} \mathcal{L}(\mathcal{X}_m, \mathcal{X}_j)$ to $\mathcal{L}(\mathcal{X}_s, \mathcal{X}_r)$, that is the operator such that for all $A = \sum_{1\leq i,j\leq 3} A_{ij} \in M$, one has $p_{rs}(A) = A_{rs}$. First, we will prove three useful lemmas.

LEMMA 3.1. *For $k \in \{1,2\}$, one has $\sqrt{n}(V_{k\cdot 3}^{(n)} - I_k) = a_{k\cdot 3}^{(n)}(H_n)$, where H_n is defined in equation (3.5) and $(a_{k\cdot 3}^{(n)})_{n\in\mathbb{N}^*}$ is a sequence of random operators from M to $\mathcal{L}(\mathcal{X}_k)$ which converges almost surely uniformly to the operator $a_{k\cdot 3}$ of $\mathcal{L}(M, \mathcal{L}(\mathcal{X}_k))$ defined by:*

$$a_{k\cdot 3}(A) = p_{kk}(A) - p_{k3}(A)V_3^{-1}V_{3k} + V_{k3}V_3^{-1}p_{33}(A)V_3^{-1}V_{3k} - V_{k3}V_3^{-1}p_{3k}(A).$$

PROOF. Using equation (3.3) and $I_k = V_{k\cdot 3} = V_k - V_{k3}V_3^{-1}V_{3k}$, we have:

$$\begin{aligned}
V_{k\cdot 3}^{(n)} - I_k &= V_k^{(n)} - V_{k3}^{(n)}V_3^{(n)^{-1}}V_{3k}^{(n)} - I_k \\
&= V_k^{(n)} - V_{k3}^{(n)}V_3^{(n)^{-1}}V_{3k}^{(n)} - V_k + V_{k3}V_3^{-1}V_{3k} \\
&= (V_k^{(n)} - V_k) - (V_{k3}^{(n)} - V_{k3})V_3^{(n)^{-1}}V_{3k}^{(n)} \\
&\quad + V_{k3}V_3^{(n)^{-1}}(V_3^{(n)} - V_3)V_3^{-1}V_{3k}^{(n)} - V_{k3}V_3^{-1}(V_{3k}^{(n)} - V_{3k}).
\end{aligned}$$

Hence $\sqrt{n}(V_{k\cdot 3}^{(n)} - I_k) = a_{k\cdot 3}^{(n)}(H_n)$, where $a_{k\cdot 3}^{(n)}$ is the random operator of $\mathcal{L}(M, \mathcal{L}(\mathcal{X}_k))$ defined by:

$$a_{k\cdot 3}^{(n)}(A) = p_{kk}(A) - p_{k3}(A)V_3^{(n)^{-1}}V_{3k}^{(n)} + V_{k3}V_3^{(n)^{-1}}p_{33}(A)V_3^{-1}V_{3k}^{(n)} - V_{k3}V_3^{-1}p_{3k}(A).$$

Then, the almost sure uniform convergence $(a_{k\cdot 3}^{(n)})_{n\in\mathbb{N}^*}$ to $a_{k\cdot 3}$ is obviously deduced from that of $V_3^{(n)}$ (resp. $V_{3k}^{(n)}$) to V_3 (resp. V_{3k}). □

LEMMA 3.2. *One has $\sqrt{n}(V_{12\cdot 3}^{(n)} - V_{12\cdot 3}) = a_{12\cdot 3}^{(n)}(H_n)$, where $(a_{12\cdot 3}^{(n)})_{n\in\mathbb{N}^*}$ is a sequence of random operators from M to $\mathcal{L}(\mathcal{X}_2, \mathcal{X}_1)$ which converges almost surely uniformly to the operator $a_{12\cdot 3}$ of $\mathcal{L}(M, \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1))$ defined by:*

$$a_{12\cdot 3}(A) = p_{12}(A) - p_{13}(A)V_3^{-1}V_{32} + V_{13}V_3^{-1}p_{33}(A)V_3^{-1}V_{32} - V_{13}V_3^{-1}p_{32}(A).$$

PROOF. Using equations (2.5) and (3.4), we can write:

$$\begin{aligned}
V_{12\cdot 3}^{(n)} - V_{12\cdot 3} &= V_{12}^{(n)} - V_{12} - V_{13}^{(n)}V_3^{(n)^{-1}}V_{32}^{(n)} + V_{13}V_3^{-1}V_{32} \\
&= (V_{12}^{(n)} - V_{12}) - (V_{13}^{(n)} - V_{13})V_3^{(n)^{-1}}V_{32}^{(n)} \\
&\quad + V_{13}V_3^{(n)^{-1}}(V_3^{(n)} - V_3)V_3^{-1}V_{32}^{(n)} - V_{13}V_3^{-1}(V_{32}^{(n)} - V_{32}).
\end{aligned}$$

Thus we have $\sqrt{n}(V_{12\cdot 3}^{(n)} - V_{12\cdot 3}) = a_{12\cdot 3}^{(n)}(H_n)$, where $a_{12\cdot 3}^{(n)} \in \mathcal{L}(M, \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1))$ is defined by:

$$a_{12\cdot 3}^{(n)}(A) = p_{12}(A) - p_{13}(A)V_3^{(n)^{-1}}V_{32}^{(n)} + V_{13}V_3^{(n)^{-1}}p_{33}(A)V_3^{-1}V_{32}^{(n)} - V_{13}V_3^{-1}p_{32}(A).$$

Then, the almost sure uniform convergence $(a_{k\cdot 3}^{(n)})_{n\in\mathbb{N}^*}$ to $a_{k\cdot 3}$ is obviously deduced from that of $V_3^{(n)}$ (resp. $V_{32}^{(n)}$) to V_3 (resp. V_{32}). □

Similarly, we also have:

LEMMA 3.3. *One has* $\sqrt{n}(V_{21\cdot 3}^{(n)} - V_{21\cdot 3}) = a_{21\cdot 3}^{(n)}(H_n)$, *where* $(a_{21\cdot 3}^{(n)})_{n\in\mathbb{N}^*}$ *is a sequence of random operators from* M *to* $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ *which converges almost surely uniformly to the operator* $a_{21\cdot 3}$ *of* $\mathcal{L}(M, \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2))$ *defined by*:

$$a_{21\cdot 3}(A) = p_{21}(A) - p_{23}(A)V_3^{-1}V_{31} + V_{23}V_3^{-1}p_{33}(A)V_3^{-1}V_{31} - V_{23}V_3^{-1}p_{31}(A).$$

These lemmas permit us to obtain the asymptotic distribution of $\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3})$. In what follows, we consider the operator:

$$\pi : A \in \mathcal{L}(\mathcal{X}_1) \mapsto \frac{1}{2}(A + A^*) \in \mathcal{L}(\mathcal{X}_1),$$

that is the orthogonal projector onto the subspace of the selfadjoint operators in $\mathcal{L}(X_1)$ and $\tilde{\otimes}$ denotes the tensor product of operators, associated with the Hilbert-Schmidt inner product: $\langle T, S\rangle_2 = \operatorname{tr}(TS^*)$.

PROPOSITION 3.2. $\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3})$ *converges in distribution, as* $n \to +\infty$, *to a r.v.* U *having a centered normal distribution in* $\mathcal{L}(\mathcal{X}_1)$ *with covariance operator given by*:

$$\Gamma = \mathbb{E}[(\pi(-V_{12\cdot 3}V_{21\cdot 3}X_{1\cdot 3}^{\otimes^2} - V_{12\cdot 3}X_{2\cdot 3}^{\otimes^2}V_{21\cdot 3} + 2V_{12\cdot 3}(X_{1\cdot 3} \otimes X_{2\cdot 3})))^{\tilde{\otimes}^2}].$$

PROOF. We have:

$$\begin{aligned}
\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3}) = {} & \sqrt{n}(V_{1\cdot 3}^{(n)-1/2} - I_1)V_{12\cdot 3}^{(n)}V_{2\cdot 3}^{(n)^{-1}}V_{21\cdot 3}^{(n)}V_{1\cdot 3}^{(n)-1/2} \\
& + [\sqrt{n}(V_{12\cdot 3}^{(n)} - V_{12\cdot 3})]V_{2\cdot 3}^{(n)^{-1}}V_{21\cdot 3}^{(n)}V_{1\cdot 3}^{(n)-1/2} \\
& - V_{12\cdot 3}V_{2\cdot 3}^{(n)^{-1}}[\sqrt{n}(V_{2\cdot 3}^{(n)} - I_2)]V_{21\cdot 3}^{(n)}V_{1\cdot 3}^{(n)-1/2} \\
& + V_{12\cdot 3}[\sqrt{n}(V_{21\cdot 3}^{(n)} - V_{21\cdot 3})]V_{1\cdot 3}^{(n)-1/2} \\
& + V_{12\cdot 3}V_{21\cdot 3}[\sqrt{n}(V_{1\cdot 3}^{(n)-1/2} - I_1)].
\end{aligned}$$

For any invertible selfadjoint nonegative operator T, one has

$$T^{-1/2} - I = -T^{-1/2}(T - I)(T^{-1/2} + I)^{-1} \tag{3.6}$$

where I denotes the identity. Then applying this equality with $T = V_{1\cdot 3}^{(n)}$ and using the three previous lemmas, we obtain:

$$\begin{aligned}
\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3}) = {} & - V_{1\cdot 3}^{(n)^{-1}}a_{1\cdot 3}^{(n)}(H_n)(V_{1\cdot 3}^{(n)^{-1/2}} + I_1)^{-1}V_{12\cdot 3}^{(n)}V_{2\cdot 3}^{(n)^{-1}}V_{21\cdot 3}^{(n)}V_{1\cdot 3}^{(n)^{-1/2}} \\
& + a_{12\cdot 3}^{(n)}(H_n)V_{2\cdot 3}^{(n)^{-1}}V_{21\cdot 3}^{(n)}V_{1\cdot 3}^{(n)^{-1/2}} - V_{12\cdot 3}V_{2\cdot 3}^{(n)^{-1}}a_{2\cdot 3}^{(n)}(H_n)V_{21\cdot 3}^{(n)}V_{1\cdot 3}^{(n)^{-1/2}} \\
& + V_{12\cdot 3}a_{21\cdot 3}^{(n)}(H_n)V_{1\cdot 3}^{(n)-1/2} \\
& - V_{12\cdot 3}V_{21\cdot 3}V_{1\cdot 3}^{(n)^{-1}}a_{1\cdot 3}^{(n)}(H_n)(V_{1\cdot 3}^{(n)^{-1/2}} + I_1)^{-1} \\
= {} & \varphi_n(H_n).
\end{aligned}$$

From the almost sure uniform convergence of $V_{12\cdot 3}^{(n)}$, $V_{1\cdot 3}^{(n)}$ and $V_{2\cdot 3}^{(n)}$, as $n \to +\infty$, to $V_{12\cdot 3}$, I_1 and I_2 respectively, we deduce that $(\varphi_n)_{n\in\mathbb{N}^*}$ almost surely uniformly converges to the operator $\varphi \in \mathcal{L}(M, \mathcal{L}(\mathcal{X}_1))$ defined by:

$$\begin{aligned}\varphi(A) = &-\frac{1}{2}[a_{1\cdot 3}(A)V_{12\cdot 3}V_{21\cdot 3} + V_{12\cdot 3}V_{21\cdot 3}a_{1\cdot 3}(A)] + a_{12\cdot 3}(A)V_{21\cdot 3} \\ &+ V_{12\cdot 3}a_{21\cdot 3}(A) - V_{12\cdot 3}a_{2\cdot 3}(A)V_{21\cdot 3}.\end{aligned}$$

Further, we have:

$$H_n = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i^{\otimes^2} - V\right] - \overline{Z}^{(n)^{\otimes^2}}.$$

By the central limit theorem $\sqrt{n}\overline{Z}^{(n)}$ converges in distribution to a r.v. having a centered normal distribution in $\mathcal{X}$ with covariance operator $\mathbb{E}(Z^{\otimes^2})$, this implies that $\overline{Z}^{(n)}$ converges in probability to 0 in $\mathcal{X}$, as $n \to +\infty$. Hence H_n has the same asymptotic distribution than $\sqrt{n}[\frac{1}{n}\sum_{i=1}^{n} Z_i^{\otimes^2} - V]$; thus, by the central limit theorem, H_n converges in distribution, as $n \to +\infty$, to a r.v. H having a centered normal distribution in M, with covariance operator defined by $\mathbb{E}[(Z^{\otimes^{2^c}})^{\tilde{\otimes}^2}]$, where $Z^{\otimes^{2^c}} := Z^{\otimes^2} - \mathbb{E}(Z^{\otimes^2})$. Moreover, we have

$$\|\varphi_n(H_n) - \varphi(H_n)\|_{\mathcal{L}(\mathcal{X}_1)} \leq \|\varphi_n - \varphi\|_\infty \|H_n\|_M,$$

where $\|\cdot\|_\infty$ is the uniform convergence norm. Since $\|H_n\|_M$ (resp. $\|\varphi_n - \varphi\|_\infty$) converges in distribution (resp. in probability), as $n \to +\infty$, to $\|H\|_M$ (resp. 0), the previous inequality shows that $\varphi_n(H_n) - \varphi(H_n)$ converges in probability, in $\mathcal{L}(\mathcal{X}_1)$, to 0 as $n \to +\infty$. Hence $\varphi_n(H_n)$ has the same asymptotic distribution than $\varphi(H_n)$, that is the distribution of $\varphi(H)$ because φ is linear. This means that $\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1.3})$ converges to the same centered normal distribution than $U = \varphi(H)$; the related covariance operator is:

$$\Gamma = \mathbb{E}[(\varphi(H))^{\tilde{\otimes}^2}] = \varphi\mathbb{E}[H^{\tilde{\otimes}^2}]\varphi^* = \varphi\mathbb{E}[(Z^{\otimes^{2^c}})^{\tilde{\otimes}^2}]\varphi^* = \mathbb{E}[(\varphi(Z^{\otimes^{2^c}}))^{\tilde{\otimes}^2}].$$

It remains to give an explicit expression of Γ. We have:

$$\begin{aligned}a_{1\cdot 3}(Z^{\otimes^{2^c}}) &= X_1^{\otimes^2} - V_1 - (X_3 \otimes X_1 - V_{13})V_3^{-1}V_{31} \\ &\quad + V_{13}V_3^{-1}(X_3^{\otimes^2} - V_3)V_3^{-1}V_{31} - V_{13}V_3^{-1}(X_1 \otimes X_3 - V_{31}) \\ &= X_1^{\otimes^2} - (V_{13}V_3^{-1}X_3) \otimes X_1 - X_1 \otimes (V_{13}V_3^{-1}X_3) \\ &\quad + (V_{13}V_3^{-1}X_3)^{\otimes^2} - V_1 + V_{13}V_3^{-1}V_{31} \\ &= X_{1\cdot 3}^{\otimes^2} - I_1, \\ a_{12\cdot 3}(Z^{\otimes^{2^c}}) &= X_2 \otimes X_1 - V_{12} - (X_3 \otimes X_1 - V_{13})V_3^{-1}V_{32} \\ &\quad + V_{13}V_3^{-1}(X_3^{\otimes^2} - V_3)V_3^{-1}V_{32} - V_{13}V_3^{-1}(X_2 \otimes X_3 - V_{32}) \\ &= X_2 \otimes X_1 - (V_{23}V_3^{-1}X_3) \otimes X_1 - X_2 \otimes (V_{13}V_3^{-1}X_3) \\ &\quad + (V_{23}V_3^{-1}X_3) \otimes (V_{13}V_3^{-1}X_3) - V_{12} + V_{13}V_3^{-1}V_{32} \\ &= X_{2\cdot 3} \otimes X_{1\cdot 3} - V_{12.3};\end{aligned}$$

similarly

$$a_{21\cdot 3}(Z^{\otimes^{2^c}}) = X_{1\cdot 3} \otimes X_{2\cdot 3} - V_{21\cdot 3},$$

and

$$a_{2\cdot 3}(Z^{\otimes^{2^c}}) = X_{2\cdot 3}^{\otimes^2} - I_2.$$

Hence

$$\begin{aligned}\varphi(Z^{\otimes^{2^c}}) = & -\frac{1}{2}[(X_{1\cdot 3}^{\otimes^2} - I_1)V_{12\cdot 3}V_{21\cdot 3} + V_{12\cdot 3}V_{21\cdot 3}(X_{1\cdot 3}^{\otimes^2} - I_1)] \\ & + (X_{2\cdot 3} \otimes X_{1\cdot 3} - V_{12\cdot 3})V_{21\cdot 3} + V_{12\cdot 3}(X_{1\cdot 3} \otimes X_{2\cdot 3} - V_{21\cdot 3}) \\ & - V_{12\cdot 3}(X_{2\cdot 3}^{\otimes^2} - I_2)V_{21\cdot 3} \\ & = \pi(-V_{12\cdot 3}V_{21\cdot 3}X_{1\cdot 3}^{\otimes^2} - V_{12\cdot 3}X_{2\cdot 3}^{\otimes^2}V_{21\cdot 3} + 2V_{12\cdot 3}(X_{1\cdot 3} \otimes X_{2\cdot 3}));\end{aligned} \tag{3.7}$$

this completes the proof. □

This proposition permits us to obtain the asymptotic distributions of the eigenelements. For $j \in \{1, \dots, s\}$, considering the operators

$$\begin{aligned} S_j &= \sum_{1 \le l \le s, l \ne j} \frac{1}{\lambda'_j - \lambda'_l} P_l, \\ \Psi_j &: T \in \mathcal{L}(\mathcal{X}_1) \mapsto P_j T S_j + S_j T P_j \in \mathcal{L}(\mathcal{X}_1), \\ \Psi'_j &: T \in \mathcal{L}(\mathcal{X}_1) \mapsto P_j T P_j \in \mathcal{L}(\mathcal{X}_1), \\ \Theta_j &: T \in \mathcal{L}(\mathcal{X}_1) \mapsto S_j T \alpha_{1\cdot 3}^{(\nu_j)} \in \mathcal{L}(\mathcal{X}_1), \end{aligned}$$

denoting by Δ the continuous map which associates to $T \in \mathcal{L}(\mathcal{X}_1)$ its complete non-increasing sequence of eigenvalues, and putting $\rho_i^{(n)} := \sqrt{\lambda_i^{(n)}}$, $\rho'_j := \sqrt{\lambda'_j}$, we have

PROPOSITION 3.3. *For $j \in \{1, \dots, s\}$, one has:*

(i) *the sequence $\sqrt{n}(P_j^{(n)} - P_j)$ converges in distribution, as $n \to +\infty$, to a r.v. having a centered normal distibution in $\mathcal{L}(\mathcal{X}_1)$ with covariance operator $\Lambda_j = \Psi_j \Gamma \Psi_j^*$.*

(ii) *the sequence $(\sqrt{n}(\lambda_i^{(n)} - \lambda'_j))_{\nu_j \le i \le \nu_j + m_j - 1}$ converges in distribution, as $n \to +\infty$, to $\Delta(\xi_j)$, where ξ_j is a r.v. having a centered normal distribution in $\mathcal{L}(\mathcal{X}_1)$ with covariance operator $\Lambda'_j = \Psi'_j \Gamma \Psi'^*_j$.*

(iii) *If $\rho'_j \ne 0$, then $(\sqrt{n}(\rho_i^{(n)} - \rho'_j))_{\nu_j \le i \le \nu_j + m_j - 1}$ converges in distribution, as $n \to +\infty$, to $\Delta(\xi'_j)$, where ξ'_j is a r.v. having a centered normal distribution in $\mathcal{L}(\mathcal{X}_1)$ with covariance operator $\Lambda''_j = (4\lambda'_j)^{-1} \Psi'_j \Gamma \Psi'^*_j$.*

(iv) *If $m_j = 1$, then $\sqrt{n}(\lambda_{\nu_j}^{(n)} - \lambda'_j)$ converges in distribution, as $n \to +\infty$, to* $\mathrm{tr}(\xi_j)$, *and for j satisfying $\rho'_j \ne 0$, $\sqrt{n}(\rho_i^{(n)} - \rho'_j)$ converges in distribution, as $n \to +\infty$, to* $\mathrm{tr}(\xi'_j)$.

(v) *If $m_j = 1$, then $\sqrt{n}(\alpha_{1\cdot 3,n}^{(\nu_j)} - \alpha_{1\cdot 3}^{(\nu_j)})$ converges in distribution, as $n \to +\infty$, to a r.v. having a centered normal distribution in $\mathcal{X}_1$ with covariance operator $\Lambda''_j = \Theta_j\, \Gamma \Theta_j^*$.*

PROOF. (i) and (ii). From Proposition 4 in Dossou-Gbete and Pousse (1991), we know that $\sqrt{n}(P_j^{(n)} - P_j)$ (resp. $(\sqrt{n}(\lambda_i^{(n)} - \lambda_j'))_{\nu_j \le i \le \nu_j+m_j-1}$) converges in distribution, as $n \to +\infty$, to $\Psi_j(U)$ (resp. $\Delta(\xi_j)$, where $\xi_j := \Psi_j'(U)$). Since $\Psi_j(U)$ (resp. ξ_j) is a linear function of U, then it has a centered normal distribution. Its covariance operator are easily shown to be Λ_j (resp. Λ_j').

(iii) If $\rho_j' \neq 0$, then from the equality

$$\sqrt{n}(\rho_i^{(n)} - \rho_j') = \sqrt{n}(\lambda_i^{(n)} - \lambda_j')(\rho_i^{(n)} + \rho_j')^{-1}$$

we can write $(\sqrt{n}(\rho_i^{(n)} - \rho_j'))_{\nu_j \le i \le \nu_j+m_j-1} = B_j^{(n)}(\eta_j^{(n)})$ where $\eta_j^{(n)} := (\sqrt{n}(\lambda_i^{(n)} - \lambda_j'))_{\nu_j \le i \le \nu_j+m_j-1}$ and $B_j^{(n)}$ is the random operator

$$(x_i)_{\nu_j \le i \le \nu_j+m_j-1} \in \mathbb{R}^{m_j} \mapsto (x_i(\rho_i^{(n)} + \rho_j')^{-1})_{\nu_j \le i \le \nu_j+m_j-1} \in \mathbb{R}^{m_j}.$$

For $i \in \{\nu_j, \dots, \nu_j + m_j - 1\}$, $\rho_i^{(n)}$ converges almost surely, as $n \to +\infty$, to ρ_j', then $B_j^{(n)}$ converges almost surely uniformly to the operator $B_j := (2\rho_j')^{-1} I_{\mathbb{R}^{m_j}}$, where $I_{\mathbb{R}^{m_j}}$ denotes the identity of $\mathbb{R}^{m_j}$. Moreover, we have:

$$\|B_j^{(n)}(\eta_j^{(n)}) - B_j(\eta_j^{(n)})\|_{\mathbb{R}^{m_j}} \le \|B_j^{(n)} - B_j\|_\infty \|\eta_j^{(n)}\|_{\mathbb{R}^{m_j}}.$$

Since $\|\eta_j^{(n)}\|_{\mathbb{R}^{m_j}}$ (resp. $\|B_j^{(n)} - B_j\|_\infty$) converges in distribution (resp. in probability), as $n \to +\infty$, to $\|\Delta(\xi_j)\|_{\mathbb{R}^{m_j}}$ (resp. 0), the previous inequality implies the convergence in probability of $B_j^{(n)}(\eta_j^{(n)}) - B_j(\eta_j^{(n)})$ to 0 as $n \to +\infty$. Hence $B_j^{(n)}(\eta_j^{(n)})$ and $B_j(\eta_j^{(n)})$ have the same limit distribution; using (ii) and the continuity of B_j, we then conclude that $(\sqrt{n}(\rho_i^{(n)} - \rho_j'))_{\nu_j \le i \le \nu_j+m_j-1}$ converges in distribution, as $n \to +\infty$, to $\Delta(\xi_j')$ where $\xi_j' := (2\rho_j')^{-1}\xi_j$. Clearly, ξ_j' has a centered normal distribution with covariance operator $(4\lambda_j')^{-1}\Psi_j'\Gamma\Psi_j'^*$.

(iv) and (v). If $m_j = 1$, then ξ_j and ξ_j' have ranks equal to one then $\Delta(\xi_j) = \mathrm{tr}(\xi_j)$ and $\Delta(\xi_j') = \mathrm{tr}(\xi_j')$. Moreover, we know from Proposition 4 in Dossou-Gbete and Pousse (1991) that $\sqrt{n}(\alpha_{1\cdot3,n}^{(\nu_j)} - \alpha_{1\cdot3}^{(\nu_j)})$ converges in distribution, as $n \to +\infty$, to $\Theta_j(U)$, that is a centered normal r.v. with covariance operator $\Lambda_j'' = \Theta_j \, \Gamma \Theta_j^*$. □

The covariance operator Γ can be expressed using the canonical variates and coefficients related to the LRCA. For $(i,j) \in \{1, \dots, p_1\}^2$, put

$$\varepsilon_{ij\cdot3} = \alpha_{1\cdot3}^{(i)} \otimes \alpha_{1\cdot3}^{(j)} + \alpha_{1\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}$$

and

$$F_{ij\cdot3} = -\rho_i^2 f_{1\cdot3}^{(i)} f_{1\cdot3}^{(j)} - \rho_i\rho_j f_{1\cdot3}^{(i)} f_{2\cdot3}^{(j)} + 2\rho_j f_{1\cdot3}^{(i)} f_{2\cdot3}^{(j)}$$

(with $\rho_i = 0$ if $i > r$); we then obtain

COROLLARY 3.1. *One has*: $\Gamma = \frac{1}{4}\sum_{1 \le i,j,k,l \le p_1} \mathbb{E}(F_{ij\cdot3}F_{kl\cdot3})\varepsilon_{ij\cdot3}\widetilde{\otimes}\varepsilon_{kl\cdot3}$.

PROOF. From (P2) and (P3) (see Section 2), it is easily seen that if $i \le r$, $V_{12\cdot3}\alpha_{2\cdot3}^{(i)} = \rho_i\alpha_{1\cdot3}^{(i)}$ and that if $i > r$, $V_{12\cdot3}\alpha_{2\cdot3}^{(i)} = 0$ (because $V_{21\cdot3}V_{12\cdot3}\alpha_{2\cdot3}^{(i)} = 0$). Then, using equation (2.8), we obtain:

$$X_{2\cdot3} \otimes X_{1\cdot3} = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} f_{1\cdot3}^{(i)} f_{2\cdot3}^{(j)} \alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)} = (X_{1\cdot3} \otimes X_{2\cdot3})^*$$

and for $k \in \{1,2\}$:

$$X_{k\cdot 3}^{\otimes^2} = \sum_{i=1}^{p_k}\sum_{j=1}^{p_k} f_{k\cdot 3}^{(i)} f_{k\cdot 3}^{(j)} \alpha_{k\cdot 3}^{(i)} \otimes \alpha_{k\cdot 3}^{(j)}.$$

Thus, using equation (2.9), we obtain

$$\text{(3.8)} \qquad V_{12\cdot 3} = \mathbb{E}(X_{2\cdot 3} \otimes X_{1\cdot 3}) = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} \delta_{ij}\rho_i \alpha_{2\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)} = \sum_{i=1}^{r} \rho_i \alpha_{2\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(i)},$$

$$V_{12\cdot 3}V_{21\cdot 3} = \sum_{i=1}^{r} \rho_i^2 \alpha_{1\cdot 3}^{(i)\otimes^2}$$

and using again equation (2.8)

$$\begin{aligned} V_{12\cdot 3}V_{21\cdot 3}X_{1\cdot 3}^{\otimes^2} &= \sum_{i=1}^{r} \sum_{1\le j,k\le p_1} \rho_i^2 f_{1\cdot 3}^{(j)} f_{1\cdot 3}^{(k)} \alpha_{1\cdot 3}^{(i)\otimes^2} \alpha_{1\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(k)} \\ &= \sum_{1\le i,k\le r} \sum_{j=1}^{p_1} \rho_i^2 f_{1\cdot 3}^{(j)} f_{1\cdot 3}^{(k)} \delta_{ki} \alpha_{1\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{p_1} \rho_i^2 f_{1\cdot 3}^{(i)} f_{1\cdot 3}^{(j)} \alpha_{1\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)}; \end{aligned}$$

moreover

$$\begin{aligned} V_{12\cdot 3}X_{2\cdot 3}^{\otimes^2}V_{21\cdot 3} = (V_{12\cdot 3}X_{2\cdot 3})^{\otimes^2} &= \left(\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} \rho_i f_{2\cdot 3}^{(j)} (\alpha_{2\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(i)}) \alpha_{2\cdot 3}^{(j)} \right)^{\otimes^2} \\ &= \left(\sum_{i=1}^{r} \rho_i f_{2\cdot 3}^{(i)} \alpha_{1\cdot 3}^{(i)} \right)^{\otimes^2} = \sum_{1\le i,j\le r} \rho_i\rho_j f_{2\cdot 3}^{(i)} f_{2\cdot 3}^{(j)} \alpha_{1\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(j)}, \end{aligned}$$

and

$$\begin{aligned} V_{12\cdot 3}(X_{1\cdot 3} \otimes X_{2\cdot 3}) &= \sum_{i=1}^{r}\sum_{j=1}^{p_1}\sum_{k=1}^{p_2} \rho_i f_{1\cdot 3}^{(j)} f_{2\cdot 3}^{(k)} (\alpha_{2\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(i)})(\alpha_{1\cdot 3}^{(j)} \otimes \alpha_{2\cdot 3}^{(k)}) \\ &= \sum_{1\le i,j\le r} \rho_j f_{1\cdot 3}^{(i)} f_{2\cdot 3}^{(j)} \alpha_{1\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(j)}. \end{aligned}$$

Thus

$$\Gamma = \mathbb{E}\left[\left(\pi \left(\sum_{1\le i,j\le r} F_{ij} \alpha_{1\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(j)} \right) \right)^{\tilde{\otimes}^2} \right] = \mathbb{E}\left[\left(\pi \left(\sum_{1\le i,j\le p_1} F_{ij} \alpha_{1\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(j)} \right) \right)^{\tilde{\otimes}^2} \right],$$

the second equality being justified by the fact that if $i > r$ or $j > r$ then $F_{ij} = 0$. Noticing that $\pi\ (\alpha_{1\cdot 3}^{(i)} \otimes \alpha_{1\cdot 3}^{(j)}) = \frac{1}{2}\varepsilon_{ij\cdot 3}$, one sees that the required result is obtained. □

3.3 *Asymptotic study of the elements associated with the null eigenvalue*

As it was noticed for the usual LCA (see Larrère (1994)), the previous results can not be exploited for statistical inference involving only the eigenelements associated with the null eigenvalue of $T_{1\cdot 3}$. Indeed, easy calculations show that the eigenvalue limiting distributions obtained in Proposition 3.3 are Dirac distributions. Then it is necessary to use another approach for having the asymptotic distributions related to these associated eigenelements. Notice that we implicitly suppose $r < p_1$ (else there does not exist a null eigenvalue).

Let P_0 be the orthogonal projector onto the eigenspace associated with the null eigenvalue of $T_{1\cdot 3}$, we have

$$P_0 = \sum_{i=r+1}^{p_1} \alpha_{1\cdot 3}^{(i)^{\otimes^2}}$$

and put $P_0^{(n)} := \sum_{i=r+1}^{p_1} \alpha_{1\cdot 3,n}^{(i)}{}^{\otimes^2}$; from the Assertion (ii) of Proposition 3.1 we have the almost sure uniform convergence of $P_0^{(n)}$ to P_0, as $n \to +\infty$. Then $P_0^{(n)} T_{1\cdot 3}^{(n)} P_0^{(n)}$ almost surely uniformly converges to $P_0 T_{1\cdot 3} P_0 = 0$, as $n \to +\infty$. Now, we can derive the limit distribution of $nP_0^{(n)} T_{1\cdot 3}^{(n)} P_0^{(n)}$. Putting $P_0' := \sum_{i=r+1}^{p_2} \alpha_{2\cdot 3}^{(i)^{\otimes^2}}$ we have

PROPOSITION 3.4. *The r.v. $nP_0^{(n)} T_{1\cdot 3}^{(n)} P_0^{(n)}$ converges in distribution, as $n \to +\infty$, to $\Psi\Psi^*$, where Ψ is a r.v. having a centered normal distribution in $\mathcal{L}(\mathcal{X}_2, \mathcal{X}_1)$, with covariance operator*:

$$\Gamma_0 = \mathbb{E}[(P_0(X_{2\cdot 3} \otimes X_{1\cdot 3})P_0')^{\tilde{\otimes}^2}].$$

PROOF. We have $nP_0^{(n)} T_{1\cdot 3}^{(n)} P_0^{(n)} = \Psi_n \Psi_n^*$, with $\Psi_n = \sqrt{n} P_0^{(n)} V_{1\cdot 3}^{(n)^{-1/2}} V_{12\cdot 3}^{(n)} \cdot V_{2\cdot 3}^{(n)^{-1/2}}$. Using equation (3.8) and the orthonormality of the $\alpha_{1\cdot 3}^{(i)}$'s, we obtain $P_0 V_{12\cdot 3} = 0$. Thus

$$\begin{aligned}
\Psi_n = {} & \sqrt{n}(P_0^{(n)} - P_0) V_{1\cdot 3}^{(n)^{-1/2}} V_{12\cdot 3}^{(n)} V_{2\cdot 3}^{(n)^{-1/2}} \\
& + P_0[\sqrt{n}(V_{1\cdot 3}^{(n)^{-1/2}} - I_1)] V_{12\cdot 3}^{(n)} V_{2\cdot 3}^{(n)^{-1/2}} \\
& + P_0[\sqrt{n}(V_{12\cdot 3}^{(n)} - V_{12\cdot 3})] V_{2\cdot 3}^{(n)^{-1/2}} \\
& + P_0 V_{12\cdot 3}[\sqrt{n}(V_{2\cdot 3}^{(n)^{-1/2}} - I_2)];
\end{aligned}$$

then using the relation in equation (3.6) with $T = V_{k\cdot 3}^{(n)}$ $(k = 1, 2)$ and the Lemmas 3.1 and 3.2, we have

$$\Psi_n = \sqrt{n}(P_0^{(n)} - P_0) V_{1\cdot 3}^{(n)^{-1/2}} V_{12\cdot 3}^{(n)} V_{2\cdot 3}^{(n)^{-1/2}} + \eta^{(n)}(H_n)$$

where $\eta^{(n)}$ is the random operator

$$\begin{aligned}
\eta^{(n)}(T) = {} & - P_0 V_{1\cdot 3}^{(n)^{-1/2}} a_{1\cdot 3}^{(n)}(T)(V_{1\cdot 3}^{(n)^{-1/2}} + I_1)^{-1} V_{12\cdot 3}^{(n)} V_{2\cdot 3}^{(n)^{-1/2}} + P_0 a_{12\cdot 3}^{(n)}(T) V_{2\cdot 3}^{(n)^{-1/2}} \\
& - P_0 V_{12\cdot 3} V_{2\cdot 3}^{(n)^{-1/2}} a_{2\cdot 3}^{(n)}(T)(V_{2\cdot 3}^{(n)^{-1/2}} + I_1)^{-1}
\end{aligned}$$

which converges almost surely uniformly to the operator η defined as

$$T \in M \mapsto -\frac{1}{2}P_0 a_{1\cdot 3}(T)V_{12\cdot 3} - \frac{1}{2}P_0 V_{12\cdot 3}a_{2\cdot 3}(T) + P_0 a_{12\cdot 3}(T) \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1).$$

One knows that (see Dossou-Gbete and Pousse (1991)) $\sqrt{n}(P_0^{(n)} - P_0) = \phi^{(n)}(\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3}))$, where $\phi^{(n)}$ converges almost surely uniformly to

$$\phi : T \in \mathcal{L}(\mathcal{X}_1) \mapsto P_0 T S_0 + S_0 T P_0 \in \mathcal{L}(\mathcal{X}_1)$$

and $S_0 := -\sum_{i=1}^r \rho_i^{-2} \alpha_{1\cdot 3}^{(i)^{\otimes 2}}$. As $\sqrt{n}(T_{1\cdot 3}^{(n)} - T_{1\cdot 3}) = \varphi_n(H_n)$ where φ_n converges almost surely uniformly to φ (see the proof of Proposition 3.2), Ψ_n converges in distribution to $\eta'(H)$ where

$$\eta'(T) = \phi(\varphi(T)) + \eta(T) = P_0\varphi(T)S_0 + S_0\varphi(T)P_0 + \eta(T); \tag{3.9}$$

since η' is linear, $\eta'(H)$ has a centered normal distribution in $\mathcal{L}(\mathcal{X}_1)$. Its covariance operator is

$$\Gamma_0 = \eta'\mathbb{E}[H^{\tilde{\otimes}^2}]\eta'^* = \eta'\mathbb{E}[(Z^{\otimes^{2^c}})^{\tilde{\otimes}^2}]\eta'^* = \mathbb{E}[(\eta'(Z^{\otimes^{2^c}}))^{\tilde{\otimes}^2}];$$

(recall that $Z^{\otimes^{2^c}} = Z^{\otimes^2} - \mathbb{E}(Z^{\otimes^2})$). Using equation (3.8) and the orthonormality of the $\alpha_{k\cdot 3}^{(i)}$'s ($k = 1, 2$) implies $V_{21\cdot 3}S_0V_{12\cdot 3} = P_0' - I_2$. From equations (3.7) and (3.9), and recalling that $P_0V_{12\cdot 3} = 0$, we obtain $\eta(Z^{\otimes^{2^c}}) = P_0(X_{2\cdot 3} \otimes X_{1\cdot 3})P_0'$; this completes the proof. □

4. Some applications

4.1 *Testing for the lack of linear relative association*

LRCA may be seen as a tool which permits to see if there is or not a linear relative association between X_1 and X_2 relative to X_3. We say that there is a lack of linear relative association, if

$$V_{12} = V_{13}V_3^{-1}V_{32}, \tag{4.1}$$

that is $V_{12\cdot 3} = 0$. One knows that (see, e.g., Timm and Carlson (1976)) when (X_1, X_2, X_3) has a normal distribution this property is equivalent to the conditional independence of X_1 and X_2, given X_3.

Following an approach which has been used for classical LCA (Cramer and Nicewander (1979), Lin (1987), Cléroux and Lazraq (1988), Dauxois and Nkiet (1997*b*), Nkiet (2000)), a class of linear relative association measures have been introduced by Dauxois and Nkiet (2002). These measures have the form $m_{/X_3}(X_1, X_2) := \Phi(\lambda)$, where Φ is a continuous symmetric function from $\mathbb{R}^{p_1}$ to $\mathbb{R}_+$ satisfying $\Phi(x) = 0 \Leftrightarrow x = 0$. These measures can be used for testing for lack of linear relative association since equation (4.1) is equivalent to $m_{/X_3}(X_1, X_2) = 0$ (see Dauxois and Nkiet (2002)). Then we consider the test of $H_0 : m_{/X_3}(X_1, X_2) = 0$ against $H_1 : m_{/X_3}(X_1, X_2) > 0$.

We take as test statistic, the r.v. $\Phi(\lambda^{(n)})$ because, by (i) of Proposition 3.1 and the continuity of Φ, it is a strongly consistent estimator of $m_{/X_3}(X_1, X_2)$. In order to derive the asymptotic distribution of $\Phi(\lambda^{(n)})$ under H_0, we suppose that the following assumptions hold:

(A5) Φ is twice differentiable on an open set $\mathcal{O}$ containing λ and $\lambda^{(n)}(\omega)$ ($\omega \in \Omega$), and it exists $M > 0$ such that for any $x \in \mathcal{O}$, one has $\|D^2\Phi(x)\| \leq M$;

(A6) putting $K_\Phi := \frac{\partial\Phi}{\partial x_1}(0)$, one has $K_\Phi \neq 0$.

Since under H_0 one has $s = 1$, $\lambda_1' = 0$, $P_0' = I_2$ $P_1 = P_0 = P_0^{(n)} = I_1$, a direct application of Theorem 1 of Dauxois and Nkiet (2000) and Proposition 3.4 gives

PROPOSITION 4.1. *Under H_0, $nK_\Phi^{-1}\Phi(\lambda^{(n)})$ converges in distribution, as $n \to +\infty$, to $R := \operatorname{tr}(\Psi\Psi^*)$, where Ψ has a centered normal distribution in $\mathcal{L}(\mathcal{X}_2, \mathcal{X}_1)$ with covariance operator equal to $\Gamma_0 = \mathbb{E}[(X_{2\cdot3} \otimes X_{1\cdot3})^{\tilde{\otimes}^2}]$.*

Then, for a given asymptotic level $\alpha \in]0,1[$, the null hypothesis is rejected when $nK_\Phi^{-1}\Phi(\lambda^{(n)}) > \mathbb{F}_Q^{-1}(\alpha)$, where $\mathbb{F}_Q$ is the distribution function of Q. Note that Q is a regular quadratic form of normal vector; then since $\mathbb{F}_Q$ is continuous (see Mathai and Provost (1992)), it is a bijective function. When (X_1, X_2, X_3) has an elliptic distribution, Q is of a more simpler form:

COROLLARY 4.1. *If (X_1, X_2, X_3) has an elliptic distribution with kurtosis κ then under H_0 the distribution of R is $(1+\kappa)\chi^2_{p_1p_2}$.*

PROOF. Since $(\alpha_{1\cdot3}^{(i)})_{1\leq i\leq p_1}$ (resp. $(\alpha_{2\cdot3}^{(j)})_{1\leq j\leq p_j}$) is an orthonormal basis of $\mathcal{X}_1$ (resp. $\mathcal{X}_2$), we can write

$$\Psi = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\langle\Psi, \alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}\rangle_2 \alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}$$

thus, putting $W_{ij} := \langle\Psi, \alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}\rangle_2$:

$$\begin{aligned}\operatorname{tr}(\Psi\Psi^*) &= \sum_{1\leq i,k\leq p_1}\sum_{1\leq j,l\leq p_2} W_{ij}W_{kl}\operatorname{tr}((\alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)})(\alpha_{1\cdot3}^{(k)} \otimes \alpha_{2\cdot3}^{(l)}))\\ &= \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} W_{ij}^2; \qquad (4.2)\end{aligned}$$

the later equality being obtained from the properties $(\alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)})(\alpha_{1\cdot3}^{(k)} \otimes \alpha_{2\cdot3}^{(l)}) = \delta_{lj}\alpha_{1\cdot3}^{(k)} \otimes \alpha_{1\cdot3}^{(i)}$ and $\operatorname{tr}(\alpha_{1\cdot3}^{(k)} \otimes \alpha_{1\cdot3}^{(i)}) = \langle\alpha_{1\cdot3}^{(k)}, \alpha_{1\cdot3}^{(i)}\rangle_{\mathcal{X}_1} = \delta_{ki}$. Each W_{ij} has a centered normal distribution because it is a linear function of Ψ. Moreover, for any $(i,k,j,l) \in \{1,\ldots,p_1\}^2 \times \{1,\ldots,p_2\}^2$ we have

$$\begin{aligned}\mathbb{E}(W_{ij}W_{kl}) &= \mathbb{E}(\langle\Psi, \alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}\rangle_2\langle\Psi, \alpha_{2\cdot3}^{(l)} \otimes \alpha_{1\cdot3}^{(k)}\rangle_2)\\ &= \mathbb{E}(\langle(\Psi\tilde{\otimes}\Psi)(\alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}), \alpha_{2\cdot3}^{(l)} \otimes \alpha_{1\cdot3}^{(k)}\rangle_2)\\ &= \langle\Gamma_0(\alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}), \alpha_{2\cdot3}^{(l)} \otimes \alpha_{1\cdot3}^{(k)}\rangle_2.\end{aligned}$$

Furthermore

$$\Gamma_0(\alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}) = \mathbb{E}(\langle X_{2\cdot3} \otimes X_{1\cdot3}, \alpha_{2\cdot3}^{(j)} \otimes \alpha_{1\cdot3}^{(i)}\rangle_2(X_{2\cdot3} \otimes X_{1\cdot3})),$$

hence

$$\mathbb{E}(W_{ij}W_{kl}) = \mathbb{E}(\langle X_{2\cdot 3}\otimes X_{1\cdot 3}, \alpha_{2\cdot 3}^{(j)}\otimes\alpha_{1\cdot 3}^{(i)}\rangle_2\langle X_{2\cdot 3}\otimes X_{1\cdot 3}, \alpha_{2\cdot 3}^{(l)}\otimes\alpha_{1\cdot 3}^{(k)}\rangle_2).$$

From the equalities $\langle x\otimes y, z\otimes t\rangle_2 = \mathrm{tr}((x\otimes y)(t\otimes z)) = \mathrm{tr}(\langle x,z\rangle(t\otimes y))$ and $\mathrm{tr}(t\otimes y) = \langle t, y\rangle$, it follows

$$\mathbb{E}(W_{ij}W_{kl}) = \mathbb{E}(f_{1\cdot 3}^{(i)}f_{1\cdot 3}^{(k)}f_{2\cdot 3}^{(j)}f_{2\cdot 3}^{(l)})$$

where $f_{k\cdot 3}^{(i)}$ is defined in equation (2.7) (recall that $V_{1\cdot 3} = I_1$ and $V_{2\cdot 3} = I_2$). Since $(f_{1\cdot 3}^{(i)}, f_{1\cdot 3}^{(k)}, f_{2\cdot 3}^{(j)}, f_{2\cdot 3}^{(l)})$ is a linear function of (X_1, X_2, X_3), it also has an elliptic distribution with kurtosis κ. Hence

$$\begin{aligned}\mathbb{E}(W_{ij}W_{kl}) = (1+\kappa)[&\mathbb{E}(f_{1\cdot 3}^{(i)}f_{1\cdot 3}^{(k)})\mathbb{E}(f_{2\cdot 3}^{(j)}f_{2\cdot 3}^{(l)}) + \mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)})\mathbb{E}(f_{1\cdot 3}^{(k)}f_{2\cdot 3}^{(l)})\\ &+ \mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(l)})\mathbb{E}(f_{1\cdot 3}^{(k)}f_{2\cdot 3}^{(j)})].\end{aligned}$$

Under H_0, all the canonical coefficients are null since $V_{12\cdot 3} = 0$; then using equations (2.9) and (2.10) we obtain $\mathbb{E}(W_{ij}W_{kl}) = (1+\kappa)(\delta_{ik}\delta_{jl})$. This shows that $\mathbb{E}(W_{ij}^2) = (1+\kappa)$ and if $(i,j)\neq(k,l)$, $\mathbb{E}(W_{ij}W_{kl}) = 0$; then the r.v. $(1+\kappa)^{-1}W_{ij}$ are independent and have the standard normal distribution. Using equation (4.2) we conclude that $R = (1+\kappa)\chi^2_{p_1p_2}$. □

Remark 4.1. This approach for testing lack of linear relative association yields a general framework containing tests which are based on what is known in the literature as partial canonical correlation coefficients. For example, the test proposed by Timm and Carlson (1976) under the assumption that (X_1, X_2, X_3) has a normal distribution appears in our context as a particular case by taking $\Phi(x) = -\sum_{i=1}^{p_1}\ln(1-x_i)$. Notice that when $p_1 = 1$ (resp. $p_1 = p_2 = 1$) our test is based on a function of the classical partial multiple correlation coefficient (resp. partial correlation coefficient).

4.2 *Testing for dimensionality in LRCA*

As in Fujikoshi and Veitch (1979) for usual LCA, we can introduce tests for determining the dimensionality of LRCA, that is the integer $d\in\{1,\dots,p_1\}$ equal to p_1 if all the relative canonical coefficients are non null or such that

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_d>\lambda_{d+1}=\cdots=\lambda_{p_1}=0$$

elsewhere. For doing that, we introduce for any $k\in\{1,\dots,p_1-1\}$ a test for $H_0^{(k)}$ against $H_1^{(k)}$, with

$$H_0^{(k)}:\lambda_k>\lambda_{k+1}=\cdots=\lambda_{p_1}=0$$

and

$$H_1^{(k)}:\lambda_k=0.$$

Notice $\lambda_{k+1},\dots,\lambda_{p_1}$ are the eigenvalues of $R_{1\cdot 3}^{(k)} = Q_kT_{1\cdot 3}Q_k$, where $Q_k := \sum_{i=k+1}^{p_1}\alpha_{1\cdot 3}^{(i)^{\otimes 2}}$. Then we take as test statistic the r.v. $\Phi(\mu_k^{(n)})$, where

$$\mu_k^{(n)} := (\lambda_{k+1}^{(n)},\dots,\lambda_{p_1}^{(n)}) = \Delta(Q_k^{(n)}T_{1\cdot 3}^{(n)}Q_k^{(n)})$$

with $Q_k^{(n)} := \sum_{i=k+1}^{p_1} \alpha_{1\cdot 3,n}^{(i)^{\otimes 2}}$, and Φ is a symmetric function from $\mathbb{R}^{p_1-k}$ to $\mathbb{R}$ satisfying (A5), (A6) and $\Phi(x) = 0 \Leftrightarrow x = 0$. Let us consider $Q'_k := \sum_{i=k+1}^{p_1} \alpha_{2\cdot 3}^{(i)^{\otimes 2}}$. Under $H_0^{(k)}$, we obviously have $Q'_k = P'_0$, $Q_k = P_0$ and $Q_k^{(n)} = P_0^{(n)}$, hence by applying again Proposition 3.4 and Theorem 1 of Dauxois and Nkiet (2000) we obtain:

PROPOSITION 4.2. *Under $H_0^{(k)}$, $nK_\Phi^{-1}\Phi(\mu_k^{(n)})$ converges in distribution, as $n \to +\infty$, to $R_k := \operatorname{tr}(\Psi_k\Psi_k^*)$, where Ψ_k has a centered normal distribution in $\mathcal{L}(\mathcal{X}_2, \mathcal{X}_1)$ with covariance operator $\Gamma_k = \mathbb{E}[(Q_k(X_{2\cdot 3} \otimes X_{1\cdot 3})Q'_k)^{\tilde{\otimes}^2}]$.*

Then, for a given asymptotic level $\alpha \in]0,1[$, the null hypothesis is rejected when $nK_\Phi^{-1}\Phi(\lambda^{(n)}) > \mathbb{F}_Q^{-1}(\alpha)$, where $\mathbb{F}_Q$ is the distribution function of Q. For determining the dimensionality, we use the previous test for $k = p_1 - 1$, $k = p_1 - 2$ and so on until $H_0^{(k)}$ is not rejected or $k = 1$ (in this case the dimensionality is 1).

When (X_1, X_2, X_3) has an elliptic distribution we have:

COROLLARY 4.2. *Under $H_0^{(k)}$, if (X_1, X_2, X_3) has an elliptic distribution with kurtosis κ, then the distribution of R_k is $(1+\kappa)\chi^2_{(p_1-k)(p_2-k)}$.*

PROOF. By similar arguments as in the proof of Corollary 4.1, we have

$$\operatorname{tr}(\Psi_k\Psi_k^*) = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} W_{ij}^{(k)^2} \tag{4.3}$$

where $W_{ij}^{(k)} := \langle \Psi_k, \alpha_{2\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)}\rangle_2$. Each $W_{ij}^{(k)}$ has a centered normal distribution because it is a linear function of Ψ_k. Moreover, for any $(i,h,j,l) \in \{1,\dots,p_1\}^2 \times \{1,\dots,p_2\}^2$ we have

$$\begin{aligned}\mathbb{E}(W_{ij}^{(k)}W_{hl}^{(k)}) &= \langle \Gamma_k(\alpha_{2\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)}), \alpha_{2\cdot 3}^{(l)} \otimes \alpha_{1\cdot 3}^{(h)}\rangle_2 \\ &= \mathbb{E}(\langle Q_k(X_{2\cdot 3} \otimes X_{1\cdot 3})Q'_k, \alpha_{2\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)}\rangle_2 \\ &\quad \times \langle Q_k(X_{2\cdot 3} \otimes X_{1\cdot 3})Q'_k, \alpha_{2\cdot 3}^{(l)} \otimes \alpha_{1\cdot 3}^{(h)}\rangle_2).\end{aligned} \tag{4.4}$$

Under $H_0^{(k)}$, the rank of $T_{1\cdot 3}$ equals k, and $Q'_k = P'_0$, $Q_k = P_0$; then, since $P_0\alpha_{1\cdot 3}^{(i)} = 0$ (resp. $P'_0\alpha_{2\cdot 3}^{(j)} = 0$) when $i \le k$ (resp. $j \le k$), we deduce from

$$\begin{aligned}&\langle Q_k(X_{2\cdot 3} \otimes X_{1\cdot 3})Q'_k, \alpha_{2\cdot 3}^{(j)} \otimes \alpha_{1\cdot 3}^{(i)}\rangle_2 \\ &\quad = \operatorname{tr}((X_{2\cdot 3} \otimes X_{1\cdot 3})P'_0(\alpha_{1\cdot 3}^{(i)} \otimes \alpha_{2\cdot 3}^{(j)})P_0) \\ &\quad = \operatorname{tr}((X_{2\cdot 3} \otimes X_{1\cdot 3})((P_0\alpha_{1\cdot 3}^{(i)}) \otimes (P'_0\alpha_{2\cdot 3}^{(j)})))\end{aligned} \tag{4.5}$$

and equation (4.4) that if $i \le k$ or $j \le k$ then $\mathbb{E}(W_{ij}^{(k)^2}) = 0$, that is $W_{ij}^{(k)} = 0$. Further, if $i > k$ and $j > k$ then $P_0\alpha_{1\cdot 3}^{(i)} = \alpha_{1\cdot 3}^{(i)}$ and $P'_0\alpha_{2\cdot 3}^{(j)} = \alpha_{2\cdot 3}^{(j)}$; hence, equation (4.5) implies

$$\mathbb{E}(W_{ij}W_{hl}) = \mathbb{E}(f_{1\cdot 3}^{(i)}f_{1\cdot 3}^{(h)}f_{2\cdot 3}^{(j)}f_{2\cdot 3}^{(l)})$$

where $f_{k\cdot 3}^{(i)}$ is defined in equation (2.7) (recall that $V_{1\cdot 3} = I_1$ and $V_{2\cdot 3} = I_2$). Using similar arguments than in the proof of Corollary 4.1, we obtain

$$\mathbb{E}(W_{ij}W_{hl}) = (1+\kappa)[\mathbb{E}(f_{1\cdot 3}^{(i)}f_{1\cdot 3}^{(h)})\mathbb{E}(f_{2\cdot 3}^{(j)}f_{2\cdot 3}^{(l)}) + \mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)})\mathbb{E}(f_{1\cdot 3}^{(h)}f_{2\cdot 3}^{(l)}) \\ + \mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(l)})\mathbb{E}(f_{1\cdot 3}^{(h)}f_{2\cdot 3}^{(j)})].$$

Since $k = r$, equation (2.9) implies that for $j > k$, we have $\mathbb{E}(f_{1\cdot 3}^{(i)}f_{2\cdot 3}^{(j)}) = 0$. Then using equation (2.10) we obtain $\mathbb{E}(W_{ij}W_{hl}) = (1+\kappa)(\delta_{ih}\delta_{jl})$, that is $\mathbb{E}(W_{ij}^2) = (1+\kappa)$ and $\mathbb{E}(W_{ij}W_{kl}) = 0$ if $(i,j) \neq (h,l)$. We conclude that if $i > k$ and $j > k$ then the r.v. $(1+\kappa)^{-1}W_{ij}$ are independent and have the standard normal distribution, and if $i \leq k$ or $j \leq k$ then they are null. The equation (4.3) can be rewritten as

$$\mathrm{tr}(\Psi_k\Psi_k^*) = \sum_{i=k+1}^{p_1}\sum_{j=k+1}^{p_2} W_{ij}^{(k)^2}$$

and this shows that $R_k = (1+\kappa)\chi^2_{(p_1-k)(p_2-k)}$. $\square$

4.3 *Testing for invariance of LRCA*

Invariance for LRCA when the related variables are transformed by linear maps have been defined in Section 2 and conditions for having this invariance property have been obtained. Nevetheless, since these conditions involve covariance operators which are unknown in practice, it is of interest to contruct a test which permits to see whether or not LRCA is invariant for a given pair of linear maps. Notice that such an approach has already been used in the literature. Indeed, in Fujikoshi (1982) and in Fujikoshi and Khatri (1990) likelihood ratio tests for additional information in LCA and for redundancy in covariate discriminant analysis under normal assumption were introduced. These tests just are particular tests for invariance when the related variables are transformed by projectors. More recently, a generalizing approach has been adopted by Nkiet (2003) who introduced a test for the invariance of LCA when the related variables are transformed by linear maps which may not be projectors, without other assumption on the distribution of these variables besides the existence of four order moments.

Here we extend for LRCA an approach used in Nkiet (2003) for LCA. Note that the results of this later work can not be applied for LRCA because we do not have an i.i.d. sample of $X_{1\cdot 3}$ and $X_{2\cdot 3}$; these r.v. are unobservable since their definitions involve covariance operators which are unknown in practice. Let A_1 and A_2 be linear maps defined on $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. Our purpose is to introduce a test for the invariance of the LRCA of X_1 and X_2 relative to X_3. Consider

$$\begin{aligned} C_1(A_1) &= \|V_{12\cdot 3} - V_{1\cdot 3}A_1^*(A_1V_{1\cdot 3}A_1^*)^\dagger A_1V_{12\cdot 3}\|^2, \\ C_2(A_2) &= \|V_{21\cdot 3} - V_{2\cdot 3}A_2^*(A_2V_{2\cdot 3}A_2^*)^\dagger A_{2\cdot 3}V_{21\cdot 3}\|^2, \\ C(A_1,A_2) &= C_1(A_1) + C_2(A_2), \end{aligned}$$

where $\|\cdot\|$ denotes the norm associated with the Hilbert-Schmidt inner product. From Proposition 2.2 it is seen that the aforementioned test is the test of the hypothesis

$$H_0 : C(A_1,A_2) = 0 \quad \text{against} \quad H_1 : C(A_1,A_2) > 0.$$

Defining for $(k,j) \in \{1,2\}^2$ with $k \neq j$

$$S_{k\cdot 3} := V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_kV_{kj\cdot 3}$$
$$S_{k\cdot 3}^{(n)} := V_{k\cdot 3}^{(n)}A_k^*(A_kV_{k\cdot 3}^{(n)}A_k^*)^\dagger A_kV_{kj\cdot 3}^{(n)}$$

and

$$C_k^{(n)}(A_k) = \|V_{kj\cdot 3}^{(n)} - S_{k\cdot 3}^{(n)}\|^2,$$

we take as test statistic the estimator $C^{(n)}(A_1,A_2) := C_1^{(n)}(A_1)+C_2^{(n)}(A_2)$ of $C(A_1,A_2)$. For defining the corresponding (asymptotic) critical region, we must derive the limit distribution of $C^{(n)}(A_1,A_2)$ under H_0. Letting $\gamma_{1\cdot 3}$ and $\gamma_{2\cdot 3}$ be the operators from M to $\mathcal{L}(\mathcal{X}_2,\mathcal{X}_1)$ and $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ respectively, defined by

$$\begin{aligned}\gamma_{k\cdot 3}(T) &= a_{k\cdot 3}(T)A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_kV_{kj\cdot 3}\\ &\quad - V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_ka_{k\cdot 3}(T)A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_kV_{kj\cdot 3}\\ &\quad + V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger a_{kj\cdot 3}(T) - a_{kj\cdot 3}(T),\end{aligned} \tag{4.6}$$

where $(k,j) \in \{1,2\}^2$ with $k \neq j$, we have

PROPOSITION 4.3. *Under H_0, the r.v. $nC^{(n)}(A_1,A_2)$ converges in distribution, as $n \to +\infty$, to $Q = \|\gamma_{1\cdot 3}(H)\|^2 + \|\gamma_{2\cdot 3}(H)\|^2$.*

PROOF. First, for $(k,j) \in \{1,2\}^2$ with $k \neq j$, we have:

$$\begin{aligned}\sqrt{n}(S_{k\cdot 3}^{(n)} - S_{k\cdot 3}) &= \sqrt{n}(V_{k\cdot 3}^{(n)} - V_{k\cdot 3})A_k^*(A_kV_{k\cdot 3}^{(n)}A_k^*)^\dagger A_kV_{kj\cdot 3}^{(n)}\\ &\quad + \sqrt{n}V_{k\cdot 3}A_k^*[(A_kV_{k\cdot 3}^{(n)}A_k^*)^\dagger - (A_kV_{k\cdot 3}A_k^*)^\dagger]A_kV_{kj\cdot 3}^{(n)}\\ &\quad + \sqrt{n}V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_k(V_{kj\cdot 3}^{(n)} - V_{kj\cdot 3}).\end{aligned} \tag{4.7}$$

It is known that for any operators T and S one has

$$T^\dagger - S^\dagger = -T^\dagger(T-S)S^\dagger + T^{\dagger^2}(T-S)\Pi_{\ker(S)} - \Pi_{\ker(T)}(T-S)S^{\dagger^2}$$

(see Theorem 3.10 in Nashed (1976)); then applying this property with $T = A_kV_{k\cdot 3}^{(n)}A_k^*$ and $S = A_kV_{k\cdot 3}A_k^*$ and using equation (4.7), we obtain $\sqrt{n}(S_{k\cdot 3}^{(n)} - S_{k\cdot 3}) = \beta_{k\cdot 3}^{(n)}(H_n)$, where $\beta_{k\cdot 3}^{(n)}$ is the random variable valued into $\mathcal{L}(M,\mathcal{L}(\mathcal{X}_{3-k},\mathcal{X}_k))$ defined by

$$\begin{aligned}\beta_{k\cdot 3}^{(n)}(T) &= a_{k\cdot 3}^{(n)}(T)A_k^*(A_kV_{k\cdot 3}^{(n)}A_k^*)^\dagger A_kV_{kj\cdot 3}^{(n)}\\ &\quad - V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}^{(n)}A_k^*)^\dagger A_ka_{k\cdot 3}^{(n)}(T)A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_kV_{kj\cdot 3}^{(n)}\\ &\quad + V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}^{(n)}A_k^*)^{\dagger^2}A_ka_{k\cdot 3}^{(n)}(T)A_k^*\Pi_{\ker(A_kV_{k\cdot 3}A_k^*)}A_kV_{kj\cdot 3}^{(n)}\\ &\quad - V_{k\cdot 3}A_k^*\Pi_{\ker(A_kV_{k\cdot 3}^{(n)}A_k^*)}A_ka_{k\cdot 3}^{(n)}(T)A_k^*(A_kV_{k\cdot 3}A_k^*)^{\dagger^2}A_kV_{kj\cdot 3}^{(n)}\\ &\quad + V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_ka_{kj\cdot 3}(T).\end{aligned}$$

From the almost sure uniform convergence of the empirical covariance operators involved in this expression and the equalities

$$V_{k\cdot 3}A_k^*\Pi_{\ker(A_kV_{k\cdot 3}A_k^*)} = L_{k\cdot 3}^*L_{k\cdot 3}A_k^*\Pi_{\ker(L_{k\cdot 3}A_k^*)} = 0,$$
$$\Pi_{\ker(A_kV_{k\cdot 3}A_k^*)}A_kV_{kj\cdot 3} = (V_{jk\cdot 3}A_k^*\Pi_{\ker(A_kV_{k\cdot 3}A_k^*)})^* = (L_{j\cdot 3}^*L_{k\cdot 3}A_k^*\Pi_{\ker(L_{k\cdot 3}A_k^*)})^* = 0,$$

it follows that $\beta_{k\cdot 3}^{(n)}$ converges almost surely uniformly to the operator $\beta_{k\cdot 3}$ defined by

$$\begin{aligned}\beta_{k\cdot 3}(T) &= a_{k\cdot 3}(T)A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_kV_{kj\cdot 3}\\ &\quad - V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_k a_{k\cdot 3}(T)A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_kV_{kj\cdot 3}\\ &\quad + V_{k\cdot 3}A_k^*(A_kV_{k\cdot 3}A_k^*)^\dagger A_k a_{kj\cdot 3}(T),\end{aligned}$$

that isZ $\beta_{k\cdot 3} = \gamma_{k\cdot 3} + a_{kj\cdot 3}$. Using this later expression and the fact that, under H_0, the equality $V_{kj\cdot 3} = S_{k\cdot 3}$ is valid, we have:

$$nC_k^{(n)}(A_k) = n\|V_{kj\cdot 3}^{(n)} - V_{kj\cdot 3} - (S_{k\cdot 3}^{(n)} - V_{kj\cdot 3})\|^2 = \|a_{kj\cdot 3}^{(n)}(H_n) - \beta_{k\cdot 3}^{(n)}(H_n)\|^2.$$

Thus:

$$nC^{(n)}(A_1, A_2) = N(L_n(H_n)),$$

where L_n is the random operator

$$u \in M \mapsto ((\beta_{1\cdot 3}^{(n)} - a_{12\cdot 3}^{(n)})(u), (\beta_{2\cdot 3}^{(n)} - a_{21\cdot 3}^{(n)})(u)) \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1) \times \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$$

and N is the continuous map

$$(v, w) \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1) \times \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2) \mapsto \|v\|^2 + \|w\|^2 \in \mathbb{R}.$$

It is easy to verify that L_n almost surely uniformly converges to the operator

$$L : u \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) \mapsto (\gamma_{1\cdot 3}(u), \gamma_{2\cdot 3}(u)) \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1) \times \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2).$$

Further, we have

$$\|L_n(H_n) - L(H_n)\|_{\mathcal{L}(\mathcal{X}_2,\mathcal{X}_1)\times\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \le \|L_n - L\|_\infty \|H_n\|_M;$$

since $\|H_n\|_M$ (resp. $\|L_n - L\|_\infty$) converges in distribution (resp. in probability), as $n \to +\infty$, to $\|H\|_M$ (resp. 0), this inequality implies the convergence in probability of $L_n(H_n) - L(H_n)$ to 0 as $n \to +\infty$. Hence $L_n(H_n)$ and $L(H_n)$ have the same limit distribution; then $L_n(H_n)$ likewise converges in distribution to $L(H)$. From the continuity of N, it comes that $nC^{(n)}(A_1, A_2)$ converges in distribution to $Q = N(L(H)) = \|\gamma_{1\cdot 3}(H)\|^2 + \|\gamma_{2\cdot 3}(H)\|^2$. □

Then, for a given (asymptotic) level $\alpha \in]0, 1[$ the null hypothesis is rejected if $nC^{(n)}(A_1, A_2) > \mathbb{F}_Q^{-1}(\alpha)$, where $\mathbb{F}_Q$ denotes the distribution function of Q. Notice that since $C^{(n)}(A_1, A_2)$ is a strongly consistent estimator of $C(A_1, A_2)$ this test is consistent. In practice, one has to replace in equation (4.6) each covariance operator by its estimator introduced in this paper.

Remark 4.2. Additional information hypothesis in canonical analysis was discussed by Fujikoshi (1982) who introduced a likelihood ratio test for this problem. Later, this test was considered by Kariya *et al.* (1987) in order to test an hypothesis related to selection of variables in the classical MANOVA model, and it was also used by Suzukawa (1997) for evaluating the effect on canonical correlation of imposing linear constraints. This test is mainly based on a normality assumption for the variates. More recently, Nkiet (2003) proposed another test for additional information derived from a test for invariance of LCA which does not require to do any assumption on the distribution of the related random variables. A similar approach can be used here for LRCA; indeed, if for $k \in \{1, 2\}$ we have the decomposition in direct sum $\mathcal{X}_k = \mathcal{X}_k^{(1)} \oplus \mathcal{X}_k^{(2)}$, a test for additional information in LRCA having the preceding property is obtained by using the test of invariance developed above with $A_k := \pi_{k1}$, where π_{k1} is the projection operator onto $\mathcal{X}_k^{(1)}$ along $\mathcal{X}_k^{(2)}$.

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