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A SEQUENTIAL SOFTWARE RELEASE POLICY

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Abstract. Most existing studies on software release policies use models based on the non-homogeneous Poisson process. In this paper, we discuss a software release policy based on a state space model. The state space model has a Gamma-Gamma type invariant conditional distribution. A cost model that removes errors in software systems and risk cost due to software failure is used. The optimal release time to minimize the expected cost in every test-debugging stage is discussed.

Key words and phrases: Kalman filter, self-exciting point process, open-loop-feed-back-optimal control problem, submartingale.

1. Introduction

An important problem in the software development process is deciding when to stop testing and release the software. This decision problem is called the optimal software release problem and has been studied widely. Okumoto and Goel (1980) addressed a cost-optimal software release policy that minimizes the total expected software cost. Yamada *et al.* (1984) considered the optimal software release problem using two cases: when the scheduled software delivery time is constant and when it is a random variable with an arbitrary distribution. Ross (1985) developed a model with an estimating and stopping rule procedure. Yamada and Osaki (1985) introduced a cost-reliability-optimal software release policy that minimizes the total expected cost and satisfies the software reliability requirement. Dalal and Mallows (1988, 1990) proposed a stopping rule based on Bayesian assumptions and suggested a graphical display for deciding when to stop testing software. Randolph and Sahinoglu (1995) discussed the stopping rule of a compound-Poisson model.

Many software cost models have been developed. Most of these software cost models were based on non-homogeneous Poisson process (NHPP) models (see for more examples, Yamada *et al.* (1995), Kimura *et al.* (1999), Xie and Hong (1999), and Pham (2000), Chapter 6). For a detailed discussion on the software release problem refer to Singpurwalla and Wilson ((1999), Chapter 6).

As Singpurwalla and Wilson (1994) acutely pointed out, the state space models have superior predictive failure data tracking abilities than many other models. Some state space models that applied software reliability were proposed. Singpurwalla and Soyer (1992) proposed a non-homogeneous autoregressive process, known as a Kalman filter, to describe reliability growth. Chen and Singpurwalla (1994) suggested a non-Gaussian Kalman filter software reliability application. This model has a Gamma-Gamma type invariant conditional distribution. Chang and Leu (1998) proposed another Kalman filter model with an exponential-Poisson type invariant conditional distribution software reliability application.

Chen and Singpurwalla (1997) unified software reliability models using self-exciting processes. All of the models based on NHPP or Kalman filters are subsumed by the family of self-exciting processes.

In this paper we present a software cost model based on a state space model. For the failure rate in every test-debugging stage to depend on the observation history, the optimal software release problem must be approached as a sequential decision problem. To avoid complexity, the optimal release time is treated as an open-loop-feedback-optimal (OLFO) control problem (Runggaldier (1993)). That is, the optimal release time is chosen based on the current observation and process history and it is assumed that no observations will be made in the future. In Section 2, the cost model based on a non-Gaussian Kalman filter model is described. Unfortunately, the optimal release time in every stage is not a closed form. However, the optimal release time can be found numerically. In Section 3, it will be proven that, under some conditions, the total testing time in the sequential decision method is finite with probability one. A numerical example is provided in Section 4.

2. The cost model

2.1 *A non-Gaussian Kalman filter model*

The non-Gaussian Kalman filter model proposed by Chen and Singpurwalla (1994) is used in our study. Accordingly, let T_n represent the time between the $(n-1)$-th and the n-th failure (n-th stage, say), θ_n the scale parameter of T_n, $D_n = \{T_1, \ldots, T_n\}$ the collection of observations until n. The model assumptions and the results are shown below.

The observation equation: $(T_n \mid \theta_n) \sim Gamma(\omega_n, \theta_n)$.

The system equation: $(C_n\theta_n/\theta_{n-1} \mid \theta_{n-1}) \sim Beta(\sigma_{n-1}, v_{n-1})$.

The initial information: $(\theta_0 \mid D_0) \sim Gamma(\sigma_0 + v_0, u_0)$,

where C_n, ω_n, v_n and σ_n are assumed to be known and non-negative. Furthermore, they are required to satisfy the condition

$$\sigma_{n-1} + \omega_n = \sigma_n + v_n, \qquad \text{for} \quad n = 2, 3, \ldots.$$

The *Results*

The posterior of θ_{n-1}: $(\theta_{n-1} \mid D_{n-1}) \sim Gamma(\sigma_{n-1} + v_{n-1}, u_{n-1})$.

The prior of θ_n: $(\theta_n \mid D_{n-1}) \sim Gamma(\sigma_{n-1}, C_n u_{n-1})$.

The 1-step forecast: $(T_n/C_n u_{n-1} \mid D_{n-1}) \sim$ Pearson Type VI $(p = \omega_n, q = \sigma_{n-1})$; (see Johnson and Kotz (1970), p. 51).

The posterior for θ_n: $(\theta_n \mid D_n) \sim Gamma(\sigma_{n-1} + \omega_n, u_n)$, where $u_n = C_n u_{n-1} + T_n$.

$T_0 \geq 0$ is chosen arbitrarily. It reflects the best assessment about the inter-failure times prior to observing any data. In the 1-step forecast for the above results the conditional density of T_n given the history of the process is

$$f_n(t_n) \equiv f(t_n \mid D_{n-1}) = \frac{\Gamma(\omega_n + \sigma_{n-1}) t_n^{\omega_n - 1} [C_n u_{n-1}]^{\sigma_{n-1}}}{\Gamma(\omega_n)\Gamma(\sigma_{n-1})[t_n + C_n u_{n-1}]^{\omega_n + \sigma_{n-1}}}. \tag{2.1}$$

The conditional density can be found by computing directly. Thus, using (2.1), we can rewrite the 1-step forecast as

$$(T_n/(T_n + C_n u_{n-1}) \mid D_{n-1}) \sim Beta(\omega_n, \sigma_{n-1}). \tag{2.2}$$

In a special case, if ω_n is a positive integer, then the reliability function in the stage is:

$$\begin{aligned} R_n(t_n) &\equiv R(t_n \mid D_{n-1}) \\ &= \frac{(C_n u_{n-1})^{\sigma_{n-1}}}{\Gamma(\sigma_{n-1})} \left[\sum_{i=1}^{\omega_n} \frac{\Gamma(\sigma_{n-1} + i - 1)}{\Gamma(i)} \frac{t_n^{i-1}}{(t_n + C_n u_{n-1})^{\sigma_{n-1}+i-1}} \right]. \end{aligned} \tag{2.3}$$

The proofs for results (2.1) and (2.3) are shown in the Appendix.

2.2 *The cost model*

The following notations are defined:
c_1: testing cost per unit time;
c_2: cost to remove one fault during the test phase;
c_3: cost due to software failure;
T: software release time;
T_n^*: optimal release time in the n-th stage;
x: mission time;
N_{total}: the number of failures during the test phase;
T_{total}: total testing time.
The expected software testing cost at the n-th stage is

$$\phi_n(t) \equiv c_1 t + c_2 F_n(t) + c_3(F_n(t + x) - F_n(t)), \tag{2.4}$$

where $F_n(t) = 1 - R_n(t)$, the conditional distribution function of T_n. It is natural to assume $c_2 < c_3$. The notation meaning can be heuristically explained as follows. At the beginning, the optimal release time t_1^* is chosen using prior information. That is, t_1^* is found to minimize $\phi_1(\cdot)$. If $T_1 > t_1^*$, then the software is released. If $T_1 < t_1^*$, the prior information is updated to the second stage, and an additional optimal release time t_2^* is chosen. The choice will depend on T_1. t_2^* is then found to minimize $\phi_2(\cdot)$. If $T_2 > t_2^*$, the software is released. If $T_2 < t_2^*$, the prior information is updated to the third stage, and a third optimal release time t_3^* is chosen, and so on ....

The proposed algorithm for determining the optimal release time does not minimize the expected total test cost. It is designed to minimize the expected test cost in every stage and assume that no observations will be made in the future. The Kalman filter model is a special case of an ∞-memory self-exciting process. It is too complex to determine the optimal decision for minimizing the expected total cost.

Note that, in every stage, the optimal release time t_n^* minimizes the expected cost (2.4). Thus we have that

$$c_1 t_n^* + c_2 F_n(t_n^*) + c_3(F(t_n^* + x) - F_n(t_n^*)) = \phi_n(t_n^*) \le \phi_n(0) = c_3 F_n(x) \le c_3.$$

It implies

$$t_n^* \le \frac{c_3}{c_1}. \tag{2.5}$$

That is, in every stage, the optimal release time is finite. The optimal release time in every stage can be determined by solving the equation

$$\phi'_n(t) = c_1 + c_2 f_n(t) + c_3(f_n(t+x) - f_n(t)) = 0. \tag{2.6}$$

Unfortunately, in general, the ϕ'_n roots do not have a closed form. Equation (2.6) is therefore solved numerically.

When will testing stop? The total testing time is

$$T_{total} = \sum_{n=1}^{m-1} T_n + T^*_m, \tag{2.7}$$

if $T_n < T^*_n$, $n \le m-1$ and $T_m > T^*_m$, for $m \ge 1$, where $\sum_{n=1}^{0} T_n \equiv 0$. The total testing time may be infinite. In the next section, it will be proven that, under some conditions, the total testing time is finite with probability one under our model assumptions.

3. The convergence of T_{total}

We note that if $\sigma_{n-1} > 1$,

$$E(T_n \mid D_{n-1}) = \frac{\omega_n}{\sigma_{n-1} - 1} C_n u_{n-1}. \tag{3.1}$$

Thus, if $\sigma_{n-1} > 1$ and $(\frac{\omega_n}{\sigma_{n-1}-1} + 1)C_{n+1} \ge 1$, for all $n \ge 1$, we have

$$\begin{aligned} E(C_{n+1}u_n \mid D_{n-1}) &= C_{n+1}E(T_n + C_n u_{n-1} \mid D_{n-1}) \\ &= C_{n+1}\left(\frac{\omega_n}{\sigma_{n-1}-1} + 1\right) C_n u_{n-1} \ge C_n u_{n-1}, \end{aligned} \tag{3.2}$$

almost surely (a.s.), for all $n \ge 1$. That is, $C_{n+1}u_n$ is a submartingale with respect to $\sigma(D_n)$, the σ-field of D_n.

Under the assumptions in Subsection 2.1, if ω_n is an integer, by (2.3), we have that the conditional reliability in every stage

$$\begin{aligned} R_n(t) &= \frac{(C_n u_{n-1})^{\sigma_{n-1}}}{\Gamma(\sigma_{n-1})}\left[\sum_{i=1}^{\omega_n} \frac{\Gamma(\sigma_{n-1}+i-1)}{\Gamma(i)} \frac{t^{i-1}}{(t+C_n u_{n-1})^{\sigma_{n-1}+i-1}}\right] \\ &\ge \frac{(C_n u_{n-1})^{\sigma_{n-1}}}{\Gamma(\sigma_{n-1})}\left[\frac{\Gamma(\sigma_{n-1})}{\Gamma(1)} \frac{t^0}{(t+C_n u_{n-1})^{\sigma_{n-1}}}\right] \\ &= \left(\frac{C_n u_{n-1}}{t + C_n u_{n-1}}\right)^{\sigma_{n-1}}. \end{aligned} \tag{3.3}$$

Then

$$F_n(t) \equiv 1 - R_n(t) \le 1 - \left(\frac{C_n u_{n-1}}{t + C_n u_{n-1}}\right)^{\sigma_{n-1}}. \tag{3.4}$$

Thus, we have Lemma 3.1 as shown below:

LEMMA 3.1. *Under our model assumptions, if $\{\omega_n\}$ are integers, $\sigma_n = \sigma > 1$, and $(\frac{\omega_n}{\sigma-1}+1)C_{n+1} \ge 1$, for all $n > 1$. For $t > 0$ is given, $(\frac{C_n u_{n-1}}{t+C_n u_{n-1}})^\sigma$ is a submartingale with respect to $\sigma(D_{n-1})$, i.e., $1 - (\frac{C_n u_{n-1}}{t+C_n u_{n-1}})^\sigma$ is a supermartingale with respect to $\sigma(D_{n-1})$.*

The lemma and theorem proofs in this section were relegated to the Appendix. Using Lemma 3.1, we have

LEMMA 3.2. *If $A \in \sigma(D_{n-1})$, we have*

$$E\left[\left(1-\left(\frac{C_n u_{n-1}}{t+C_n u_{n-1}}\right)^{\sigma}\right) I_A\right] \le \left(1-\left(\frac{C_1 u_0}{t+C_1 u_0}\right)^{\sigma}\right) E[I_A], \qquad n \ge 1.$$

LEMMA 3.3. *Under the assumptions in Lemma 3.1, we have, for all $n \ge 1$,*

$$P(T_1 < T_1^*, \ldots, T_n < T_n^*) \le \left(1-\left(\frac{C_1 u_0}{(c_3/c_1)+C_1 u_0}\right)^{\sigma}\right)^{n}. \tag{3.5}$$

The upper bound of $E(N_{total})$ and $E(T_{total})$ is found by using Lemma 3.3.

THEOREM 3.1. *Under the assumptions of Lemma 3.1, $E(N_{total}) \le (1+\frac{c_3}{c_1 C_1 u_0})^{\sigma}-1$ and $E(T_{total}) \le (\frac{c_3}{c_1})/(\frac{C_1 u_0}{c_3/c_1+C_1 u_0})^{\sigma}$.*

The model will be modified after each stage. The remained testing time and the number of failures after each stage are more important than the initial test stage. Note that, if D_n is given, the bounds of the expected remaining testing time $E(T_{total,n} \mid D_n)$, and the expected number of failures after n-th stage $E(N_{total,n} \mid D_n)$, can be shown below.

COROLLARY 3.1. *Under the assumptions in Lemma 3.1, if D_n is given, then*

$$E(N_{total,n} \mid D_n) \le \left(1+\frac{c_3}{c_1 C_{n+1} u_n}\right)^{\sigma}-1, \qquad a.s.$$

and

$$E(T_{total,n} \mid D_n) \le \frac{c_3/c_1}{\left(\dfrac{C_{n+1} u_n}{c_3/c_1 + C_{n+1} u_n}\right)^{\sigma}}, \qquad a.s.$$

Remark. Some conditions are set in Lemma 3.1 such that $(\frac{C_n u_{n-1}}{t+C_n u_{n-1}})^{\sigma}$ is a submartingale with respect to $\sigma(D_{n-1})$. This is a lower bound of the reliability function $R_n(t)$. Using (2.5), we know that the optimal release time in every stage is bounded. This implies

$$R_n(t_n^*) \ge \left(\frac{C_n u_{n-1}}{(c_3/c_1)+C_n u_{n-1}}\right)^{\sigma}, \qquad \text{a.s.} \tag{3.6}$$

Then Lemma 3.3 and Theorem 3.1 are proved. In general, other state space models could be considered in the software release problem. As in the above discussion the total testing time is proven finite with probability one when the following conditions are satisfied:

(i) There exists a sequence of positive variables, $\{X_1, X_2, \ldots, X_n, \ldots\}$, such that

$$R_n(t_n^*) \ge X_n, \qquad \text{a.s. for all} \quad n \ge 1.$$

(ii) X_n is a submartingale with respect to $\sigma(D_{n-1})$.

These conditions imply that there is growth in the lower bound for the reliability function.

4. A numerical example

The "system 40" data of Musa (1979) is used to illustrate the workings of our cost model. In our study cases below, all of the processes are stopped at the 53rd stage. As the discussion in Chen and Singpurwalla (1994), we set $\sigma_n = \omega_n = \upsilon_n = 2$ and $C_n = C = 0.425$. Thus the model satisfies the assumptions in Lemma 3.1. Using (2.1) and (2.3), we have the conditional density of T_n

$$f_n(t) = \frac{6t[Cu_{n-1}]^2}{[t + Cu_{n-1}]^4}. \tag{4.1}$$

The distribution function is

$$F_n(t) = 1 - \frac{(Cu_{n-1})^2(3t + Cu_{n-1})}{(t + Cu_{n-1})^3}. \tag{4.2}$$

Using (2.6), the optimal release time can be found in every stage by solving the following equation

$$\phi_n'(t) = c_1 + (c_2 - c_3) \cdot \frac{6t(Cu_{n-1})^2}{(t + Cu_{n-1})^4} + c_3 \frac{6(t + x)(Cu_{n-1})^2}{(t_n + x + Cu_{n-1})^4} = 0. \tag{4.3}$$

The impact of the cost coefficients and the mission time on the release time are studied next. Let us study the following cases:

Case 1: $c_1 = 0.1$, $c_2 = 10$, $c_3 = 100000$, and $x = 30000$.
Case 2: $c_1 = 0.1$, $c_2 = 10$, $c_3 = 100000$, and $x = 3000$.
Case 3: $c_1 = 0.1$, $c_2 = 10$, $c_3 = 10000$, and $x = 30000$.
Case 4: $c_1 = 0.1$, $c_2 = 1$, $c_3 = 100000$, and $x = 30000$.
Case 5: $c_1 = 0.01$, $c_2 = 10$, $c_3 = 100000$, and $x = 30000$.

In every stage, 10000 processes were generated to predict the future behavior of the model and estimate the expected total cost. Note that, using (2.2), the n-th inter-failure time can be generated using the following equation:

$$T_n = Cu_{n-1}\beta/(1 - \beta), \tag{4.4}$$

where β is generated from $Beta(2,2)$. We set $t_0 = 320$, for the value is chosen arbitrarily. The optimal release time is listed after the 1st stage. The release time in every stage is shown in Table 1. Note that, only Case 5 will release the software between the 52nd and 53rd failure. The other cases will release the software before the 7th failure. The data between the 9th and 50th stages are skipped. Using Table 1, increasing the value of c_3 or x will result in a longer testing time in every stage. Decreasing the value of c_1 or c_2 will result in a longer testing time in every stage.

The expected total cost in every stage is shown in Table 2. The expected total cost in stage n is defined as follows:

$$\begin{aligned} &c_1 E(T_1 + \cdots + T_n + \cdots + T_{n+K-1} + T^*_{n+K} \mid D_n) + c_2 E(n + K - 1 \mid D_n) \\ &\quad + c_3 P(T^*_{n+K} < T_{n+K} < T^*_{n+K} + x \mid D_n), \end{aligned} \tag{4.5}$$

where $K \geq 1$, a.s. That is, $n + K$ is the last testing stage. Using Table 2, increasing the value of c_3 or x will result in a larger testing cost. Because the c_1 and c_2 are values smaller than c_3, the impact of these values is not significant.

Table 1. The impact of the cost coefficients and the mission time on the release time.

n	Failure time t_n	Optimal release time t_n^*				
		Case 1	Case 2	Case 3	Case 4	Case 5
1	14390					
2	9000	46609.28	27825.87	18640.32	46611.29	97986.93
3	2880	47660.24	28265.38	18998.40	47662.23	100221.18
4	5700	36870.05	23449.16	39259.89	36871.49	77561.55
5	21800	37586.09	23788.96	15123.73	37587.57	79049.09
6	26800	61837.02	33457.90	—	61840.20	131196.64
7	113540	73079.30	36271.74		73083.66	157345.72
8	112137	—	—		—	266799.56
...	...					...
51	31365					102848.67
52	24313					158000.95
53	298890					162542.11

Table 2. The impact of the cost coefficients and the mission time on the expected total cost.

n	Expected total cost in every stage				
	Case 1	Case 2	Case 3	Case 4	Case 5
1					
2	71374.99	38923.22	8511.42	71253.16	72144.20
3	71800.46	39399.04	9476.45	71706.49	70955.31
4	78139.22	44768.68	9895.51	77137.01	78218.44
5	76937.30	44948.89	13070.71	77904.79	77808.99
6	68883.84	29228.36	—	68268.18	63327.52
7	57517.27	21044.21		57462.46	56959.00
8	—	—		—	40519.03
...					...
51					66897.98
52					65881.27
53					38453.57

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Appendix

PROOF OF (2.1). From the observation equation and the prior of θ_n for the results, we have
$g_n(t_n \mid \theta_n) = \frac{t_n^{\omega_n-1}\theta_n^{\omega_n}\exp(-\theta_n t_n)}{\Gamma(\omega_n)}$, and

$$\pi_n(\theta_n \mid D_{n-1}) = \frac{(C_n u_{n-1})^{\sigma_{n-1}}\theta_n^{\sigma_{n-1}-1}\exp(-C_n u_{n-1}\theta_n)}{\Gamma(\sigma_{n-1})},$$

where $g_n(t_n \mid \theta_n)$ is the density of $(T_n \mid \theta_n)$ and $\pi_n(\theta_n \mid D_{n-1})$ is the prior density of θ_n. Thus, we have

$$\begin{aligned} f_n(t_n) \equiv f(t_n \mid D_{n-1}) &= \int_0^\infty g_n(t_n \mid \theta_n)\cdot \pi_n(\theta_n \mid D_{n-1})d\theta_n \\ &= \frac{\Gamma(\omega_n+\sigma_{n-1})t_n^{\omega_n-1}[C_n u_{n-1}]^{\sigma_{n-1}}}{\Gamma(\omega_n)\Gamma(\sigma_{n-1})[t_n+C_n u_{n-1}]^{\omega_n+\sigma_{n-1}}}. \end{aligned}$$

Remark. The random variable X has a beta prime distribution, a standard form of Pearson type VI distribution, if its density is

$$h(x) = \frac{\Gamma(p+q)x^{p-1}}{\Gamma(p)\Gamma(q)[x+1]^{p+q}}.$$

Now let $Y_n = \frac{T_n}{C_n u_{n-1}}$, the density of $(Y_n \mid D_{n-1})$ is

$$h(y_n \mid D_{n-1}) = \frac{\Gamma(\omega_n+\sigma_{n-1})y_n^{\omega_n-1}}{\Gamma(\omega_n)\Gamma(\sigma_{n-1})[y_n+1]^{\omega_n+\sigma_{n-1}}}.$$

Thus, $(Y_n \mid D_{n-1}) = (T_n/C_n u_{n-1} \mid D_{n-1})$ has a beta prime distribution with $p=\omega_n$, $q=\sigma_{n-1}$.

PROOF OF (2.3). If $\omega_n = 1$, the result is trivial. Thus, we consider $\omega_n \geq 2$. By (2.1),

$$R_n(t_n) = \int_{t_n}^\infty f_n(t)dt = \frac{\Gamma(\omega_n+\sigma_{n-1})[C_n u_{n-1}]^{\sigma_{n-1}}}{\Gamma(\omega_n)\Gamma(\sigma_{n-1})}\int_{t_n}^\infty \frac{t^{\omega_n-1}}{[t+C_n u_{n-1}]^{\omega_n+\sigma_{n-1}}}dt.$$

We can use integration by parts to find $\int_{t_n}^\infty \frac{t^{\omega_n-1}}{(t+C_n u_{n-1})^{\omega_n+\sigma_{n-1}}}dt$.

Let $y(t) = t^{\omega_n-1}$ and $z'(t) = (t+C_n u_{n-1})^{-(\omega_n+\sigma_{n-1})}$, we have

$$\begin{aligned} &\int_{t_n}^\infty \frac{t^{\omega_n-1}}{(t+C_n u_{n-1})^{\omega_n+\sigma_{n-1}}}dt \\ &= \left[\frac{-t^{\omega_n-1}}{(\omega_n+\sigma_{n-1}-1)(t+C_n u_{n-1})^{\omega_n+\sigma_{n-1}-1}}\right]_{t_n}^\infty \\ &\quad + \int_{t_n}^\infty \frac{(\omega_n-1)t^{\omega_n-2}}{(\omega_n+\sigma_{n-1}-1)(t+C_n u_{n-1})^{\omega_n+\sigma_{n-1}-1}}dt \\ &= \frac{t_n^{\omega_n-1}}{(\omega_n+\sigma_{n-1}-1)(t_n+C_n u_{n-1})^{\omega_n+\sigma_{n-1}-1}} \\ &\quad + \frac{\omega_n-1}{\omega_n+\sigma_{n-1}-1}\int_{t_n}^\infty \frac{t^{\omega_n-2}}{(t+C_n u_{n-1})^{\omega_n+\sigma_{n-1}-1}}dt. \end{aligned}$$

Now, using integration by parts $\omega_n - 1$ times,

$$\int_{t_n}^\infty \frac{t^{\omega_n-1}}{(t+C_n u_{n-1})^{\omega_n+\sigma_{n-1}}}dt$$

$$
\begin{aligned}
&= \frac{t_n^{\omega_n - 1}}{(\omega_n + \sigma_{n-1} - 1)(t_n + C_n u_{n-1})^{\omega_n + \sigma_{n-1} - 1}} \\
&\quad + \sum_{i=2}^{\omega_n} \frac{t_n^{\omega_n - i} \prod_{j=2}^{i} (\omega_n - j + 1)}{(t_n + C_n u_{n-1})^{\omega_n + \sigma_{n-1} - i} \prod_{j=1}^{i} (\omega_n + \sigma_{n-1} - j)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
R_n(t_n) &= \frac{\Gamma(\omega_n + \sigma_{n-1})[C_n u_{n-1}]^{\sigma_{n-1}}}{\Gamma(\omega_n)\Gamma(\sigma_{n-1})} \int_{t_n}^{\infty} \frac{t^{\omega_n - 1}}{(t + C_n u_{n-1})^{\omega_n + \sigma_{n-1}}} dt \\
&= \frac{(C_n u_{n-1})^{\sigma_{n-1}}}{\Gamma(\sigma_{n-1})} \left[\sum_{i=1}^{\omega_n} \frac{\Gamma(\sigma_{n-1} + i - 1)}{\Gamma(i)} \frac{t_n^{i-1}}{(t_n + C_n u_{n-1})^{\sigma_{n-1} + i - 1}} \right].
\end{aligned}
$$

PROOF OF LEMMA 3.1. Now if $t > 0$ is fixed, for $n \geq 1$,

$$
\begin{aligned}
&E\left(\frac{C_{n+1} u_n}{t + C_{n+1} u_n} \mid D_{n-1} \right) - \frac{C_n u_{n-1}}{t + C_n u_{n-1}} \\
&\quad = E\left(\frac{C_{n+1} u_n}{t + C_{n+1} u_n} - \frac{C_n u_{n-1}}{t + C_n u_{n-1}} \mid D_{n-1} \right) \\
&\quad = E\left(\frac{t(C_{n+1} u_n - C_n u_{n-1})}{(t + C_{n+1} u_n)(t + C_n u_{n-1})} \mid D_{n-1} \right) \\
&\quad = E\left(\frac{t(C_{n+1} u_n - C_n u_{n-1})^+}{(t + C_{n+1} u_n)(t + C_n u_{n-1})} - \frac{t(C_{n+1} u_n - C_n u_{n-1})^-}{(t + C_{n+1} u_n)(t + C_n u_{n-1})} \mid D_{n-1} \right) \\
&\quad \geq E\left(\frac{t(C_{n+1} u_n - C_n u_{n-1})^+}{(t + C_{n+1} u_n)(t + C_n u_{n-1})} \mid D_{n-1} \right) \\
&\quad \quad - E\left(\frac{t(C_{n+1} u_n - C_n u_{n-1})^-}{t^2} \mid D_{n-1} \right) \\
&\quad = E\left(\frac{t(C_{n+1} u_n - C_n u_{n-1})^+}{(t + C_{n+1} u_n)(t + C_n u_{n-1})} \mid D_{n-1} \right) \\
&\quad \quad - \frac{1}{t} E((C_{n+1} u_n - C_n u_{n-1})^- \mid D_{n-1}),
\end{aligned}
$$

by (3.2), $E((C_{n+1}u_n - C_n u_{n-1})^- \mid D_{n-1}) = 0$, a.s. Thus, we have $\frac{C_{n+1}u_n}{t+C_{n+1}u_n}$ is also a submartingale with respect to $\sigma(D_n)$. Moreover, the function $g(z) = z^\sigma$ is an increasing convex function for $z > 0$ when $\sigma > 1$. Then using Jensen's inequality, $(\frac{C_{n+1}u_n}{t+C_{n+1}u_n})^\sigma$ is a submartingale with respect to $\sigma(D_n)$.

PROOF OF LEMMA 3.2. Note that if $n = 1$, it is trivially. Now for $n \geq 2$,

$$
\begin{aligned}
&E\left[\left(\left(1 - \left(\frac{C_n u_{n-1}}{t + C_n u_{n-1}} \right)^\sigma \right) - \left(1 - \left(\frac{C_{n-1} u_{n-2}}{t + C_{n-1} u_{n-2}} \right)^\sigma \right) \right)^+ I_A \mid D_{n-2} \right] \\
&\quad \leq E\left[\left(\left(1 - \left(\frac{C_n u_{n-1}}{t + C_n u_{n-1}} \right)^\sigma \right) - \left(1 - \left(\frac{C_{n-1} u_{n-2}}{t + C_{n-1} u_{n-2}} \right)^\sigma \right) \right)^+ \mid D_{n-2} \right] \\
&\quad = 0, \quad \text{a.s.}
\end{aligned}
$$

Then we have

$$E\left[\left(1-\left(\frac{C_n u_{n-1}}{t+C_n u_{n-1}}\right)^{\sigma}\right) I_A\right] \le E\left[\left(1-\left(\frac{C_{n-1} u_{n-2}}{t+C_{n-1} u_{n-2}}\right)^{\sigma}\right) I_A\right].$$

By induction, we have the result.

PROOF OF LEMMA 3.3. Note that, u_0 is given in our model. Using (3.4), $n = 1$ holds. Thus, we consider $n \ge 2$ below.

$$\begin{aligned}
P(T_1 < T_1^*, \ldots, T_n < T_n^*) &= E[I_{[T_1<T_1^*,\ldots,T_n<T_n^*]}] \\
&= E[E[I_{[T_1<T_1^*,\ldots,T_n<T_n^*]} \mid D_{n-1}]] \\
&= E[E[I_{[T_n<T_n^*]} \mid D_{n-1}] I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*]}] \\
&= E[F_n(T_n^*) I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*]}] \\
&\le E\left[F_n\left(\frac{c_3}{c_1}\right) I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*]}\right] \qquad \text{(by (2.5))} \\
&= E\left[\left(1-R_n\left(\frac{c_3}{c_1}\right)\right) I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*]}\right] \\
&\le E\left[\left(1-\left(\frac{C_n u_{n-1}}{(c_3/c_1)+C_n u_{n-1}}\right)^{\sigma}\right) I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*]}\right] \\
&\qquad\qquad \text{(by (3.4), set } t = c_3/c_1) \\
&\le \left(1-\left(\frac{C_1 u_0}{(c_3/c_1)+C_1 u_0}\right)^{\sigma}\right) E[I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*]}] \\
&\qquad\qquad \text{(using Lemma 3.2)} \\
&= \left(1-\left(\frac{C_1 u_0}{(c_3/c_1)+C_1 u_0}\right)^{\sigma}\right) F(T_1 < T_1^*, \ldots, T_{n-1} < T_{n-1}^*).
\end{aligned}$$

By induction, we have the result.

PROOF OF THEOREM 3.1. Let $A_1 = \{T_1 > T_1^*\}$, $A_n = \{T_1 < T_1^*, \ldots, T_{n-1} < T_{n-1}^*, T_n > T_n^*\}$, $n \ge 2$, note that the sets $\{A_n\}$ are disjoint. Thus, we have $\bigcup_{i=n}^{\infty} A_i = \{T_1 < T_1^*, \ldots, T_n < T_n^*\}$, for $n \ge 2$. Then

$$\begin{aligned}
N_{total} &= \sum_{n=2}^{\infty} [(n-1) I_{[T_1<T_1^*,\ldots,T_{n-1}<T_{n-1}^*,T_n>T_n^*]}] \\
&= \sum_{n=2}^{\infty} (n-1) I_{A_n} \\
&= I_{\bigcup_{i=2}^{\infty} A_i} + I_{\bigcup_{i=3}^{\infty} A_i} + \cdots \\
&= \sum_{n=2}^{\infty} I_{\bigcup_{i=n}^{\infty} A_i} \\
&= \sum_{n=1}^{\infty} I_{[T_1<T_1^*,\ldots,T_n<T_n^*]}.
\end{aligned}$$

$$E(N_{total}) = \sum_{n=1}^{\infty} nP(T_1 < T_1^*, \ldots, T_n < T_n^*, T_{n+1} > T_{n+1}^*)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} P(T_1 < T_1^*, \dots, T_n < T_n^*) \\
&\leq \sum_{n=1}^{\infty} \left(1 - \left(\frac{C_1 u_0}{(c_3/c_1) + C_1 u_0}\right)^{\sigma}\right)^{n} \qquad \text{(using Lemma 3.3)} \\
&= \frac{1 - \left(\frac{C_1 u_0}{(c_3/c_1) + C_1 u_0}\right)^{\sigma}}{\left(\frac{C_1 u_0}{(c_3/c_1) + C_1 u_0}\right)^{\sigma}} = \left(1 + \frac{c_3}{c_1 C_1 u_0}\right)^{\sigma} - 1.
\end{aligned}
$$

By (2.7), we have

$$
\begin{aligned}
E(T_{total}) &= E\left(T_1^* I_{[T_1 > T_1^*]} + \sum_{n=2}^{\infty}\left[\left(\sum_{i=1}^{n-1} T_i + T_n^*\right) I_{[T_1 < T_1^*, \dots, T_{n-1} < T_{n-1}^*, T_n > T_n^*]}\right]\right) \\
&\leq E\left(T_1^* I_{[T_1 > T_1^*]} + \sum_{n=2}^{\infty}\left[\left(\sum_{i=1}^{n} T_i^*\right) I_{[T_1 < T_1^*, \dots, T_{n-1} < T_{n-1}^*, T_n > T_n^*]}\right]\right) \\
&\leq E\left(\frac{c_3}{c_1} I_{[T_1 > T_1^*]} + \sum_{n=2}^{\infty}\left[\left(\sum_{i=1}^{n} \frac{c_3}{c_1}\right) I_{[T_1 < T_1^*, \dots, T_{n-1} < T_{n-1}^*, T_n > T_n^*]}\right]\right) \\
&= E\left(\frac{c_3}{c_1} I_{[T_1 > T_1^*]} + \sum_{n=2}^{\infty}\left[n \frac{c_3}{c_1} I_{[T_1 < T_1^*, \dots, T_{n-1} < T_{n-1}^*, T_n > T_n^*]}\right]\right) \\
&= \frac{c_3}{c_1} E\left(I_{[T_1 > T_1^*]} + \sum_{n=2}^{\infty}[n I_{[T_1 < T_1^*, \dots, T_{n-1} < T_{n-1}^*, T_n > T_n^*]}]\right) \\
&= \frac{c_3}{c_1}[1 + P(T_1 < T_1^*) + \cdots + P(T_1 < T_1^*, \dots, T_n < T_n^*) + \cdots] \\
&\leq \frac{c_3/c_1}{\left(\frac{C_1 u_0}{c_3/c_1 + C_1 u_0}\right)^{\sigma}} \qquad \text{(using Lemma 3.3).}
\end{aligned}
$$

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