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# UNIMODALITY OF UNIFORM GENERALIZED ORDER STATISTICS, WITH APPLICATIONS TO MEAN BOUNDS

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**Abstract.** We prove that uniform generalized order statistics are unimodal for an arbitrary choice of model parameters. The result is applied to establish optimal lower and upper bounds on the expectations of generalized order statistics based on nonnegative samples in the population mean unit of measurement. The bounds are attained by two-point distributions.

*Key words and phrases*: Generalized order statistics, uniform sample, nonnegative sample, unimodal distribution, expectation, optimal bound.

## 1. Introduction and notation

Let $U$ denote a standard uniform random variable, and let $X$ have a distribution function $G$. If $X$ is nonnegative, then its mean can be written as

$$\mu = \mathrm{E}X = \int_0^{G^{-1}(1)} [1 - G(x)]dx, \tag{1.1}$$

where

$$G^{-1}(x) = \inf\{y : G(y) \geq x\}, \qquad x \in (0, 1], \qquad G^{-1}(0) = G^{-1}(0+).$$

Note that both $\mu$ and $G^{-1}(1)$ may be infinite. Following Kamps (1995), for given $n \in \mathcal{N}$ and positive parameters $\gamma_1, \dots, \gamma_n$, we define uniform generalized order statistics $U_1, \dots, U_n$ which have the common density function

$$f^{U_1,\dots,U_n}(u_1, \dots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j\right) \left(\prod_{j=1}^{n-1} (1 - u_j)^{\gamma_j - \gamma_{j+1} - 1}\right) (1 - u_n)^{\gamma_n - 1} \tag{1.2}$$

supported on the cone $0 \leq u_1 \leq \cdots \leq u_n < 1$ of $\mathcal{R}^n$. It is shown in Cramer and Kamps (2003) that the distribution of $U_1, U_2, \dots, U_n$ is identical with that of $1 - B_{\gamma_1}, 1 - B_{\gamma_1} B_{\gamma_2}, \dots, 1 - \prod_{i=1}^n B_{\gamma_i}$, where $B_{\gamma_i}$, $i = 1, \dots, n$, are independent random variables with respective Beta distributions $B(\gamma_i, 1)$, $i = 1, \dots, n$, i.e., power distributions with exponents $\gamma_i$. This explains the fact that $0 \leq U_1 \leq \cdots \leq U_n \leq 1$ almost surely. Another consequence of the representation is that the marginal density function of the $r$-th uniform generalized order statistic can be written in terms of a particular Meijer's **G**-function, i.e.,

$$f_r(t) = f^{U_r}(t) = \left(\prod_{i=1}^r \gamma_i\right) \mathbf{G}^{r,0}_{r,r}\left[1 - t \,\middle|\, \begin{matrix} \gamma_1, \dots, \gamma_r \\ \gamma_1 - 1, \dots, \gamma_r - 1 \end{matrix}\right], \qquad t \in [0, 1) \tag{1.3}$$

(see also Mathai ((1993), pp. 83–84)). Obviously, $f_r(t)$ is positive and continuous for $0 < t < 1$. Subsequently, the marginal distribution function of the $r$-th uniform generalized order statistic is denoted by $F_r$.

Generalized order statistics based on some distribution function $G$ and parameters $\gamma_1, \dots, \gamma_n > 0$ were introduced by Kamps (1995) via the quantile transformation

$$X_r = G^{-1}(U_r), \qquad 1 \le r \le n.$$

Accordingly, the cumulative distribution function of $X_r$ is given by $G_r = F_r \circ G$. If $G(0-) = 0$, then

$$\mathrm{E}X_r = \int_0^{G^{-1}(1)} [1 - F_r(G(x))]dx = \int_0^1 G^{-1}(x) f_r(x)dx. \tag{1.4}$$

In fact, the latter representation holds true for arbitrary $G$ with a finite mean.

In this paper, we establish sharp lower and upper bounds for the expectation of generalized order statistics (1.4) based on an arbitrary distribution function $G$ with a nonnegative support and finite mean (1.1) in terms of the mean unit of measurement. This is an extension of results in Papadatos (1997), where ordinary order statistics of independent identically distributed samples were studied. Rychlik (1993) presented analogous bounds for arbitrary linear combinations of order statistics based on dependent identically distributed samples. Moriguti (1953), Nagaraja (1978), Raqab (1997), and Balakrishnan *et al.* (2001) derived sharp mean-variance bounds on the expectations of order statistics, record values, $k$-th record values, and progressively type II censored order statistics, respectively, under an additional assumption that $G$ has a finite second moment. A comprehensive study of mean-variance bounds for order and record statistics from general and restricted populations is presented in Rychlik (2001). The problems of moments existence for order statistics and records were examined by Sen (1959), Nagaraja (1978), and Lin (1987).

By (1.4), it is clear that the bounds on the expectations of generalized order statistics depend on properties of the density functions given in (1.3). The crucial ones are unimodality and behavior at the boundaries of the support interval $[0, 1]$. These are established in Section 2. In Section 3 we derive sharp mean bounds on the generalized order statistics based on nonnegative random variables. Applications of the results of Section 2 to mean-variance bounds on generalized order statistics are presented in Cramer *et al.* (2002).

## 2. Unimodality of uniform generalized order statistics

Since we consider the expectation of a single generalized order statistic which has a representation

$$X_r = G^{-1}\left(1 - \prod_{i=1}^{r} B_{\gamma_i}\right),$$

we can assume without loss of generality that $\gamma_1 \ge \cdots \ge \gamma_r > 0$. Suppose that the parameters have $1 \le \ell \le r$ distinct values $\delta_1 > \cdots > \delta_\ell$ with respective multiplicities $d_1, \dots, d_\ell$ ($\ell \in \{1, \dots, r\}$). Precisely, we arrange the parameters in descending order

$$\begin{aligned}\gamma_1 &= \cdots = \gamma_{d_1} = \delta_1 > \cdots \\ &> \gamma_{d_1+d_2+\cdots+d_{i-1}+1} = \cdots = \gamma_{d_1+d_2+\cdots+d_i} = \delta_i > \cdots \\ &> \gamma_{d_1+d_2+\cdots+d_{\ell-1}+1} = \cdots = \gamma_{d_1+d_2+\cdots+d_\ell} = \delta_\ell.\end{aligned}$$

In the sequel, we use some relations satisfied by Meijer's **G**-functions that can be found in Mathai (1993) (see also Cramer and Kamps (2003)).

LEMMA 2.1. *Let $r \geq 2$ and $z \in [0, 1)$.*

(i) $$\mathbf{G}_{1,1}^{1,0}\left[z\middle|\begin{matrix}\gamma_1\\ \gamma_1-1\end{matrix}\right] = z^{\gamma_1-1}$$

(ii) $$(\gamma_r-\gamma_1)\mathbf{G}_{r,r}^{r,0}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_r\\ \gamma_1-1,\dots,\gamma_r-1\end{matrix}\right]$$
$$= \mathbf{G}_{r-1,r-1}^{r-1,0}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_{r-1}\\ \gamma_1-1,\dots,\gamma_{r-1}-1\end{matrix}\right] - \mathbf{G}_{r-1,r-1}^{r-1,0}\left[z\middle|\begin{matrix}\gamma_2,\dots,\gamma_r\\ \gamma_2-1,\dots,\gamma_r-1\end{matrix}\right]$$

(iii) $$\frac{d}{dz}\mathbf{G}_{r,r}^{r,0}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_r\\ \gamma_1-1,\dots,\gamma_r-1\end{matrix}\right]$$
$$= \frac{1}{z}\left((\gamma_r-1)\mathbf{G}_{r,r}^{r,0}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_r\\ \gamma_1-1,\dots,\gamma_r-1\end{matrix}\right] - \mathbf{G}_{r-1,r-1}^{r-1,0}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_{r-1}\\ \gamma_1-1,\dots,\gamma_{r-1}-1\end{matrix}\right]\right)$$

(iv) $$z^a\mathbf{G}_{r,r}^{r,0}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_r\\ \gamma_1-1,\dots,\gamma_r-1\end{matrix}\right] = \mathbf{G}_{r,r}^{r,0}\left[z\middle|\begin{matrix}\gamma_1+a,\dots,\gamma_r+a\\ \gamma_1+a-1,\dots,\gamma_r+a-1\end{matrix}\right], \qquad a \in \mathbb{R}.$$

PROOF. (i) Mathai ((1993), p. 130); (ii) Cramer and Kamps (2003); (iii) Mathai ((1993), p. 94, Property 2.14) in connection with Mathai ((1993), p. 70, Property 2.2); (iv) Mathai ((1993), p. 69, Property 2.1).

In Lemma 2.2, we determine limits of the density function (1.3) of the $r$-th uniform generalized order statistic at the ends of the support interval $[0, 1)$. For $U_1 = 1 - B_{\gamma_1}$, we simply have

$$f_1(t) = \gamma_1(1-t)^{\gamma_1-1}, \qquad 0 < t < 1, \tag{2.1}$$

and the conclusions are trivial.

LEMMA 2.2. *Let $r \geq 2$.*
(i) $f_r(0) = 0$.
(ii) *If* $\gamma_r < 1$ *then* $\lim_{t\to 1-} f_r(t) = \infty$.
(iii) *If* $\gamma_r = 1$ *and* $\gamma_{r-1} = 1$ *then* $\lim_{t\to 1-} f_r(t) = \infty$.
(iv) *If* $\gamma_r = 1$ *and* $\gamma_{r-1} > 1$ *then* $\lim_{t\to 1-} f_r(t) = \prod_{i=1}^{r-1} \frac{\gamma_i}{\gamma_i - 1} \in (1, \infty)$.
(v) *If* $\gamma_r > 1$ *then* $\lim_{t\to 1-} f_r(t) = f_r(0) = 0$.

PROOF. (i) If $\gamma_1 = \dots = \gamma_r$ (e.g., in the case of $k$-th record values), we have

$$f_r(t) = \frac{\gamma_1^r}{(r-1)!}(1-t)^{\gamma_1-1}[-\ln(1-t)]^{r-1}, \tag{2.2}$$

which vanishes at 0. If $\gamma_1 > \gamma_2$, using Lemma 2.1(i) and (ii), we find

$$f_2(t) = \frac{\gamma_1\gamma_2}{\gamma_1 - \gamma_2}[(1-t)^{\gamma_2-1} - (1-t)^{\gamma_1-1}], \tag{2.3}$$

which is 0 at 0. For the case $\gamma_1 > \gamma_r$ with $r \geq 3$, the result follows by induction with respect to $r$. It suffices to combine (1.3) with the recurrence relation of Lemma 2.1(ii).

(iii) If $\gamma_1 = \cdots = \gamma_r = 1$, then the statement immediately follows from (2.2). Suppose that $\gamma_r = \gamma_{r-1} = 1$ and $\gamma_1 > 1$ such that $\ell \geq 2$ and $d_\ell \geq 2$. Due to Cramer and Kamps (2003), the density function (1.3) can be written as

$$f_r(t) = \sum_{i=1}^{\ell}\sum_{j=1}^{d_i} c_{ij}(1-t)^{\delta_i-1}(-\ln(1-t))^{d_i-j}. \tag{2.4}$$

If $1 \leq i < \ell$, then $\delta_i > 1$, and

$$\lim_{t\to 1-}(1-t)^{\delta_i-1}[-\ln(1-t)]^{d_i-j} = 0.$$

Hence, we only need to consider the summands of (2.4) with $i = \ell$. Using $(-\ln(1-t))^{1-j} \to 0$, $t \to 1-$, for $j \geq 2$, we find from $c_{\ell 1} > 0$ (cf. Cramer and Kamps (2003), Theorem 3.4)

$$\lim_{t\to 1-} f_r(t) = \lim_{t\to 1-}[-\ln(1-t)]^{d_\ell-1}\left[c_{\ell 1} + \underbrace{\sum_{j=2}^{d_\ell} c_{\ell j}(-\ln(1-t))^{1-j}}_{\to 0}\right] = +\infty. \tag{2.5}$$

(iv) We have $\ell \geq 2$ and $d_\ell = 1$. Proceeding as the the proof of (iii), we combine (2.4) with (2.5), and obtain

$$\lim_{t\to 1-} f_r(t) = c_{\ell 1}.$$

The final representation of the right-hand side follows from a result of Cramer and Kamps ((2003), Theorem 3.4).

(ii) By (1.3) and Lemma 2.1(iv), for $a = 1 - \gamma_r > 0$, we obtain

$$\begin{aligned} f_r(t) &= \left(\prod_{i=1}^{r}\gamma_i\right)\frac{1}{(1-t)^a}\mathbf{G}^{r,0}_{r,r}\left[1-t\,\middle|\,\begin{matrix}\gamma_1+a,\dots,\gamma_r+a\\ \gamma_1+a-1,\dots,\gamma_r+a-1\end{matrix}\right] \\ &= \left(\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+a}\right)\frac{1}{(1-t)^a}\left(\prod_{i=1}^{r}(\gamma_i+a)\right) \\ &\quad\times \mathbf{G}^{r,0}_{r,r}\left[1-t\,\middle|\,\begin{matrix}\gamma_1+a,\dots,\gamma_r+a\\ \gamma_1+a-1,\dots,\gamma_r+a-1\end{matrix}\right]. \end{aligned} \tag{2.6}$$

The latter product is the density function of a uniform generalized order statistic with parameters $\gamma_1 + a \geq \cdots \geq \gamma_r + a = 1$, which for $t \to 1-$ tends to a positive, possibly infinite value, as we have proven in (iii) and (iv). Since $\lim_{t\to 1-}(1-t)^{-a} = +\infty$, the same holds for (2.6).

(v) If $\gamma_1 = \cdots = \gamma_r > 1$, we immediately conclude the claim from (2.2). For $\gamma_1 > \gamma_r > 1$, the proof is carried out by induction. If $r = 2$, the result easily follows from (2.3).

Suppose now that this holds for some $r \geq 2$ and arbitrary parameters $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r > 1$. Consider a uniform generalized order statistic with parameters $\gamma_1 \geq \cdots \geq \gamma_{r+1} > 1$ such that $\gamma_1 > \gamma_{r+1}$. In particular, the assertion is true for the density functions $f_r$ and $\tilde{f}_r$ of the uniform generalized order statistics with parameters $\gamma_1 \geq \cdots \geq \gamma_r > 1$ and $\gamma_2 \geq \cdots \geq \gamma_{r+1} > 1$, respectively. Since $\gamma_1 > \gamma_{r+1}$ we obtain from the recurrence relation of Lemma 2.1(ii)

$$f_{r+1}(t) = \frac{1}{\gamma_{r+1} - \gamma_1}[\gamma_{r+1} f_r(t) - \gamma_1 \tilde{f}_r(t)],$$

which implies the final conclusion.

THEOREM 2.1. *The density function of each uniform generalized order statistic is unimodal.*

*For $r = 1$, the density function is strictly increasing, constant and strictly decreasing for $\gamma_1 < 1$, $= 1$ and $> 1$, respectively.*

*For $r \geq 2$, we have the following. If $\gamma_r \leq 1$, then the density function is strictly increasing. Otherwise it is strictly unimodal with a mode in $(0,1)$.*

PROOF. For $r = 1$, the assertions are easily concluded from formula (2.1). If $r \geq 2$ and $\gamma_r \leq 1$, then, by Lemma 2.1(iii), we find

$$\frac{d}{dt} f_r(t) = f_r'(t) = \frac{1}{1-t}[(1-\gamma_r) f_r(t) + \gamma_r f_{r-1}(t)] > 0, \qquad t \in (0,1),$$

and so $f_r(t)$ is increasing, because both density functions $f_r$ and $f_{r-1}$ are positive in $(0,1)$.

If $\gamma_1 = \gamma_2 > 1$, we differentiate (2.2) with $r = 2$, and conclude that

$$f_2'(t) = \gamma_1^2 (1-t)^{\gamma_1 - 2}[1 + (\gamma_1 - 1)\ln(1-t)]$$

has the unique zero

$$t = 1 - \exp\left(-\frac{1}{\gamma_1 - 1}\right) \in (0,1).$$

For $\gamma_1 > \gamma_2 > 1$, differentiating (2.3) yields

$$f_2'(t) = -\frac{\gamma_1 \gamma_2}{\gamma_1 - \gamma_2}[(\gamma_2 - 1)(1-t)^{\gamma_2 - 2} - (\gamma_1 - 1)(1-t)^{\gamma_1 - 2}]$$

with the unique solution

$$t = 1 - \left(\frac{\gamma_2 - 1}{\gamma_1 - 1}\right)^{1/(\gamma_1 - \gamma_2)} \in (0,1).$$

Since $f_2(0) = f_2(1) = 0$ in both cases, the density function $f_2$ is unimodal.

Suppose now that $f_r$ is strictly unimodal with a mode $z_r \in (0,1)$ and $f_r'(t) \neq 0$, $t \neq z_r$, for some $r \geq 2$ and arbitrary parameters $\gamma_1 \geq \cdots \geq \gamma_r > 1$. We show that the same holds for $f_{r+1}$ with $\gamma_1 \geq \cdots \geq \gamma_r \geq \gamma_{r+1} > 1$. We use some arguments from the proof of Lemma 2.7 in Balakrishnan *et al.* (2001). First we define the function

$$g_r(t) = (1-t)^{2-\gamma_r} f_r'(t), \qquad t \in (0,1). \tag{2.7}$$

By Lemma 2.1(iii), we obtain

$$g_r(t) = -\left(\prod_{i=1}^{r}\gamma_i\right)(1-t)^{1-\gamma_r}$$
$$\times\left\{(\gamma_r-1)\mathbf{G}^{r,0}_{r,r}\left[1-t\,\middle|\,\begin{matrix}\gamma_1,\dots,\gamma_r\\ \gamma_1-1,\dots,\gamma_r-1\end{matrix}\right]\right.$$
$$\left.-\mathbf{G}^{r-1,0}_{r-1,r-1}\left[1-t\,\middle|\,\begin{matrix}\gamma_1,\dots,\gamma_{r-1}\\ \gamma_1-1,\dots,\gamma_{r-1}-1\end{matrix}\right]\right\}.$$

Differentiating the function, applying Lemma 2.1(iii) again and some calculations, yield

$$g_r'(t) = \gamma_r(1-t)^{m_{r-1}}g_{r-1}(t), \qquad t\in(0,1), \tag{2.8}$$

where $m_{r-1} = \gamma_{r-1}-\gamma_r-1$.

Note that by assumption $f_r'(t)$ is positive for $t<z_r$ and negative for $t>z_r$. By (2.7) and (2.8), the same holds for $g_r(t)$ and $g_{r+1}'(t)$, respectively. Therefore, $g_{r+1}$ is strictly increasing in $(0,z_r)$ and strictly decreasing in $(z_r,1)$.

The function $f_{r+1}(t)$ is positive for $t\in(0,1)$, and tends to 0 at the ends of the interval (cf. Lemma 2.2(v)). This implies

$$\lim_{t\to 1-} f_{r+1}'(t) \le 0 \le \lim_{t\to 0+} f_{r+1}'(t).$$

By the definition of $g_{r+1}$ this means that $\varepsilon_0$ and $\varepsilon_1$ exist such that $g_{r+1}(t)>0$, $t\in(0,\varepsilon_0)$, and $g_{r+1}(t)<0$, $t\in(\varepsilon_1,1)$. Since $g_{r+1}$ is strictly increasing up to $z_r$ and then strictly decreasing, it has exactly one zero $z_{r+1}$ in the interval $(0,1)$. Moreover, $z_r<z_{r+1}$. Since the zeros in $(0,1)$ of $g_{r+1}$ and $f_{r+1}'$ coincide we find that $f_{r+1}'$ has exactly one zero in $(0,1)$, i.e., $z_{r+1}$. Hence, $f_{r+1}'(t)$ is positive for $0<t<z_{r+1}$ and negative for $z_{r+1}<t<1$. Consequently, $f_{r+1}$ has at most one local extremum. From Lemma 2.2(v) we know that $f_{r+1}(0)=f_{r+1}(1)=0$ such that $f_{r+1}$ has at least one local maximum in $(0,1)$. This proves the assertion.

As a by-product of the above proof, we deduce that the sequence of modes $z_r$ of density functions $f_r$ is increasing if $\gamma_r>1$.

## 3. Mean bounds

Suppose that $X$ has a distribution function $G$ such that $G(0-)=0$ and a finite mean (1.1). Then, by (1.4), we have

$$EX_r = \int_0^{G^{-1}(1)} H_r(G(x))[1-G(x)]dx, \tag{3.1}$$

where

$$H_r(u) = \frac{1-F_r(u)}{1-u}, \qquad u\in[0,1),$$

is a positive, continuous function, bounded except for in a neighborhood of 1, and

$$0 \le \lim_{u\to 1-} H_r(u) = f_r(1) \le +\infty.$$

By (3.1) and (1.1), we have

$$A_r\mu \le EX_r \le B_r\mu, \tag{3.2}$$

where

$$A_r = \inf\{H_r(u) : u \in [0,1)\}, \qquad B_r = \sup\{H_r(u) : u \in [0,1)\}.$$

This yields the relation

$$0 \le A_r \le H_r(0) = 1 \le B_r \le +\infty.$$

We now prove that the bounds (3.2) are sharp.

Let $X$ be a two-point random variable such that

$$P(X=0) = a, \qquad P\left(X = \frac{\mu}{1-a}\right) = 1-a, \qquad a \in [0,1),$$

with the distribution function

$$G(x) = \begin{cases} 0, & x < 0, \\ a, & 0 \le x < \dfrac{\mu}{1-a}, \\ 1, & \dfrac{\mu}{1-a} \le x. \end{cases} \tag{3.3}$$

Then $X$ is nonnegative almost surely, $EX = \mu$, and

$$EX_r = \int_0^{G^{-1}(1)} H_r(a)[1-G(x)]dx = H_r(a)\mu,$$

which leads to $A_r \le H_r(a) \le B_r$ for all $a \in [0,1)$.

Note that $a = 0$ implies that $X$ is concentrated at $\mu$. If $A_r = H_r(a)$ for some $a \in [0,1)$, then the lower bound in (3.2) is attained by the two-point distribution function (3.3). The analogous statement holds if $B_r = H_r(a)$ for some $a \in [0,1)$.

If $A_r < H_r(a)$ for all $a \in [0,1)$, the lower bound is given by $A_r = f_r(1)$. This yields

$$\forall \varepsilon > 0 \quad \exists \alpha_\varepsilon \in [0,1) \quad \forall u \in [\alpha_\varepsilon, 1) \quad H_r(u) \le f_r(1) + \varepsilon. \tag{3.4}$$

Let $X_\varepsilon$ be a random variable with distribution function

$$G_\varepsilon(x) = \begin{cases} 0, & x < 0, \\ \alpha_\varepsilon, & 0 \le x < \dfrac{\mu}{1-\alpha_\varepsilon}, \\ 1, & x \ge \dfrac{\mu}{1-\alpha_\varepsilon}. \end{cases} \tag{3.5}$$

Then $X_\varepsilon \ge 0$, $EX_\varepsilon = \mu$ and

$$E(X_\varepsilon)_r = \int_0^{G_\varepsilon^{-1}(1)} H_r(\alpha_\varepsilon)[1-G_\varepsilon(x)]dx \le [f_r(1)+\varepsilon]\mu. \tag{3.6}$$

This means that the bound $A_r = f_r(1)$ is attained in limit by sequences of two-point distribution functions (3.5) with $\varepsilon = \varepsilon_n \searrow 0$. If $+\infty > B_r = f_r(1) > H_r(a)$ for all

$a \in [0,1)$, we get the same conditions for attainability of the upper bound, as we repeat the above arguments, replacing $+\varepsilon$ by $-\varepsilon$, and reversing the inequality signs in (3.4) and (3.6).

If $B_r = f_r(1) = +\infty$, then for every arbitrarily large $M > 0$ there is $\alpha_M < 1$ such that for all $\alpha_M \le u < 1$, we have $H_r(u) \ge M$. Taking a two-point random variable $X_M$ with distribution function

$$G_M(x) = \begin{cases} 0, & x < 0, \\ \alpha_M, & 0 \le x < \dfrac{\mu}{1-\alpha_M}, \\ 1, & x \ge \dfrac{\mu}{1-\alpha_M}, \end{cases} \tag{3.7}$$

we have $EX_M = \mu$, and

$$E(X_M)_r = \int_0^{G_M^{-1}(1)} H_r(\alpha_M)[1 - G_M(x)]dx \ge M \cdot \mu.$$

Summing up, we have proved that if the global extremum (either lower or upper) of $H_r$ is attained at some point $a \in [0,1)$, then the respective bound is attained by (3.3). Otherwise it is attained in limit by sequences of distributions (3.5) and (3.7) with $\varepsilon = \varepsilon_n \searrow 0$ and $M = M_n \nearrow +\infty$, respectively. In any case, it suffices to take into account two-point distributions only.

Note that it is possible to take sequences of distribution functions different from (3.5) and (3.7). It is sufficient to consider any nonnegative random variables $X_n$ with mean $\mu$ and such that $P(X_n = 0) \ge \alpha_{\varepsilon_n}$ ($\alpha_{M_n}$, respectively). If $H_r(a)$ amounts to either $A_r$ or $B_r$ for all $a \in A \subset [0,1)$, and $A$ contains more than a single point, then some distributions different from (3.3) attain the respective bound in (3.2). The necessary and sufficient condition is

$$G(x) \in H_r^{-1}(A_r) \qquad \forall x \in [0, G^{-1}(1)).$$

Now we specify the bounds (3.2) for the generalized order statistic $X_r$ with arbitrarily chosen parameters $r \ge 1$ and $\gamma_1 \ge \cdots \ge \gamma_r > 0$.

THEOREM 3.1. *Let $X \ge 0$ have a finite mean $\mu$.*

(i) *If $r = 1$ and $\gamma_1 = 1$, then*

$$EX_r = \mu.$$

(ii) *If $r = 1$ and $\gamma_1 > 1$ then*

$$0 \le EX_r \le \mu.$$

*The lower bound is attained in limit by the sequences of the form (3.5) with $\varepsilon \searrow 0$. The upper bound is attained by the degenerate distribution concentrated at $\mu$.*

(iii) *If $r \ge 2$ and $\gamma_r = 1 < \gamma_{r-1}$, then*

$$\mu \le EX_r \le \left( \prod_{i=1}^{r-1} \frac{\gamma_i}{\gamma_i - 1} \right) \mu.$$

*The former inequality becomes equality for $X$ concentrated at $\mu$. The latter one holds in limit for* (3.5).

(iv) *If either $\gamma_r < 1$ with $r \geq 1$ or $\gamma_r = \gamma_{r-1} = 1$ with $r \geq 2$, then*

$$\mu \leq EX_r \leq +\infty.$$

*The conditions of attainability of the bounds coincide with those of the previous case, with* (3.5) *replaced by* (3.7).

(v) *If $\gamma_r > 1$ with $r \geq 2$, then*

$$0 \leq EX_r \leq f_r(a)\mu$$

*for a unique $a \in (0, 1)$ satisfying*

$$1 - F_r(a) = (1 - a) f_r(a). \tag{3.8}$$

*The lower bound is attained by sequences of* (3.5) *with $\varepsilon \searrow 0$. The upper bound is attained by* (3.3) *with $a$ defined in* (3.8).

PROOF. It suffices to determine the bounds only. The conditions of attainability can be deduced from the arguments preceding the theorem. The first statement is trivial, because for $\gamma_1 = 1$ the distributions of $X$ and $X_r$ are identical.

(ii) If $\gamma_1 > 1$, then, $H_1(u) = (1 - u)^{\gamma_1 - 1}$ is strictly decreasing with $B_1 = H_1(0) = 1$ and $A_1 = f_1(1) = 0$.

If either (iii) or (iv) hold, then $f_r$ is strictly decreasing by Theorem 2.1 with $f_r(0) < 1 < f_r(1) \leq +\infty$, and so $F_r$ is strictly convex. Therefore, $H_r$ is strictly increasing, and $A_r = H_r(0) = 1 < B_r = f_r(1)$. Under conditions (iii), we use Lemma 2.2(iv) for evaluating the upper bound. Otherwise it is infinite.

(v) By Theorem 2.1, the distribution function $F_r$ is first strictly convex and then strictly concave. Relation $f_r(0) = 0$ implies that $F_r(x) < x$ in a neighborhood of 0, and furthermore $B_r > H_r(0) = 1$. Moreover, there is a unique point $a$ less than the mode $z_r$ of $f_r$ such that the line with slope $H_r(a)$ is tangent to the graph of $F_r$. This leads to the maximal slope $H_r(a)$. We have $B_r = H_r(a)$, and the condition is

$$H_r(a) = \frac{1 - F_r(a)}{1 - a} = F_r'(a) = f_r(a)$$

so that $B_r = f_r(a)$. The lower bound is here $A_r = f_r(1) = 0$. This ends the proof.

The bounds of Papadatos (1997) for ordinary order statistics can be deduced from Theorem 3.1(ii), (iii), and (v) in the cases of the sample minima, maxima and nonextreme order statistics, respectively. Similar conclusions hold for progressive type II censored order statistics. Suppose that $r \geq 2$. Then we have $\gamma_1 > \cdots > \gamma_{r-1} > \gamma_r \geq 1$ where $\gamma_r = 1$ holds if $r = n$ and no items are withdrawn from the experiment after the last failure. In that case we obtain the bound

$$\mu \leq EX_n \leq \left( \prod_{i=1}^{n-1} \frac{\gamma_i}{\gamma_i - 1} \right).$$

Otherwise, case (v) holds and equation (3.8) is a polynomial equation in $a$ which corresponds to that one in Balakrishnan *et al.* ((2001), eq. (12)).

For record values, we apply evaluations of Theorem 3.1(iv). The infinite upper bound is not a surprise here, because Naragaja (1978) constructed distributions with finite means such that the expectations of respective records are infinite. For $k$-th records, we refer to Theorem 3.1(v).

It is evident that the bounds of this section can be extended to arbitrary linear combinations of generalized order statistics. Then we have

$$E\sum_{r=1}^{n} c_r X_r = \int_0^{G^{-1}(1)} \sum_{r=1}^{n} c_r H_r(G(x))[1-G(x)]dx,$$

and look for the extrema of the function

$$\sum_{r=1}^{n} c_r H_r(u) = \sum_{r=1}^{n} c_r \frac{1-F_r(u)}{1-u}, \qquad u \in [0,1).$$

The solutions strongly depend on the choice of the coefficients $c_r$, $1 \le r \le n$. In fact, we can also evaluate combinations of generalized order statistics coming from different models, e.g., ordinary order statistics, progressive type II censored order statistics, and record values.

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