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WAITING TIME PROBLEMS FOR A TWO-DIMENSIONAL PATTERN*

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Abstract. We consider waiting time problems for a two-dimensional pattern in a sequence of i.i.d. random vectors each of whose entries is 0 or 1. We deal with a two-dimensional pattern with a general shape in the two-dimensional lattice which is generated by the above sequence of random vectors. A general method for obtaining the exact distribution of the waiting time for the first occurrence of the pattern in the sequence is presented. The method is an extension of the method of conditional probability generating functions and it is very suitable for computations with computer algebra systems as well as usual numerical computations. Computational results applied to computation of exact system reliability are also given.

Key words and phrases: Waiting time problem, two-dimensional pattern, probability generating function, discrete distribution, conditional distribution, reliability, consecutive system.

1. Introduction

This paper is concerned with the following waiting time problem for a two-dimensional pattern. Let $\boldsymbol{X}_i$, $i = 1, 2, \dots$ be a sequence of m-dimensional i.i.d. random column vectors whose entries are $\{0, 1\}$-valued i.i.d. random variables. Here, we assume that

$$P(X_{i,j} = 1) = p(= 1 - q),$$

where $X_{i,j}$ means the j-th element of $\boldsymbol{X}_i$. Suppose we are given a two-dimensional pattern D of 1's of a finite width whose height is at most m.

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Example 1. We let $m = 5$ and let D be the following pattern of 1's with a diamond shape,

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & \\
 & & 1 & &
\end{array}. \tag{1.1}
$$

We are interested in the first occurrence of D in the sequence of $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$. The following array is a realization of occurrence of D,

101101011110101110100111111010010**1**01
0111101001101100111110100110101**111**1
10101000110110111000010110110**11111**
110101111010010110011001011010**111**0
1011100101011110111101111010110**1**10,

which shows that in this realization the pattern D has occurred for the first time in the 34th trial.

In this paper we derive the exact distribution of the waiting time for the first occurrence of a two-dimensional pattern by using the method of conditional probability generating functions (p.g.f.'s).

Our motivation for investigating such a complicated problem is provided by two-dimensional engineering consecutive systems. Salvia and Lasher (1990) introduced two-dimensional consecutive systems with an example of a group of connector pins for an electronic device which includes some redundancy in its design, such that the connection is good unless a square of size 2 (4 pins) is defective. Though they restricted the discussion to square patterns in the paper, they noted that it is desired to determine the probability of detecting patterns of an arbitrary shape.

Our result can be applied to determine the exact reliability of the above two-dimensional consecutive system which operates unless the components corresponding to a given pattern of an arbitrary shape are defective, when the column size of the pattern is the same as that of the system.

When the column size of the pattern is less than the size of the system, we have to consider more general problem such as the sooner waiting time or the first waiting time problems, which will be treated in Section 3. In order to treat precisely a pattern D whose column size is less than m, we must determine its scanning rectangular window R which covers D. Generally, the width of R is the same as that of D, but the height of R can be greater than or equal to that of D. The scanning rectangular window R scans in the sequence of m-dimensional random column vectors and checks within the window whether the pattern occurs or not. If the height of R is less than m, R can move not only horizontally freely but also vertically by $m-$(height of R) steps. For example, when

$m = 5$ and $D = \begin{matrix} 1 \\ 111 \\ 1 \end{matrix}$, we can consider the following six scanning rectangular windows:

$$R_1 = \begin{bmatrix} * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix}, \quad R_2 = \begin{bmatrix} * & * & * \\ * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \\ * & * & * \end{bmatrix}, \quad R_3 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \end{bmatrix},$$

$$R_4 = \begin{bmatrix} * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \\ * & * & * \end{bmatrix}, \quad R_5 = \begin{bmatrix} * & * & * \\ * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \end{bmatrix}, \quad R_6 = \begin{bmatrix} * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \end{bmatrix},$$

where $*$ means "1" or "0" whichever occurs there. Of course, the meaning of occurrence of the pattern is determined by the selection of the scanning rectangular window. For example, when we select R_1 for the scanning window, we think the pattern D occurs if and only if D occurs in the highest position (between the first and the third rows). However, when we select R_6, we think the pattern D occurs if D occurs in any position, i.e., between the first and the third rows, between the second and the fourth rows, or between the third and the fifth rows. The follwing array is a realization of the first occurrence of the pattern for the scanning rectangular windows R_2, R_4, R_5 and R_6,

1011011001001000101001001100100100110
010010100111001000011000101010111101**1**
100000011001110111100000010100100**111**
111000001010000100000111000101010**1**0
1000110011010001001010111100110101000.

In this case the pattern occurs between the second row and the fourth row.

Over the last few decades, waiting time problems for occurrences of a pattern in a univariate random sequence have been investigated by many authors (e.g., Blom and Thorburn (1982), Fu (1996), Koutras (1997), and Uchida (1998)). In particular, the finite Markov chain imbedding technique studied by Fu and Koutras (1994) gave us possibilities of deriving some exact waiting time distributions even in some dependent random sequences (see also Balakrishnan and Koutras (2002)). However, the problem in the present paper is much more complicated than ever. Hence, since it is difficult to give an intuitive conditioning, we use the method of conditional p.g.f.'s with an idea of systematic conditioning, which leads an algorithm to generate a linear system of equations of conditional p.g.f.'s. By using computer algeba systems, we can derive the exact distributions by solving the system of linear equations of the conditional p.g.f.'s.

2. Waiting time in the case that the height of the window is m

When the height of the scanning rectangular window is equal to the column size of the random vectors, the window can not move vertically, and the waiting time problem becomes relatively simple. First, here we explain in this case our basic idea to derive the exact distribution of the waiting time for the first occurrence of a two-dimensional pattern D.

Suppose we are given a two-dimensional pattern D and the scanning rectangular window R which covers D. Then, $R = [\boldsymbol{r}_1\ \boldsymbol{r}_2\ \cdots\ \boldsymbol{r}_\ell]$ is an $m \times \ell$ matrix of "1" or "$*$" elements. For example, we may suppose that D is given by (1.1). Then, the scanning rectangular window R becomes uniquely

$$R = \begin{bmatrix} * & * & 1 & * & * \\ * & 1 & 1 & 1 & * \\ 1 & 1 & 1 & 1 & 1 \\ * & 1 & 1 & 1 & * \\ * & * & 1 & * & * \end{bmatrix} = [\boldsymbol{r}_1\ \boldsymbol{r}_2\ \cdots\ \boldsymbol{r}_5].$$

For each $i = 1, 2, \dots, \ell$, let $R(i)$ be the subrectangular from the left

$$R(i) = [\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_i].$$

When we treat the waiting time problem for the first occurrence of the two-dimensional pattern D, the state space for the finite Markov chain imbedding technique or the totality of conditions for the method of conditional p.g.f.'s can be constructed by considering whether each subrectangular $R(i)$ currently holds or not just before the next trial. In other words, suppose that we have just observed the j-th trial. Then, in order to know the next "state" after obsering the next trial $\boldsymbol{X}_{j+1}$, we have to remember whether $R(i)$ occurs in $[\boldsymbol{X}_{j-i+1}, \dots, \boldsymbol{X}_j]$ or not for all $i = 1, \dots, \ell$. When we dealt with the corresponding problem for a one-dimensional pattern we considered only what is the longest subpattern occurred currently. This is because we can see whether each shorter subpattern currently occurs if a subpattern occurs currently. However, for two-dimensional patterns, we can not expect such a simple situation. Therefore, we denote by a $\{0,1\}$-vector $\boldsymbol{a} = (a_1, a_2, \dots, a_\ell)$ of length ℓ the current state, where

$$a_i = \begin{cases} 0 & \text{if } R(i) \text{ does not occur currently} \\ 1 & \text{if } R(i) \text{ occurs currently.} \end{cases}$$

Of course, $\boldsymbol{a} \in \{0,1\}^\ell$, however, all the element of $\{0,1\}^\ell$ are not necessarily possible as current states. Hence, we define by $S(D, R)$ the set of possible $\boldsymbol{a}$'s as a current state for the waiting time problem. $S(D, R)$ depends on D and R. For example, if D is the pattern given by (1.1) and $m = 5$, then R is the unique window given above and

$$S(D,R) = \left\{ \begin{array}{l} (0,0,0,0,0), (1,0,0,0,0), (1,1,0,0,0), (1,1,1,0,0), \\ (1,1,0,1,0), (1,1,1,1,0), (1,0,0,0,1), (1,1,0,0,1), \\ (1,1,0,1,1), (1,1,1,0,1), (1,1,1,1,1) \end{array} \right\}.$$

Let $S_1(D, R) = \{\boldsymbol{a} \in S(D, R) \mid a_\ell = 1\}$ and $S_0(D, R) = S(D, R) \setminus S_1(D, R)$.

Before presenting a general result, we explain intuitively how to derive the exact distribution of the waiting time for the first occurrence of a pattern with examples.

Let D be the pattern given in Example 1 and $m = 5$. For any $\boldsymbol{a} \in S_0(D, R)$, suppose that we observe currently $\boldsymbol{a}$. Then we denote by $\phi(\boldsymbol{a}; t)$ the conditional p.g.f.

of the waiting time for the first occurrence of D from this time. Mathematically, we can define $\phi(\boldsymbol{a};t)$ as follows: Let W be the waiting time and $\phi(\boldsymbol{a};t) = E[t^{W-j} \mid A(j,\boldsymbol{a})]$, where the event $A(j,\boldsymbol{a})$ is that $W > j$ and at the j-th trial the current state is $\boldsymbol{a}$. In the example, since the height of the window is $m(=5)$, the occurrence of the event $A(j,\boldsymbol{a})$ means that the indicator of occurrence of $R(i)$ in $[\boldsymbol{X}_{j-i+1},\dots,\boldsymbol{X}_j]$ is a_i for $i=1,\dots,\ell(=5)$. Note that the above conditional expectation does not depend on j by virtue of the stationarity of the sequence $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n,\dots$. Therefore, the notation $\phi(\boldsymbol{a};t)$ does not include j. Suppose that we observe currently $\boldsymbol{a} = (0,0,0,0,0)$. Then, if we observe the next trial, the state may change from $(0,0,0,0,0)$ to $(1,0,0,0,0)$. Otherwise, it may remain as it is and there are no possibilities to change to other states at the next trial. The probability that the state changes from $(0,0,0,0,0)$ to $(1,0,0,0,0)$ is p, that is, the probability that we observe $(*,*,1,*,*)'$. And, with probability $q(=1-p)$, the state remains unchanged. Hence, we have the next relation:

$$\phi((0,0,0,0,0);t) = pt\phi((1,0,0,0,0);t) + qt\phi((0,0,0,0,0);t).$$

Similarly, we see that the state $(1,0,0,0,0)$ changes $(1,1,0,0,0)$ and $(0,0,0,0,0)$ by one step with probabilities p^3 and q, respectively, and it remains unchanged with probability $(p-p^3)$, because if we observe $(*,1,1,1,*)'$ in the next trial, the state changes from $(1,0,0,0,0)$ to $(1,1,0,0,0)$, and if we observe $(*,*,0,*,*)'$ in the next trial, the state changes from $(1,0,0,0,0)$ to $(0,0,0,0,0)$. Then, we have

$$\phi((1,0,0,0,0);t) = p^3t\phi((1,1,0,0,0);t) + (p-p^3)t\phi((1,0,0,0,0);t) + qt\phi((0,0,0,0,0);t).$$

Similarly, considering next step from every state in $S_0(D,R)$, we obtain the next equations:

$$\begin{aligned}
\phi((1,1,0,0,0);t) &= p^5t\phi((1,1,1,0,0);t) + (p^3-p^5)t\phi((1,1,0,0,0);t)\\
&\quad + (p-p^3)t\phi((1,0,0,0,0);t) + qt\phi((0,0,0,0,0);t)\\
\phi((1,1,1,0,0);t) &= p^5t\phi((1,1,1,1,0);t) + (p^3-p^5)t\phi((1,1,0,1,0);t)\\
&\quad + (p-p^3)t\phi((1,0,0,0,0);t) + qt\phi((0,0,0,0,0);t)\\
\phi((1,1,1,1,0);t) &= pt + qt\phi((0,0,0,0,0);t)\\
\phi((1,1,0,1,0);t) &= pt + qt\phi((0,0,0,0,0);t).
\end{aligned}$$

By solving the above system of equations of conditional p.g.f.'s, we have the p.g.f. of the distribution of the waiting time for D, where the unconditional p.g.f. is $\phi((0,0,0,0,0);t)$. By expanding the p.g.f. with respect to t, we obtain the probability distribution.

Example 2. Let $D = \begin{matrix} & & 1 & & \\ & 1 & & 1 & \\ 1 & & & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix}$ and $m=5$. Then the unique scanning rectangular window R becomes

$$R = \begin{bmatrix} * & * & 1 & * & * \\ * & 1 & * & 1 & * \\ 1 & * & * & * & 1 \\ * & 1 & * & 1 & * \\ * & * & 1 & * & * \end{bmatrix}.$$

In this example, we see that

$$S(D,R)=\left\{\begin{array}{l}(0,0,0,0,0),(0,0,0,1,0),(0,0,1,0,0),(0,0,1,1,0),\\(0,1,0,0,0),(0,1,0,1,0),(0,1,1,0,0),(0,1,1,1,0),\\(1,0,0,0,0),(1,0,0,0,1),(1,0,0,1,0),(1,0,0,1,1),\\(1,0,1,0,0),(1,0,1,0,1),(1,0,1,1,0),(1,0,1,1,1),\\(1,1,0,0,0),(1,1,0,0,1),(1,1,0,1,0),(1,1,0,1,1),\\(1,1,1,0,0),(1,1,1,0,1),(1,1,1,1,0),(1,1,1,1,1)\end{array}\right\},$$

and

$$S_0(D,R)=\left\{\begin{array}{l}(0,0,0,0,0),(0,0,0,1,0),(0,0,1,0,0),(0,0,1,1,0),\\(0,1,0,0,0),(0,1,0,1,0),(0,1,1,0,0),(0,1,1,1,0),\\(1,0,0,0,0),(1,0,0,1,0),(1,0,1,0,0),(1,0,1,1,0),\\(1,1,0,0,0),(1,1,0,1,0),(1,1,1,0,0),(1,1,1,1,0)\end{array}\right\}.$$

Similarly as Example 1, considering the next step for every $\boldsymbol{a}\in S_0(D,R)$, we obtain the following system of equations of conditional p.g.f.'s:

$$\begin{aligned}
\phi((0,0,0,0,0);t) &= pt\phi((1,0,0,0,0);t)+qt\phi((0,0,0,0,0);t),\\
\phi((1,0,0,0,0);t) &= p^3t\phi((1,1,0,0,0);t)+p^2qt\phi((0,1,0,0,0);t)\\
&\quad+(p-p^3)t\phi((1,0,0,0,0);t)+(q-p^2q)t\phi((0,0,0,0,0);t),\\
\phi((1,1,0,0,0);t) &= p^5t\phi((1,1,1,0,0);t)+(p^4-p^5)t\phi((0,1,1,0,0);t)\\
&\quad+(p^3-p^5)t\phi((1,0,1,0,0);t)+(p^2q-p^4q)t\phi((0,0,1,0,0);t)\\
&\quad+(p^3-p^5)t\phi((1,1,0,0,0);t)+(p^2q-p^4q)t\phi((0,1,0,0,0);t)\\
&\quad+p(1-p^2)^2t\phi((1,0,0,0,0);t)+q(1-p^2)^2t\phi((0,0,0,0,0);t),\\
\phi((0,1,0,0,0);t) &= p^3t\phi((1,0,1,0,0);t)+p^2qt\phi((0,0,1,0,0);t)\\
&\quad+(p-p^3)t\phi((1,0,0,0,0);t)+q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((1,1,1,0,0);t) &= p^5t\phi((1,1,1,1,0);t)+p^4qt\phi((0,1,1,1,0);t)\\
&\quad+p^3(1-p^2)t\phi((1,1,0,1,0);t)+p^2q(1-p^2)t\phi((0,1,0,1,0);t)\\
&\quad+p^3(1-p^2)t\phi((1,0,1,0,0);t)+p^2q(1-p^2)t\phi((0,0,1,0,0);t)\\
&\quad+p(1-p^2)^2t\phi((1,0,0,0,0);t)+q(1-p^2)^2t\phi((0,0,0,0,0);t),\\
\phi((0,1,1,0,0);t) &= p^5t\phi((1,0,1,1,0);t)+p^4qt\phi((0,0,1,1,0);t)\\
&\quad+p^3(1-p^2)t\phi((1,0,0,1,0);t)+p^2q(1-p^2)t\phi((0,0,0,1,0);t)\\
&\quad+p^3(1-p^2)t\phi((1,0,1,0,0);t)+p^2q(1-p^2)t\phi((0,0,1,0,0);t)\\
&\quad+p(1-p^2)^2t\phi((1,0,0,0,0);t)+q(1-p^2)^2t\phi((0,0,0,0,0);t),\\
\phi((1,0,1,0,0);t) &= p^3t\phi((1,1,0,1,0);t)+p^2qt\phi((0,1,0,1,0);t)\\
&\quad+p(1-p^2)t\phi((1,0,0,0,0);t)+q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((0,0,1,0,0);t) &= p^3t\phi((1,0,0,1,0);t)+p^2qt\phi((0,0,0,1,0);t)\\
&\quad+p(1-p^2)t\phi((1,0,0,0,0);t)+q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((1,1,1,1,0);t) &= pt+p^4qt\phi((0,1,1,1,0);t)+p^2q(1-p^2)t\phi((0,1,0,1,0);t)\\
&\quad+p^2q(1-p^2)t\phi((0,0,1,0,0);t)+q(1-p^2)^2t\phi((0,0,0,0,0);t),\\
\phi((0,1,1,1,0);t) &= pt+p^4qt\phi((0,0,1,1,0);t)+p^2q(1-p^2)t\phi((0,0,0,1,0);t)
\end{aligned}$$

$$
\begin{aligned}
&\quad + p^2q(1-p^2)t\phi((0,0,1,0,0);t) + q(1-p^2)^2t\phi((0,0,0,0,0);t),\\
\phi((1,0,1,1,0);t) &= pt + p^2qt\phi((0,1,0,1,0);t) + q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((0,0,1,1,0);t) &= pt + p^2qt\phi((0,0,0,1,0);t) + q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((1,1,0,1,0);t) &= pt + p^4qt\phi((0,1,1,0,0);t) + p^2q(1-p^2)t\phi((0,0,1,0,0);t)\\
&\quad + p^2q(1-p^2)t\phi((0,1,0,0,0);t) + q(1-p^2)^2t\phi((0,0,0,0,0);t),\\
\phi((0,1,0,1,0);t) &= pt + p^2qt\phi((0,0,1,0,0);t) + q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((1,0,0,1,0);t) &= pt + p^2qt\phi((0,1,0,0,0);t) + q(1-p^2)t\phi((0,0,0,0,0);t),\\
\phi((0,0,0,1,0);t) &= pt + qt\phi((0,0,0,0,0);t).
\end{aligned}
$$

In the above examples, we searched for possible states in the next trial from every state $\boldsymbol{a} \in S_0(D,R)$ and calculated the corresponding transition probabilities. However, it is not necessarily easy to construct the system of equations of conditional p.g.f.'s for every pattern D by the above method.

Here, we present a general method for constructing the system of equations of conditional p.g.f.'s for an arbitrarily given pattern D. Let $V(m)(=\{0,1\}^m)$ be the totality of m-dimensional column vectors whose entries are 0 or 1. For $i = 1,\dots,\ell$, let $\boldsymbol{u}_i$ be the 0,1-vector obtained by substituting 0 for $*$ of $\boldsymbol{r}_i$, where $\boldsymbol{r}_i$ is the column vector of the scanning rectangular window R. We define a mapping $f_D : S_0(D,R) \times V(m) \to S(D,R)$ by $f_D(\boldsymbol{a},\boldsymbol{e}) = \boldsymbol{b}(=(b_1,b_2,\dots,b_\ell))$, where

$$
b_1 = \begin{cases} 1 & \text{if} \quad \boldsymbol{e} - \boldsymbol{u}_1 \geq \boldsymbol{0} \\ 0 & \text{otherwise,} \end{cases}
$$

and for $i = 2,\dots,\ell$,

$$
b_i = \begin{cases} 1 & \text{if} \quad \boldsymbol{e} - \boldsymbol{u}_i \geq \boldsymbol{0} \text{ and } a_{i-1} = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Then, we obtain the following theorem.

THEOREM 2.1. *For a given pattern D and its scanning rectangular window R, the conditional p.g.f.'s of the waiting time for the first occurrrence of the pattern D in the sequence of i.i.d. $V(m)$-valued random vectors satisfy the following system of linear equations: for every $\boldsymbol{a} \in S_0(D,R)$,*

$$
\phi(\boldsymbol{a};t) = \sum_{\boldsymbol{e}\in V(m)} p^{N_1(\boldsymbol{e})}(1-p)^{m-N_1(\boldsymbol{e})}t\phi(f_D(\boldsymbol{a},\boldsymbol{e});t),
$$

where $N_1(\boldsymbol{e})$ means the number of 1's in $\boldsymbol{e}$, and for every $\boldsymbol{a} \in S_1(D,R)$, $\phi(\boldsymbol{a};t) = 1$.

PROOF. Suppose that we are observing $\boldsymbol{a} \in S_0(D,R)$ currently. Then, if we observe $\boldsymbol{e} \in V(m)$ in the next trial, our state changes from $\boldsymbol{a}$ to $f_D(\boldsymbol{a},\boldsymbol{e}) \in S(D,R)$ by the definition of f_D. Since the probability that we observe $\boldsymbol{e}$ is $p^{N_1(\boldsymbol{e})}(1-p)^{m-N_1(\boldsymbol{e})}$, we obtain the above equation. To be precise, since the mapping f_D is not necessarily one-to-one, for given $\boldsymbol{a}$ and $\boldsymbol{e}$, the transition probability from $\boldsymbol{a}$ to $f_D(\boldsymbol{a},\boldsymbol{e})$ is given by

$$
\sum_{\boldsymbol{e}'\in V(\boldsymbol{a},\boldsymbol{e})} p^{N_1(\boldsymbol{e}')}(1-p)^{m-N_1(\boldsymbol{e}')},
$$

where $V(\boldsymbol{a},\boldsymbol{e}) = \{\boldsymbol{e}' \in V(m) \mid f_D(\boldsymbol{a},\boldsymbol{e}') = f_D(\boldsymbol{a},\boldsymbol{e})\}$. It is clear that the transition probability can be obtained naturally by simplifying the right hand side of the above equation. If $\boldsymbol{a} \in S_1(D,R)$, then $a_\ell = 1$ and hence this means that the pattern has just occurred. Therefore, we have $\phi(\boldsymbol{a};t) = 1$ for every $\boldsymbol{a} \in S_1(D,R)$. This completes the proof.

Remark. Theorem 2.1 gives explicitly a general form of the system of equations of conditional p.g.f.'s of the waiting time for any given two-dimensional pattern D. Hence, by using this result we can also give an algorithm for constructing the system of equations of conditional p.g.f.'s in computer algebra systems.

Actually, in complicated problems treated in the present paper, construction of the state set $S(D,R)$ is one of difficult problems. However, we have to note that the above mapping f_D is useful for constructing the state set $S(D,R)$. We briefly give here an algorithm to obtain the set $S(D,R)$. First, we let $A_0 = \{(\underbrace{0,\dots,0}_{\ell})\}$. For $i = 1,2,\dots,$ we construct recursively

$$A_i = A_{i-1} \cup (\cup_{\boldsymbol{a}\in A_{i-1}} \cup_{\boldsymbol{e}\in V(m)} \{f_D(\boldsymbol{a},\boldsymbol{e})\}).$$

Then, there exists a number i_0 such that $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{i_0} = A_{i_0+1} = \cdots$, since the number of elements of A_i is finite (at most 2^ℓ). Thus, we can obtain $S(D,R) = A_{i_0}$ systematically.

In fact, we have treated Examples 1 and 2 by using the above systematic method. We can use the method for all two-dimensional patterns. However, the constructed set $S(D,R)$ in this way is not necessarily minimal for every pattern. For example, one of the referees indicated that in Example 1 the states $(1,1,1,1,0)$ and $(1,1,0,1,0)$ need not be distinguished. We can see the fact from the equality of the right hand sides of the last two equations of the conditional p.g.f.'s. Though reduction of the state space may contribute to reduction of time for solving the corresponding system of equations, minimality of the state space is not necessarily needed for calculation. Since it seems difficult to construct minimal state space systematically for all two-dimensional patterns, we do not study the problem of constructing minimal state space in the present paper.

3. Waiting time in the case that the height of the window is less than m

In this section we treat the case that the height of R is less than m. Let h be the height of R. Then, the scanning rectangular window R can take $(m-h+1)$ possible vertical positions. For example, we may suppose that $m = 5$, $D = \begin{matrix} 1 \\ 111 \\ 1 \end{matrix}$, and

$$R = \begin{bmatrix} * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \end{bmatrix}.$$

In this case, R can take three possible vertical positions, since the height of R is three. In Section 2, we represented every state by an ℓ-dimensional 0,1-vector $\boldsymbol{a} \in \{0,1\}^\ell$. However, in this general case, we represent every state by a list of $(m-h+1)$ ℓ-dimensional

0, 1-vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{m-h+1})$, where $\boldsymbol{a}_j = (a_{j,1}, \dots, a_{j,\ell})$ shows the current state in the sequence of h-dimensional column vectors which consist of the j-th, $(j+1)$-th, $\dots$, and $(j+h-1)$-th rows. Though we can see that $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{m-h+1}) \in \{0,1\}^{\ell(m-h+1)}$, every element of $\{0,1\}^{\ell(m-h+1)}$ is not necessarily possible as a current state. Let $\boldsymbol{S}(D, R)$ denote the set of all $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{m-h+1})$-vectors that can be observed as a current state. $\boldsymbol{S}(D, R)$ depends on D and R. For the above example, we see that

$$\boldsymbol{S}(D,R) = \left\{ \begin{array}{l}
((1,1,0),(1,0,0),(1,1,0)),((0,0,0),(0,0,0),(0,0,0)),\\
((1,1,0),(1,1,1),(1,0,0)),((0,0,0),(0,0,0),(1,0,0)),\\
((0,0,0),(1,0,0),(0,0,0)),((0,0,0),(1,0,0),(1,0,0)),\\
((1,0,0),(0,0,0),(0,0,0)),((1,0,1),(1,0,0),(0,0,0)),\\
((1,0,0),(0,0,0),(1,0,0)),((1,0,0),(1,0,0),(0,0,0)),\\
((1,0,1),(0,0,0),(0,0,0)),((1,0,1),(1,1,1),(1,1,0)),\\
((1,0,0),(1,0,0),(1,0,0)),((0,0,0),(1,0,0),(1,0,1)),\\
((0,0,0),(1,0,0),(1,1,1)),((1,0,1),(1,0,1),(0,0,0)),\\
((0,0,0),(0,0,0),(1,0,1)),((1,0,1),(0,0,0),(1,0,0)),\\
((1,0,1),(0,0,0),(1,0,1)),((1,0,1),(1,1,0),(1,0,1)),\\
((1,1,0),(1,0,1),(0,0,0)),((1,0,1),(1,1,0),(1,1,1)),\\
((1,0,1),(1,1,1),(1,0,0)),((1,0,0),(1,1,1),(1,0,1)),\\
((0,0,0),(1,0,0),(1,1,0)),((1,0,1),(1,1,1),(1,0,1)),\\
((1,0,1),(1,1,1),(1,1,1)),((1,1,1),(1,0,0),(0,0,0)),\\
((1,1,1),(1,1,1),(1,1,0)),((1,1,1),(1,1,1),(1,0,0)),\\
((1,1,1),(1,1,0),(1,0,1)),((1,1,1),(1,1,0),(1,1,1)),\\
((1,0,0),(1,0,0),(1,1,0)),((1,1,1),(1,0,1),(0,0,0)),\\
((0,0,0),(1,0,1),(1,0,1)),((0,0,0),(1,0,1),(1,1,1)),\\
((1,1,0),(1,1,1),(1,1,1)),((1,1,0),(1,1,1),(1,0,1)),\\
((1,1,0),(1,1,0),(1,0,0)),((1,1,1),(1,1,1),(1,1,1)),\\
((1,0,0),(1,1,0),(1,0,0)),((1,1,1),(1,1,1),(1,0,1)),\\
((1,0,0),(1,1,1),(1,1,1)),((1,0,0),(1,1,0),(1,1,0)),\\
((1,1,0),(1,1,0),(1,1,0)),((1,1,1),(1,1,0),(1,1,0)),\\
((1,1,0),(1,1,1),(1,1,0)),((1,1,0),(1,0,0),(0,0,0)),\\
((1,1,0),(1,0,0),(1,0,0)),((1,0,1),(1,1,0),(1,0,0)),\\
((1,0,1),(1,1,0),(1,1,0)),((1,1,1),(1,1,0),(1,0,0)),\\
((1,0,0),(0,0,0),(1,0,1)),((1,0,0),(1,1,0),(1,0,1)),\\
((1,0,0),(1,1,0),(1,1,1)),((1,1,0),(1,1,0),(1,1,1)),\\
((1,1,0),(1,1,0),(1,0,1)),((0,0,0),(1,0,1),(0,0,0)),\\
((0,0,0),(1,0,1),(1,0,0)),((0,0,0),(1,0,1),(1,1,0)),\\
((1,0,0),(1,0,1),(0,0,0)),((1,0,0),(1,1,1),(1,0,0)),\\
((1,0,0),(1,1,1),(1,1,0))
\end{array} \right\}.$$

Let $\boldsymbol{S}_1(D,R) = \{((a_{1,1},\ldots,a_{1,\ell}),\ldots,(a_{m-h+1,1},\ldots,a_{m-h+1,\ell})) \in \boldsymbol{S}(D,R) \mid a_{j,\ell} = 1$ for some $j\}$ and $\boldsymbol{S}_0(D,R) = \boldsymbol{S}(D,R) \setminus \boldsymbol{S}_1(D,R)$. Then, we obtain

$$\boldsymbol{S}_0(D,R) = \left\{ \begin{array}{l} ((1,1,0),(1,0,0),(1,1,0)),((0,0,0),(0,0,0),(0,0,0)), \\ ((0,0,0),(0,0,0),(1,0,0)),((0,0,0),(1,0,0),(0,0,0)), \\ ((0,0,0),(1,0,0),(1,0,0)),((1,0,0),(0,0,0),(0,0,0)), \\ ((1,0,0),(0,0,0),(1,0,0)),((1,0,0),(1,0,0),(0,0,0)), \\ ((1,0,0),(1,0,0),(1,0,0)),((0,0,0),(1,0,0),(1,1,0)), \\ ((1,0,0),(1,0,0),(1,1,0)),((1,1,0),(1,1,0),(1,0,0)), \\ ((1,0,0),(1,1,0),(1,0,0)),((1,0,0),(1,1,0),(1,1,0)), \\ ((1,1,0),(1,1,0),(1,1,0)),((1,1,0),(1,0,0),(0,0,0)), \\ ((1,1,0),(1,0,0),(1,0,0)) \end{array} \right\}.$$

For each $\boldsymbol{e} = (e_1,\ldots,e_m)'$ and for $i = 1,2,\ldots,m-h+1$, we define $\boldsymbol{e}(i) \equiv (e_i,e_{i+1},\ldots,e_{i+h-1})' \in V(h)$. Similarly as in Section 2 we define a mapping $\boldsymbol{f}_D : \boldsymbol{S}_0(D,R) \times V(m) \to \boldsymbol{S}(D,R)$ by $\boldsymbol{f}_D((\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{m-h+1}),\boldsymbol{e}) = (\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_{m-h+1})$, where for $j = 1,\ldots,m-h+1$, $\boldsymbol{b}_j = (b_{j,1},\ldots,b_{j,\ell})$ is defined as

$$b_{j,1} = \begin{cases} 1 & \text{if} \quad \boldsymbol{e}(j) - \boldsymbol{u}_1 \geq \boldsymbol{0} \\ 0 & \text{otherwise,} \end{cases}$$

and for $i = 2,\ldots,\ell$,

$$b_{j,i} = \begin{cases} 1 & \text{if} \quad \boldsymbol{e}(j) - \boldsymbol{u}_i \geq \boldsymbol{0} \text{ and } a_{j,i-1} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, similarly as in the proof of Theorem 2.1, we obtain the result.

THEOREM 3.1. *For a given pattern D and its scanning rectangular window R, the conditional p.g.f.'s of the waiting time for the first occurrrence of the pattern D in the sequence of i.i.d. $V(m)$-valued random vectors satisfy the following system of linear equations: for every $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{m-h+1}) \in \boldsymbol{S}_0(D,R)$,*

$$\phi((\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{m-h+1});t) = \sum_{\boldsymbol{e} \in V(m)} p^{N_1(\boldsymbol{e})}(1-p)^{m-N_1(\boldsymbol{e})} t \phi(\boldsymbol{f}_D((\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{m-h+1}),\boldsymbol{e});t),$$

where $N_1(\boldsymbol{e})$ means the number of 1's in $\boldsymbol{e}$, and for every $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{m-h+1}) \in \boldsymbol{S}_1(D,R)$, $\phi((\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{m-h+1});t) = 1$.

In this case, we can also obtain $S(D,R)$ systematically by using the mapping $\boldsymbol{f}_D$ similarly as the previous section.

4. Computational results

Since Theorems 2.1 and 3.1 give general forms of the systems of equations of conditional p.g.f.'s of the waiting time for any given two-dimensional pattern and scanning rectangular window, we can interpret these results as algorithms for constructing the conditional p.g.f.'s. Practically, we can obtain the p.g.f.'s of the waiting time problems by using computer algebra systems.

Example 3. (Continuation of Example 1.) By solving the linear system of equations given in Example 1, we obtain

$$\phi((0,0,0,0,0);t)$$
$$= \frac{p^{13}t^5}{1-t+p^5t-p^5t^2+p^8t^2-p^8t^3+p^{11}t^3-p^{11}t^4+p^{12}t^4-p^{12}t^5+p^{13}t^5}.$$

By extracting the coefficient of t^i for $i = 1, 2, \ldots$ in the power series expansion of the above p.g.f., we can obtain exact probability value $P(W = i)$ as a function of p. Because the formulas are very long, we give here only Fig. 1, which shows the waiting time distribution with $p = 0.85$.

Example 4. (Continuation of Example 2.) By solving the linear system of equations given in Example 2, we obtain the p.g.f. $\phi((0,0,0,0,0);t)$. However, since the formula of $\phi((0,0,0,0,0);t)$ is a little long, we refrain from giving here the whole formula. Actually, it is a ratio between two polynomials: The numerator is a polynomial with 82 terms. When we regard it as a polynomial with respect to t, its degree is 16 and the leading coefficient is

$$p^{56} - 7p^{55} + 21p^{54} - 35p^{53} + 35p^{52} - 21p^{51} + 7p^{50} - p^{49}.$$

The denominator with 166 terms is a polynomial with respect to t with degree 16 and with leading coefficient

$$p^{56} - 8p^{55} + 28p^{54} - 56p^{53} + 70p^{52} - 56p^{51} + 28p^{50} - 8p^{49} + p^{48}.$$

Though the formula of the p.g.f. is relatively long, there is no problem for the power series expansion of the p.g.f. Since the probability function of the waiting time distribution as a function of p is too long to write here, we also give here only Fig. 2, which shows the waiting time distribution with $p = 0.8$.

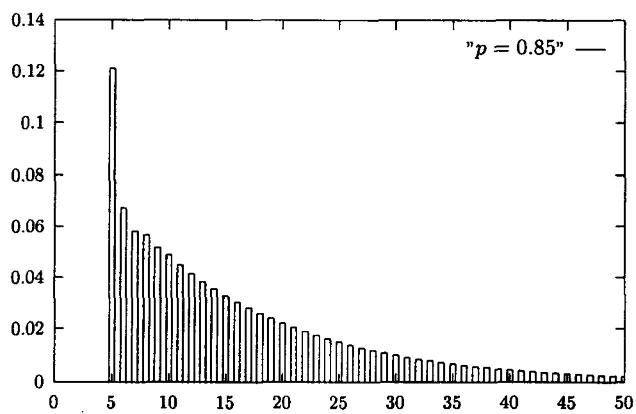

Fig. 1. The distribution of the waiting time of Example 3 with $p = 0.85$.

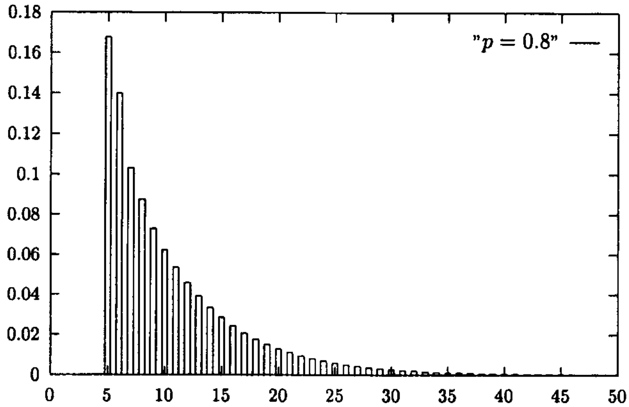

Fig. 2. The distribution of the waiting time of Example 4 with $p = 0.8$.

Example 5. Let us consider the example given in Section 3, i.e., the example that the height of R is less than m with $m = 5$, $D = \begin{matrix} 1 \\ 111 \\ 1 \end{matrix}$, and

$$R = \begin{bmatrix} * & 1 & * \\ 1 & 1 & 1 \\ * & 1 & * \end{bmatrix}.$$

By using Theorem 3.1, we can construct the system of linear equations of conditional p.g.f.'s. Further, we can obtain its solution $\phi(((0,0,0),(0,0,0),(0,0,0));t)$ by using computer algebra systems. However, since the formula of $\phi(((0,0,0),(0,0,0),(0,0,0));t)$ is rather long, we also refrain from giving here the whole formula. Actually, it is a ratio between two polynomials: The numerator is a polynomial with 104 terms. When we regard it as a polynomial with respect to t, its degree is 12 and the leading coefficient is

$$p^{43} - 10p^{42} + 45p^{41} - 119p^{40} + 203p^{39} - 231p^{38} + 175p^{37} - 85p^{36} + 24p^{35} - 3p^{34}.$$

The denominator with 134 terms is a polynomial with respect to t with degree 12 and with leading coefficient

$$p^{43} - 10p^{42} + 45p^{41} - 120p^{40} + 210p^{39} - 252p^{38} + 210p^{37} - 120p^{36} + 45p^{35} - 10p^{34} + p^{33}.$$

Though the formula of the p.g.f. is long, the power series expansion of the p.g.f. can be performed successfully. However, since the long formula of the probability function of the waiting time distribution as a function of p may not be suitable for giving here, we give only Fig. 3, which shows the waiting time distribution with $p = 0.6$.

As direct applications of this example, let us consider the following two-dimensional lattice system of $5 \times n$ components. Suppose that components of the system fails independently with probability p and the system fails if and only if there is at least one pattern $D = \begin{matrix} 1 \\ 111 \\ 1 \end{matrix}$ of failed components. In particular, we consider the next three systems, System 1, System 2 and System 3 for $n = 10, 20$ and 30, respectively. We denote by W the

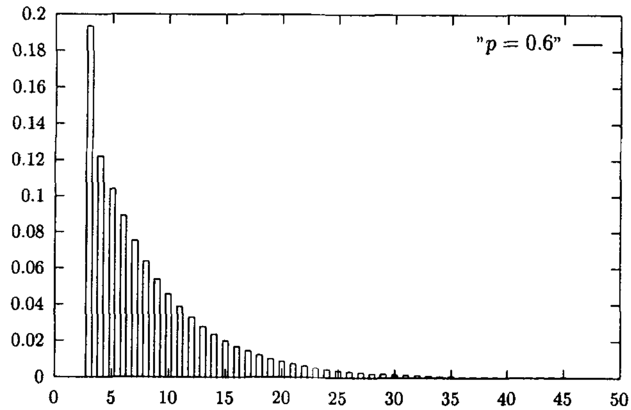

Fig. 3. The distribution of the waiting time of Example 5 with $p = 0.6$.

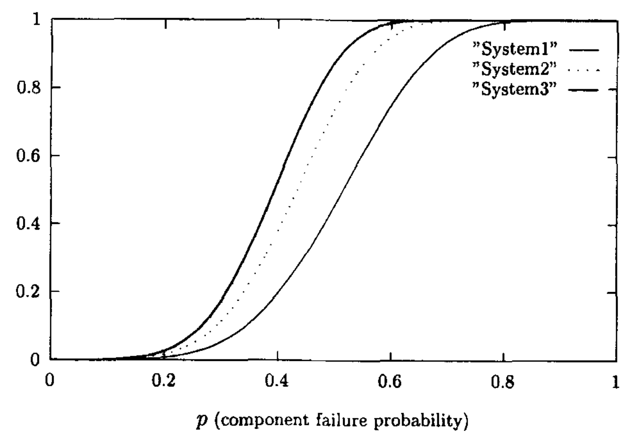

Fig. 4. System failure probabilities of the three 2-dimensional consecutive systems.

waiting time for the pattern. Then, since the probability $P(W \leq n)$ is the system failure probability, it is easily calculated from the p.g.f. of W, $\phi((0,0,0),(0,0,0),(0,0,0));t)$. By summimg the coefficients of t^i for $i = 0,1,\ldots,n$ in the power series expansion of the above p.g.f., we obtain the exact system failure probabilities of the systems. Since the exact formulae are very long, we give only Fig. 4, which shows the system failure probabilities as functions of component failure probability for the three systems.

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