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ESTIMATION OF THE EIGENVALUES OF NONCENTRALITY PARAMETER IN MATRIX VARIATE NONCENTRAL BETA DISTRIBUTION

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Abstract. We consider the problem of estimating the eigenvalues of noncentrality parameter matrix in a matrix variate noncentral beta distribution, also known as multivariate noncentral F distribution. A decision theoretic approach is taken with square error as the loss function. We propose two types of new estimators and show their superior performance theoretically as well as numerically.

Key words and phrases: Unbiased estimator, empirical Bayes estimator, zonal polynomial, orthogonally invariant estimator, Monte Carlo simulations.

1. Introduction

Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be independent $p \times p$ random matrices of which distributions are given by

$$\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma}, \boldsymbol{\Delta}), \qquad \boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma}), \qquad n_1 \geq p,\ n_2 \geq p.$$

While $\boldsymbol{S}_1$ has a noncentral Wishart distribution with n_1 degrees of freedom, scale parameter $\boldsymbol{\Sigma}$ and noncentrality parameter matrix $\boldsymbol{\Delta}$, $\boldsymbol{S}_2$ has a central Wishart distribution with n_2 degrees of freedom and scale parameter $\boldsymbol{\Sigma}$. In some multivariate linear models such as MANOVA, discriminant analysis, the eigenvalues of $\boldsymbol{\Delta}$ need to be estimated. Leung and Muirhead (1987) and Leung and Lo (1996) deal with this estimation problem. A motivation has been discussed (Leung and Muirhead (1987) and Leung and Lo (1996)) in connection with choosing the standard invariant tests (LR test, Hotelling-Lawley trace, Bartlett-Nanda-Pillai trace, Roy's maximum root) whose powers depend on the eigenvalues of a noncentrality matrix. Another interesting motivation appears in the problem of constructing modified AIC or C_p in selection of variables for multivariate regression. In general, the AIC and the C_p have been proposed as approximately unbiased estimators for their risks or underlying criterion functions. In order to reduce bias of the formal AIC and C_p, we need to estimate the bias which depends on a noncentrality matrix or its eigenvalues. For its references, see, e.g. Fujikoshi and Satoh (1997).

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As Leung and Muirhead (1987) and Leung and Lo (1996) proposed, we transform the distributions as follows. Let

$$\boldsymbol{A} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}_1\boldsymbol{\Sigma}^{-1/2}, \qquad \boldsymbol{B} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}_2\boldsymbol{\Sigma}^{-1/2}.$$

Then $\boldsymbol{A}$ and $\boldsymbol{B}$ are independently distributed as

$$\boldsymbol{A} \sim W_p(n_1, \boldsymbol{I}_p, \boldsymbol{\Omega}), \qquad \boldsymbol{B} \sim W_p(n_2, \boldsymbol{I}_p),$$

where

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1/2}$$

has the same eigenvalues as $\boldsymbol{\Delta}$. Though $\boldsymbol{A}$ and $\boldsymbol{B}$ are not observable, the eigenvalues of

$$\boldsymbol{F} = \boldsymbol{A}^{1/2}\boldsymbol{B}^{-1}\boldsymbol{A}^{1/2}$$

are observable since they are the same as those of $\boldsymbol{S}_1\boldsymbol{S}_2^{-1}$.

Now we have the newly formulated estimation problem. $\boldsymbol{F}$ is distributed as so-called "multivariate noncentral F" distribution. (Sometimes this name is confusing since there is a vector variate F distribution. We think "matrix variate noncentral beta distribution" is preferable. For the detailed classification of the matrix variate distributions, see Gupta and Nagar (1999).) We use the notation

$$\boldsymbol{F} \sim F_p(n_1, n_2, \boldsymbol{I}_p, \boldsymbol{\Omega}). \tag{1.1}$$

The density of $\boldsymbol{F}$ is given by

$$\begin{aligned} &K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right)|\boldsymbol{F}|^{n_1/2-(p+1)/2}|\boldsymbol{I}_p + \boldsymbol{F}|^{-(n_1+n_2)/2} \\ &\qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p + \boldsymbol{F})^{-1}\right), \end{aligned} \tag{1.2}$$

where

$$K_0 = \frac{\Gamma_p\left(\dfrac{n_1+n_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}. \tag{1.3}$$

The hypergeometric function ${}_1F_1$ is defined by

$$\begin{aligned} &{}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p + \boldsymbol{F})^{-1}\right) \\ &\quad = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{\left(\dfrac{n_1+n_2}{2}\right)_\kappa}{\left(\dfrac{n_1}{2}\right)_\kappa}C_\kappa\left(\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p + \boldsymbol{F})^{-1}\right), \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} &(a)_\kappa = \prod_{j=1}^{p}(a - (j-1)/2)_{k_j}, \\ &(a - (j-1)/2)_{k_j} = (a - (j-1)/2)(a - (j-1)/2 + 1)\cdots(a - (j-1)/2 + k_j - 1) \end{aligned}$$

for κ, a partition of k, i.e.;

$$\kappa = (k_1, \ldots, k_p), \quad k_1 \geq \cdots \geq k_p \geq 0, \quad k_1 + \cdots + k_p = k.$$

For this density, see Gupta and Nagar ((1999), p. 192). We estimate the eigenvalues of $\boldsymbol{\Omega}$,

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p), \quad \omega_1 \geq \cdots \geq \omega_p > 0$$

from the eigenvalues of F,

$$\boldsymbol{l} = (l_1, \ldots, l_p), \quad l_1 \geq \ldots \geq l_p > 0.$$

Instead of estimating $\boldsymbol{\omega}$ directly, Leung and Muirhead (1987) and Leung and Lo (1996) estimate $\boldsymbol{\Omega}$ first with orthogonally invariant estimators. In general, an orthogonally invariant estimator has the form,

$$\widehat{\boldsymbol{\Omega}} = \boldsymbol{H}\boldsymbol{\Psi}(\boldsymbol{L})\boldsymbol{H}', \quad \boldsymbol{\Psi}(\boldsymbol{L}) = \mathrm{diag}(\psi_1(\boldsymbol{L}), \ldots, \psi_p(\boldsymbol{L})), \tag{1.5}$$

where

$$\boldsymbol{F} = \boldsymbol{H}\boldsymbol{L}\boldsymbol{H}' \tag{1.6}$$

is the spectral decomposition of $\boldsymbol{F}$ with an orthogonally invariant matrix $\boldsymbol{H}$ and $\boldsymbol{L} = \mathrm{diag}(l_1, \ldots, l_p)$. They evaluate the estimator $\widehat{\boldsymbol{\Omega}}$ of $\boldsymbol{\Omega}$ using the loss functions

$$L_1(\widehat{\boldsymbol{\Omega}}, \boldsymbol{\Omega}) = \mathrm{tr}(\widehat{\boldsymbol{\Omega}} - \boldsymbol{\Omega})^2, \tag{1.7}$$

$$L_2(\widehat{\boldsymbol{\Omega}}, \boldsymbol{\Omega}) = \mathrm{tr}(\widehat{\boldsymbol{\Omega}}\boldsymbol{\Omega}^{-1} - \boldsymbol{I}_p)^2. \tag{1.8}$$

After getting a superior estimator $\widehat{\boldsymbol{\Omega}}$, they use $\psi_i(\boldsymbol{L})$ in (1.5) as an estimator of ω_i. It should be noted that $\widehat{\boldsymbol{\Omega}}$ itself in (1.5) is of no use since $\boldsymbol{H}$ is not observable. It is used only for the evaluation of $(\psi_1(\boldsymbol{L}), \ldots, \psi_p(\boldsymbol{L}))$ as an estimator of $\boldsymbol{\omega}$.

Actually they conceived estimators of $\boldsymbol{\Omega}$ that dominate the unbiased estimator

$$\widehat{\boldsymbol{\Omega}}^{(u)} = (n_2 - p - 1)\boldsymbol{F} - n_1\boldsymbol{I}_p, \tag{1.9}$$

(see (3.1) of Leung and Muirhead (1987)). Their estimators have the form

$$\alpha\widehat{\boldsymbol{\Omega}}^{(u)} \tag{1.10}$$

and

$$\widehat{\boldsymbol{\Omega}}^{(l)} = \alpha\widehat{\boldsymbol{\Omega}}^{(u)} + \frac{\beta}{\mathrm{tr}\,\boldsymbol{F}}\boldsymbol{I}_p, \tag{1.11}$$

where α and β are some positive constants. They proved the following theoretically.

1. $\alpha\widehat{\boldsymbol{\Omega}}^{(u)}$ dominates $\widehat{\boldsymbol{\Omega}}^{(u)}$ w.r.t. $L_1(\widehat{\boldsymbol{\Omega}}, \boldsymbol{\Omega})$ if $n_2 > p + 3$ and $\max(0, \alpha_l) \leq \alpha < 1$, where

$$\alpha_l = \frac{n_2 - p - 5}{n_2 - p - 1}$$

(Theorem 3.1 in Leung and Muirhead (1987)).

2. $\alpha\widehat{\mathbf{\Omega}}^{(u)}$ dominates $\widehat{\mathbf{\Omega}}^{(u)}$ w.r.t. $L_2(\widehat{\mathbf{\Omega}}, \mathbf{\Omega})$ if $n_2 > p+3$ and $\max(0, \alpha_l) \le \alpha < 1$, where

$$\alpha_l = \frac{(n_2-p)(n_2-p-4)}{(n_2-p-1)(n_2-1)} - \frac{p}{n_2-1} - \frac{1}{(n_2-p-1)(n_2-1)}$$

(Theorem 3.3 in Leung and Lo (1996)).

3. $\widehat{\mathbf{\Omega}}^{(l)}$ dominates $\alpha\widehat{\mathbf{\Omega}}^{(u)}$ w.r.t. $L_1(\widehat{\mathbf{\Omega}}, \mathbf{\Omega})$ if $n_2 > p+3$, $pn_1 > 4$, $0 < \alpha \le \alpha^{(u)}$ and $0 < \beta < \beta^{(u)}$, where

$$\alpha^{(u)} = 1 + \frac{2}{p(n_2-p-1)}$$

and

$$\beta^{(u)} = \frac{4\alpha(n_1+n_2-p-1)(pn_1-4)}{p(n_2-p+3)(n_2-p+1)}$$

(Theorem 3.2 in Leung and Muirhead (1987)).

4. $\widehat{\mathbf{\Omega}}^{(l)}$ dominates $\alpha\widehat{\mathbf{\Omega}}^{(u)}$ w.r.t. $L_2(\widehat{\mathbf{\Omega}}, \mathbf{\Omega})$ if $n_1 > 4$, $n_2 > p-1$, $0 < \alpha < \alpha^{(u)}$ and $0 < \beta < \beta^{(u)}$, where

$$\alpha^{(u)} = 1 + \frac{2}{p(n_2-p+1)}$$

and

$$\beta^{(u)} = \frac{4\alpha(n_1-4)}{n_2-p+3}$$

(Theorem 3.5 in Leung and Lo (1996)).

Consequently the estimators, (1.9), (1.10) and (1.11) respectively give rise to the following estimators of $\boldsymbol{\omega}$;

$$\widehat{\boldsymbol{\omega}}^{(u)} = (\widehat{\omega}_1^{(u)}, \dots, \widehat{\omega}_p^{(u)}), \tag{1.12}$$

$$\alpha\widehat{\boldsymbol{\omega}}^{(u)} = (\alpha\widehat{\omega}_1^{(u)}, \dots, \alpha\widehat{\omega}_p^{(u)}), \tag{1.13}$$

$$\widehat{\boldsymbol{\omega}}^{(l)} = (\widehat{\omega}_1^{(l)}, \dots, \widehat{\omega}_p^{(l)}), \tag{1.14}$$

with

$$\widehat{\omega}_i^{(u)} = (n_2-p-1)l_i - n_1, \qquad i = 1, \dots, p \tag{1.15}$$

and

$$\widehat{\omega}_i^{(l)} = \alpha((n_2-p-1)l_i - n_1) + \frac{\beta}{\operatorname{tr} F}, \qquad i = 1, \dots, p, \tag{1.16}$$

where α and β are some constants in the aforementioned intervals. They proposed specific values of α and β as optimal in the sense that they maximize certain risk difference lower bounds. Leung and Muirhead (1987) gives

$$\begin{cases} \alpha_1 = \dfrac{n_2-p-3}{n_2-p-1} \\ \beta_1 = \dfrac{2(n_2-p-3)(n_1+n_2-p-1)(pn_1-4)}{p(n_2-p-1)(n_2-p+3)(n_2-p+1)} \end{cases} \tag{1.17}$$

for the loss function $L_1(\widehat{\mathbf{\Omega}}, \mathbf{\Omega})$. Leung and Lo (1996) gives

$$(1.18) \qquad \begin{cases} \alpha_2 = \dfrac{(n_2-p)(n_2-p-3)}{(n_2-p-1)(n_2-1)} \\ \beta_2 = \dfrac{2(n_2-p)(n_2-p-3)(n_1-4)}{(n_2-p-1)(n_2-p+3)(n_2-1)} \end{cases}$$

for the loss function $L_2(\widehat{\mathbf{\Omega}}, \mathbf{\Omega})$. Let

$$\widehat{\boldsymbol{\omega}}^{(lj)} = (\widehat{\omega}_1^{(lj)}, \ldots, \widehat{\omega}_p^{(lj)}), \qquad j = 1, 2$$

denote the specific estimator given by (1.14) and (1.16) with (1.17) or (1.18) respectively.

In this paper we directly evaluate the risk of an estimator $\widehat{\boldsymbol{\omega}} = (\widehat{\omega}_1, \ldots, \widehat{\omega}_p)$ using the quadratic loss function,

$$(1.19) \qquad L_3(\widehat{\boldsymbol{\omega}}, \boldsymbol{\omega}) = \sum_{i=1}^{p} (\widehat{\omega}_i - \omega_i)^2.$$

In Section 2, we propose a new estimator and prove its dominance over the estimator $\widehat{\boldsymbol{\omega}}^{(u)}$. In Section 3, we propose another estimator, which is an empirical Bayes estimator. In Section 4, we carry out Monte Carlo simulation to compare these new estimators with those in the past literature.

We note that the derivation of new estimators here is based on the same idea as in Gupta *et al.* (2002), which deals with the estimation of noncentrality parameter of noncentral Wishart distribution.

2. First estimator

We consider a new type of estimator $\widehat{\boldsymbol{\omega}}^{(*)} = (\widehat{\omega}_1^{(*)}, \ldots, \widehat{\omega}_p^{(*)})$ defined by

$$(2.1) \qquad \widehat{\omega}_i^{(*)} = \widehat{\omega}_i^{(u)} + b|\boldsymbol{F}|^{-h} = (n_2-p-1)l_i - n_1 + b|\boldsymbol{F}|^{-h}, \qquad i = 1, \ldots, p,$$

where h and b are some positive constants. This type of estimator is also treated in Gupta *et al.* (2002). In Section 2 of that paper, we can find some motivation behind the use of this type of estimator.

We will prove that $\alpha\widehat{\boldsymbol{\omega}}^{(*)}$ dominates $\alpha\widehat{\boldsymbol{\omega}}^{(u)}$ under a certain condition on α. Since the distribution of $\boldsymbol{l}$ only depends on $\boldsymbol{\omega}$, we assume that $\mathbf{\Omega}$ is a diagonal matrix, $\mathrm{diag}(\omega_1, \ldots, \omega_p)$. For the proof, we need the following lemmas. Hereafter let M denote the constant value as

$$M = \frac{1}{n_2+2h-p-1} \frac{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \operatorname{etr}\left(-\frac{1}{2}\mathbf{\Omega}\right).$$

LEMMA 2.1. *If*

$$(2.2) \qquad \frac{-n_2+p-1}{2} < h < \frac{n_1-p+1}{2},$$

then we have

$$E[|\boldsymbol{F}|^{-h}] = M(n_2 + 2h - p - 1)_1F_1\left(\frac{n_1}{2} - h; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\right).$$

PROOF. From (1.2), we have

$$\begin{aligned} E[|\boldsymbol{F}|^{-h}] &= K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \int_{F>0} |\boldsymbol{F}|^{n_1/2-h-(p+1)/2} |\boldsymbol{I}_p + \boldsymbol{F}|^{-(n_1+n_2)/2} \\ &\qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p + \boldsymbol{F})^{-1}\right) d\boldsymbol{F}. \end{aligned}$$

The transformation,

$$\boldsymbol{U} = (\boldsymbol{I}_p + \boldsymbol{F})^{-1/2}\boldsymbol{F}(\boldsymbol{I}_p + \boldsymbol{F})^{-1/2}$$

with Jacobian (see e.g. p. 14, (xvii) of Gupta and Nagar (1999))

$$J(\boldsymbol{F} \to \boldsymbol{U}) = |\boldsymbol{I}_p - \boldsymbol{U}|^{-(p+1)}$$

leads to

$$\begin{aligned} E[|\boldsymbol{F}|^{-h}] &= K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \int_{0<U<I_p} |\boldsymbol{U}|^{n_1/2-h-(p+1)/2} |\boldsymbol{I}_p - \boldsymbol{U}|^{n_2/2+h-(p+1)/2} \\ &\qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{U}\right) d\boldsymbol{U} \\ &= K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \frac{\Gamma_p\left(\frac{n_1}{2} - h\right)\Gamma_p\left(\frac{n_2}{2} + h\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}\right)} \\ &\qquad \times {}_2F_2\left(\frac{n_1+n_2}{2}, \frac{n_1}{2} - h; \frac{n_1}{2}, \frac{n_1+n_2}{2}; \frac{1}{2}\boldsymbol{\Omega}\right) \\ &= M(n_2 + 2h - p - 1)_1F_1\left(\frac{n_1}{2} - h; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\right). \end{aligned}$$

For the second equation, see e.g. Theorem 1.6.3 of Gupta and Nagar (1999). The third equation comes from the fact

$${}_2F_2\left(\frac{n_1+n_2}{2}, \frac{n_1}{2} - h; \frac{n_1}{2}, \frac{n_1+n_2}{2}; \frac{1}{2}\boldsymbol{\Omega}\right) = {}_1F_1\left(\frac{n_1}{2} - h; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\right).$$

Note that (2.2) is a necessary and sufficient condition for the convergence of $E[|\boldsymbol{F}|^{-h}]$. This is obvious from the first equation. □

LEMMA 2.2. *Suppose*

$$\frac{-n_2 + p + 1}{2} < h < \frac{n_1 - p + 1}{2}. \tag{2.3}$$

Then

$$E[|\boldsymbol{F}|^{-h} \operatorname{tr} \boldsymbol{F}] = -\left(\frac{n_2}{2} + h - \frac{p+1}{2}\right)^{-1} K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) g'(1),$$

where

$$g(x) = \int_{0<U<I_p} |\boldsymbol{I_p} - x\boldsymbol{U}|^{n_2/2+h-(p+1)/2}|\boldsymbol{U}|^{n_1/2-h-(p+1)/2} \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{U}\right) d\boldsymbol{U}, \qquad x \le 1.$$

PROOF. First we note that (2.3) is a necessary and sufficient condition for the convergence of both $g(x)$ and $E[|\boldsymbol{F}|^{-h}\operatorname{tr}\boldsymbol{F}]$.

Similarly to the proof of Lemma 2.1, we have

$$\begin{aligned} E[|\boldsymbol{F}|^{-h}\operatorname{tr}\boldsymbol{F}] &= K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right)\int_{F>0} |\boldsymbol{F}|^{n_1/2-h-(p+1)/2}|\boldsymbol{I_p}+\boldsymbol{F}|^{-(n_1+n_2)/2}\operatorname{tr}\boldsymbol{F} \\ &\qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I_p}+\boldsymbol{F})^{-1}\right) d\boldsymbol{F} \\ &= K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \\ &\quad \times \int_{0<U<I_p} |\boldsymbol{U}|^{n_1/2-h-(p+1)/2}|\boldsymbol{I_p}-\boldsymbol{U}|^{n_2/2+h-(p+1)/2}\operatorname{tr}(\boldsymbol{U}^{-1}-\boldsymbol{I_p})^{-1} \\ &\qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{U}\right) d\boldsymbol{U}. \end{aligned}$$

Let $\boldsymbol{d} = (d_1, \ldots, d_p)$, $d_1 \ge \cdots \ge d_p \ge 0$, be the eigenvalues of $\boldsymbol{U}$. Then we have

$$|\boldsymbol{I_p} - x\boldsymbol{U}|^{\alpha} = \prod_{i=1}^{p}(1-xd_i)^{\alpha},$$

and

$$\frac{d}{dx}|\boldsymbol{I_p} - x\boldsymbol{U}|^{\alpha} = -\alpha\prod_{i=1}^{p}(1-xd_i)^{\alpha}\sum_{i=1}^{p}\frac{d_i}{1-xd_i}.$$

Therefore we have

$$\left(\frac{d}{dx}|\boldsymbol{I_p} - x\boldsymbol{U}|^{\alpha}\right)\bigg|_{x=1} = -\alpha|\boldsymbol{I_p}-\boldsymbol{U}|^{\alpha}\operatorname{tr}(\boldsymbol{U}^{-1}-\boldsymbol{I_p})^{-1}.$$

Using this relationship, we have

$$\begin{aligned} (2.4)\qquad & E[|\boldsymbol{F}|^{-h}\operatorname{tr}\boldsymbol{F}] \\ &= -\left(\frac{n_2}{2}+h-\frac{p+1}{2}\right)^{-1} K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \\ &\quad \times \int_{0<U<I_p}\left(\frac{d}{dx}|\boldsymbol{I_p}-x\boldsymbol{U}|^{n_2/2+h-(p+1)/2}\right)\bigg|_{x=1}|\boldsymbol{U}|^{n_1/2-h-(p+1)/2} \\ &\qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{U}\right) d\boldsymbol{U} \\ &= -\left(\frac{n_2}{2}+h-\frac{p+1}{2}\right)^{-1} K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) g'(1). \qquad \square \end{aligned}$$

LEMMA 2.3. *Suppose that the assumption* (2.3) *is satisfied. Then we have*

$$g'(1) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{pn_1}{2} + ph - k \right) \sum_{\kappa} \frac{\left(\frac{n_1+n_2}{2}\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} g_\kappa(1),$$

where

$$g_\kappa(x) = \int_{0<U<xI_p} |\boldsymbol{I}_p - \boldsymbol{U}|^{n_2/2+h-(p+1)/2} |\boldsymbol{U}|^{n_1/2-h-(p+1)/2} C_\kappa \left(\frac{1}{2} \boldsymbol{\Omega} \boldsymbol{U} \right) d\boldsymbol{U}.$$

PROOF. The transformation, $x\boldsymbol{U} \to \boldsymbol{U}$, leads to

$$\begin{aligned} g(x) &= x^{-p(n_1/2-h)} \int_{0<U<xI_p} |\boldsymbol{I}_p - \boldsymbol{U}|^{n_2/2+h-(p+1)/2} |\boldsymbol{U}|^{n_1/2-h-(p+1)/2} \\ &\qquad \times {}_1F_1 \left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \frac{1}{2x} \boldsymbol{\Omega} \boldsymbol{U} \right) d\boldsymbol{U} \\ &= \sum_{k=0}^{\infty} \frac{x^{-pn_1/2+ph-k}}{k!} \sum_{\kappa} \frac{\left(\frac{n_1+n_2}{2}\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} g_\kappa(x). \end{aligned} \tag{2.5}$$

Therefore we have

$$g'(1) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{\left(\frac{n_1+n_2}{2}\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} \left\{ \left(-\frac{pn_1}{2} + ph - k \right) g_\kappa(1) + g'_\kappa(1) \right\}.$$

We prove that

$$g'_\kappa(1) = 0. \tag{2.6}$$

We have, with some constant K_1,

$$\begin{aligned} g_\kappa(x) &= K_1 \int_{\mathcal{D}_x} \prod_{i<j} (d_i - d_j) \prod_{i=1}^{p} (1-d_i)^{n_2/2+h-(p+1)/2} \prod_{i=1}^{p} d_i^{n_1/2-h-(p+1)/2} \\ &\qquad \times \int_{\mathcal{O}(p)} C_\kappa \left(\frac{1}{2} \boldsymbol{\Omega} \boldsymbol{H} \boldsymbol{D} \boldsymbol{H}' \right) d\mu(\boldsymbol{H}) d\boldsymbol{d}, \end{aligned}$$

where

$$\boldsymbol{D} = \operatorname{diag}(d_1, \ldots, d_p), \qquad \mathcal{D}_x = \{ \boldsymbol{d} \mid 0 < d_p < \cdots < d_1 < x \},$$

and μ is the invariant probability measure on $\mathcal{O}(p)$, the group of $p \times p$ orthogonal matrices (see e.g. (22), p. 105 of Muirhead (1982)). Using a fundamental formula on the integration of a zonal polynomial,

$$\int_{\mathcal{O}(p)} C_\kappa \left(\frac{1}{2} \boldsymbol{\Omega} \boldsymbol{H} \boldsymbol{D} \boldsymbol{H}' \right) d\mu(\boldsymbol{H}) d\boldsymbol{d} = \frac{C_\kappa \left(\frac{1}{2} \boldsymbol{\Omega} \right) C_\kappa(\boldsymbol{D})}{C_\kappa(\boldsymbol{I}_p)},$$

we have

$$g_\kappa(x) = K_1 \frac{\mathcal{C}_\kappa\left(\frac{1}{2}\boldsymbol{\Omega}\right)}{\mathcal{C}_\kappa(\boldsymbol{I}_p)} \int_0^\infty \int_{d_p}^\infty \cdots \int_{d_3}^\infty \int_{d_2}^x f(d_1,\ldots,d_p) dd_1 dd_2 \cdots dd_{p-1} dd_p,$$

where

$$f(d_1,\ldots,d_p) = \prod_{i<j}(d_i - d_j) \prod_{i=1}^p (1-d_i)^{n_2/2+h-(p+1)/2} \prod_{i=1}^p d_i^{n_1/2-h-(p+1)/2} \mathcal{C}_\kappa(\boldsymbol{D}).$$

Let

$$g_\kappa^*(x) = \int_{d_2}^x f(d_1,\ldots,d_p) dd_1.$$

Then we have

$$g_\kappa'(1) = K_1 \frac{\mathcal{C}_\kappa\left(\frac{1}{2}\boldsymbol{\Omega}\right)}{\mathcal{C}_\kappa(\boldsymbol{I}_p)} \int_0^\infty \int_{d_p}^\infty \cdots \int_{d_3}^\infty (g_\kappa^*)'(1) dd_2 \cdots dd_{p-1} dd_p.$$

Note that

$$(g_\kappa^*)'(x) = f(x, d_2, \ldots, d_p)$$

and that

$$(g_\kappa^*)'(1) = f(1, d_2, \ldots, d_p) = 0,$$

since $n_2/2 + h - (p+1)/2 > 0$. This completes the proof of (2.6). □

LEMMA 2.4. *Suppose that the assumption* (2.3) *is satisfied. Then*

$$E[|\boldsymbol{F}|^{-h} \operatorname{tr} \boldsymbol{F}] = M \left\{ p(n_1 - 2h)_1F_1\left(\frac{n_1}{2} - h; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\right) + 2\sum_{k=0}^\infty \frac{1}{k!} \sum_\kappa \frac{\left(\frac{n_1}{2} - h\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} k \mathcal{C}_\kappa\left(\frac{1}{2}\boldsymbol{\Omega}\right) \right\}.$$

PROOF. From Lemmas 2.2 and 2.3, we have

$$E[|\boldsymbol{F}|^{-h} \operatorname{tr} \boldsymbol{F}] = \left(\frac{n_2}{2} + h - \frac{p+1}{2}\right)^{-1} K_0 \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right) \times \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{pn_1}{2} - ph + k\right) \sum_\kappa \frac{\left(\frac{n_1+n_2}{2}\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} g_\kappa(1). \tag{2.7}$$

Note that

$$g_\kappa(1) = \int_{0<U<I_p} |I_p - U|^{n_2/2+h-(p+1)/2} |U|^{n_1/2-h-(p+1)/2} C_\kappa\left(\frac{1}{2}\Omega U\right) dU \tag{2.8}$$
$$= \frac{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}\right)} \frac{\left(\frac{n_1}{2}-h\right)_\kappa}{\left(\frac{n_1+n_2}{2}\right)_\kappa} C_\kappa\left(\frac{1}{2}\Omega\right)$$

(see e.g. Lemma 1.5.3. of Gupta and Nagar (1999)). From (1.3), (2.7) and (2.8), we have

$$E[|F|^{-h}\operatorname{tr} F] = \left(\frac{n_2}{2} + h - \frac{p+1}{2}\right)^{-1} K_0 \operatorname{etr}\left(-\frac{1}{2}\Omega\right) \frac{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}\right)}$$
$$\times \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{pn_1}{2} - ph + k\right) \sum_\kappa \frac{\left(\frac{n_1}{2}-h\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} C_\kappa\left(\frac{1}{2}\Omega\right)$$
$$= M\left\{p(n_1-2h)\,{}_1F_1\left(\frac{n_1}{2}-h; \frac{n_1}{2}; \frac{1}{2}\Omega\right)\right.$$
$$\left. + 2\sum_{k=0}^{\infty}\frac{1}{k!}\sum_\kappa \frac{\left(\frac{n_1}{2}-h\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa} kC_\kappa\left(\frac{1}{2}\Omega\right)\right\}. \qquad \square$$

In order to evaluate $E[|F|^{-h}\operatorname{tr} F]$, the next lemma is useful.

LEMMA 2.5. *Suppose*

$$\frac{p-1}{2} \le h < \frac{n_1-p+1}{2}. \tag{2.9}$$

Then

$$E[|F|^{-h}\operatorname{tr} F] \le M\left\{(p(n_1-2h) + \operatorname{tr}\Omega)\,{}_1F_1\left(\frac{n_1}{2}-h; \frac{n_1}{2}; \frac{1}{2}\Omega\right)\right\}.$$

PROOF. First suppose $h > (p-1)/2$. Since $2 \le p \le n_2$, we have $(-n_2+p+1)/2 \le (p-1)/2$. Therefore the condition (2.3) is satisfied. Using the result of Lemma 2.4 and the relationship

$$kC_\kappa\left(\frac{1}{2}\Omega\right) = \sum_{i=1}^{p} \lambda_i \frac{\partial}{\partial\lambda_i} C_\kappa(\Lambda)\bigg|_{\Lambda=(1/2)\Omega}, \qquad \Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_p)$$

(see, e.g. (7), p. 267 of Muirhead (1982)), we have

$$E[|F|^{-h}\operatorname{tr} F] = M\left\{p(n_1-2h)\,{}_1F_1\left(\frac{n_1}{2}-h; \frac{n_1}{2}; \frac{1}{2}\Omega\right)\right. \tag{2.10}$$

$$+2\sum_{i=1}^{p}\lambda_i\frac{\partial}{\partial\lambda_i}{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\mathbf{\Lambda}\right)\Bigg|_{\mathbf{\Lambda}=(1/2)\mathbf{\Omega}}\Bigg\}.$$

Now we evaluate

$$\sum_{i=1}^{p}\lambda_i\frac{\partial}{\partial\lambda_i}{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\mathbf{\Lambda}\right)\Bigg|_{\mathbf{\Lambda}=(1/2)\mathbf{\Omega}}.$$

Since $h > (p-1)/2$, we have the following integral representation for the hypergeometric function ${}_1F_1$ (see, e.g. Gupta and Nagar (1999), Corollary 1.6.3.1).

$$\begin{aligned}{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\mathbf{\Lambda}\right) &= \frac{\Gamma_p\left(\frac{n_1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p(h)}\\ &\quad\times\int_{0<X<I_p}|\boldsymbol{X}|^{(n_1-p-1)/2-h}|\boldsymbol{I_p}-\boldsymbol{X}|^{h-(1/2)(p+1)}\operatorname{etr}(\mathbf{\Lambda}\boldsymbol{X})d\boldsymbol{X}.\end{aligned}$$

Using this expression, we have

$$\begin{aligned}(2.11)\qquad &\sum_{i=1}^{p}\lambda_i\frac{\partial}{\partial\lambda_i}{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\mathbf{\Lambda}\right)\Bigg|_{\mathbf{\Lambda}=(1/2)\mathbf{\Omega}}\\ &= \frac{\Gamma_p\left(\frac{n_1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p(h)}\\ &\quad\times\int_{0<X<I_p}|\boldsymbol{X}|^{(n_1-p-1)/2-h}|\boldsymbol{I_p}-\boldsymbol{X}|^{h-(1/2)(p+1)}\\ &\qquad\qquad\times\operatorname{tr}(\mathbf{\Lambda}\boldsymbol{X})\operatorname{etr}(\mathbf{\Lambda}\boldsymbol{X})d\boldsymbol{X}\Bigg|_{\mathbf{\Lambda}=(1/2)\mathbf{\Omega}}\\ &= \frac{1}{2}\frac{\Gamma_p\left(\frac{n_1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p(h)}\\ &\quad\times\int_{0<X<I_p}|\boldsymbol{X}|^{(n_1-p-1)/2-h}|\boldsymbol{I_p}-\boldsymbol{X}|^{h-(1/2)(p+1)}\\ &\qquad\qquad\times\operatorname{tr}(\mathbf{\Omega}\boldsymbol{X})\operatorname{etr}\left(\frac{1}{2}\mathbf{\Omega}\boldsymbol{X}\right)d\boldsymbol{X}.\end{aligned}$$

Since $\boldsymbol{X} < \boldsymbol{I_p}$, we have $(\boldsymbol{X})_{ii} \le 1$. Therefore

$$(2.12)\qquad \operatorname{tr}\mathbf{\Omega}\boldsymbol{X} = \sum_{i=1}^{p}\omega_i(\boldsymbol{X})_{ii} \le \sum_{i=1}^{p}\omega_i = \operatorname{tr}\mathbf{\Omega}.$$

From (2.11) and (2.12), we have

$$(2.13)\qquad \sum_{i=1}^{p}\lambda_i\frac{\partial}{\partial\lambda_i}{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\mathbf{\Lambda}\right)\Bigg|_{\mathbf{\Lambda}=(1/2)\mathbf{\Omega}}$$

$$\leq \frac{1}{2} \frac{\Gamma_p\left(\frac{n_1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p(h)} \operatorname{tr} \boldsymbol{\Omega}$$
$$\times \int_{0<X<I_p} |\boldsymbol{X}|^{(n_1-p-1)/2-h} |\boldsymbol{I}_p - \boldsymbol{X}|^{h-(1/2)(p+1)} \operatorname{etr}\left(\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{X}\right) d\boldsymbol{X}$$
$$= \frac{1}{2} \operatorname{tr} \boldsymbol{\Omega}_1 F_1\left(\frac{n_1}{2}-h; \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega}\right).$$

If we substitute (2.13) in (2.10), we get the inequality of Lemma 2.5. If we consider the limit values on the both sides as $h \to (p-1)/2$, we notice the inequality also holds true when $h = (p-1)/2$. □

Now we prove the main theorem. Consider the estimator,

$$\alpha\widehat{\boldsymbol{\omega}}^{(*)} = (\alpha\widehat{\omega}_1^{(*)}, \dots, \alpha\widehat{\omega}_p^{(*)}),$$

where

$$0 < \alpha < 1 + \frac{n_1 - p + 1}{2(n_2 - p - 1)}. \tag{2.14}$$

The estimator $\alpha\widehat{\boldsymbol{\omega}}^{(*)}$ has the following dominance property over $\alpha\widehat{\boldsymbol{\omega}}^{(u)}$.

THEOREM 2.1. *Suppose $n_1 > 3p-3$, $n_2 > p+1$. Then $\alpha\widehat{\boldsymbol{\omega}}^{(*)}$ dominates $\alpha\widehat{\boldsymbol{\omega}}^{(u)}$, if*

$$\max\left(\frac{p-1}{2}, \frac{(\alpha-1)(n_2-p-1)}{2}\right) < h < \frac{n_1-p+1}{4}, \tag{2.15}$$

and

$$0 < b \leq 4h \frac{n_1+n_2-p-1}{n_2+2h-p-1} \frac{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}. \tag{2.16}$$

PROOF. First we note that the condition on α in (2.14) and $n_1 > 3p-3$, $n_2 > p+1$ guarantee the existence of h that satisfies (2.15).

Risk difference between $\alpha\widehat{\boldsymbol{\omega}}^{(*)}$ and $\alpha\widehat{\boldsymbol{\omega}}^{(u)}$ w.r.t. the quadratic loss function L_3, (1.19) is given by

$$\begin{aligned}
&E[L_3(\alpha\widehat{\boldsymbol{\omega}}^{(*)}, \boldsymbol{\omega}) - L_3(\alpha\widehat{\boldsymbol{\omega}}^{(u)}, \boldsymbol{\omega})] \\
&= E\left[\sum_{i=1}^{p} \{(\alpha\widehat{\omega}_i^{(*)} - \omega_i)^2 - (\alpha\widehat{\omega}_i^{(u)} - \omega_i)^2\}\right] \\
&= E\left[\sum_{i=1}^{p} \{(\alpha\widehat{\omega}_i^{(*)} - \alpha\widehat{\omega}_i^{(u)} + \alpha\widehat{\omega}_i^{(u)} - \omega_i)^2 - (\alpha\widehat{\omega}_i^{(u)} - \omega_i)^2\}\right] \\
&= E\left[\sum_{i=1}^{p} (\alpha\widehat{\omega}_i^{(*)} - \alpha\widehat{\omega}_i^{(u)})^2 + 2\sum_{i=1}^{p} (\alpha\widehat{\omega}_i^{(*)} - \alpha\widehat{\omega}_i^{(u)})(\alpha\widehat{\omega}_i^{(u)} - \omega_i)\right] \\
&= \alpha^2 b^2 p E[|\boldsymbol{F}|^{-2h}] + 2\alpha^2 b(n_2-p-1) E[|\boldsymbol{F}|^{-h} \operatorname{tr} \boldsymbol{F}] \\
&\quad - 2\alpha^2 n_1 b p E[|\boldsymbol{F}|^{-h}] - 2\alpha b \operatorname{tr} \boldsymbol{\Omega} E[|\boldsymbol{F}|^{-h}].
\end{aligned} \tag{2.17}$$

First we evaluate $E[|\boldsymbol{F}|^{-2h}]$. Since (2.15) guarantees the condition

$$\frac{-n_2+p-1}{2} < 2h < \frac{n_1-p+1}{2},$$

we can use Lemma 2.1 and we have

$$\begin{aligned}(2.18)\qquad E[|\boldsymbol{F}|^{-2h}] &= M(n_2+2h-p-1)\\ &\quad\times \frac{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}\,{}_1F_1\left(\frac{n_1}{2}-2h;\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\right)\\ &\le M(n_2+2h-p-1)\\ &\quad\times \frac{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}\,{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\right).\end{aligned}$$

The condition (2.15) is sufficient for (2.2) and (2.9). Therefore we have, using (2.18), Lemma 2.1 and Lemma 2.5,

$$\begin{aligned}&E[L_3(\alpha\widehat{\boldsymbol{\omega}}^{(*)},\boldsymbol{\omega})-L_3(\alpha\widehat{\boldsymbol{\omega}}^{(u)},\boldsymbol{\omega})]\\ &\le \alpha b M{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\right)\\ &\quad\times\left\{\alpha bp(n_2+2h-p-1)\frac{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}\right.\\ &\qquad+2\alpha(n_2-p-1)(p(n_1-2h)+\operatorname{tr}\boldsymbol{\Omega})\\ &\qquad\left.-2\alpha n_1p(n_2+2h-p-1)-2(n_2+2h-p-1)\operatorname{tr}\boldsymbol{\Omega}\right\}\\ &=\alpha b M{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\right)\\ &\quad\times\left\{\alpha bp(n_2+2h-p-1)\frac{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}\right.\\ &\qquad-4\alpha ph(n_1+n_2-p-1)\\ &\qquad\left.+(2(\alpha-1)(n_2-p-1)-4h)\operatorname{tr}\boldsymbol{\Omega}\right\}.\end{aligned}$$

Because of (2.15), the last term in the brace is negative. Since

$$\alpha b M{}_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\right)>0,$$

we have

$$(2.19)\qquad E[L_3(\alpha\widehat{\boldsymbol{\omega}}^{(*)},\boldsymbol{\omega})-L_3(\alpha\widehat{\boldsymbol{\omega}}^{(u)},\boldsymbol{\omega})]$$

$$< \alpha^2 bpM_1F_1\left(\frac{n_1}{2}-h;\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\right)$$
$$\times\left\{b(n_2+2h-p-1)\frac{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}\right.$$
$$\left.-4h(n_1+n_2-p-1)\right\}.$$

The value in the brace on the right side of (2.19) is negative because of (2.16). □

The right side of (2.19) is minimized when

$$b=2h\frac{n_1+n_2-p-1}{n_2+2h-p-1}\frac{\Gamma_p\left(\frac{n_1}{2}-h\right)\Gamma_p\left(\frac{n_2}{2}+h\right)}{\Gamma_p\left(\frac{n_1}{2}-2h\right)\Gamma_p\left(\frac{n_2}{2}+2h\right)}(\equiv b(h)).$$

It is not obvious which value of h minimizes the right side of (2.19). Therefore we propose here the following two estimators. When $\alpha \le 1$ (notice $\alpha_1, \alpha_2 < 1$), we can substitute (2.15) with

$$\frac{p-1}{2}\le h<\frac{n_1-p+1}{4}.$$

Using $h=(n_1-p)/4$, which is close to the upper bound, and $b=b((n_1-p)/4)$, we have

$$\widehat{\boldsymbol{\omega}}^{(*1)}=(\widehat{\omega}_1^{(*1)},\dots,\widehat{\omega}_p^{(*1)}).$$

On the other hand, if we use the lower bound value $h=(p-1)/2$ and $b=b((p-1)/2)$, we have

$$\widehat{\boldsymbol{\omega}}^{(*2)}=(\widehat{\omega}_1^{(*2)},\dots,\widehat{\omega}_p^{(*2)}).$$

So far we considered the estimation of ω_i, $i=1,\dots,p$. In multivariate analysis, the estimation of $\operatorname{tr}\boldsymbol{\Omega}$ is sometimes needed. The estimator $\widehat{\boldsymbol{\omega}}^{(u)}$ naturally leads to an estimator of $\operatorname{tr}\boldsymbol{\Omega}$,

$$\widehat{\operatorname{tr}\boldsymbol{\Omega}}^{(u)}=\sum_{i=1}^p\widehat{\omega}_i^{(u)}=(n_2-p-1)\operatorname{tr}\boldsymbol{F}-n_1p,$$

while the new estimator $\widehat{\boldsymbol{\omega}}^{(*)}$ leads to another estimator

$$\widehat{\operatorname{tr}\boldsymbol{\Omega}}^{(*)}=\sum_{i=1}^p\widehat{\omega}_i^{(*)}=\widehat{\operatorname{tr}\boldsymbol{\Omega}}^{(u)}+bp|\boldsymbol{F}|^{-h}.$$

On comparing the risks of $\alpha\widehat{\operatorname{tr}\boldsymbol{\Omega}}^{(u)}$ and $\alpha\widehat{\operatorname{tr}\boldsymbol{\Omega}}^{(*)}$ with α in (2.14) w.r.t. the quadratic loss,

$$(\widehat{\operatorname{tr}\boldsymbol{\Omega}}-\operatorname{tr}\boldsymbol{\Omega})^2, \tag{2.20}$$

we have the following result.

THEOREM 2.2. *Suppose all the conditions in Theorem* 2.1 *are satisfied. Then* $\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(*)}$ *dominates* $\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(u)}$.

PROOF. The risk difference between $\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(u)}$ and $\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(*)}$ with respect to the quadratic loss (2.20) equals to

$$\begin{aligned}
&E[(\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(*)} - \mathrm{tr}\,\boldsymbol{\Omega})^2] - E[(\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(u)} - \mathrm{tr}\,\boldsymbol{\Omega})^2] \\
&\quad = E[(\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(*)} - \widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(u)})^2] \\
&\qquad + 2E[(\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(*)} - \widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(u)})(\widehat{\alpha \mathrm{tr}\,\boldsymbol{\Omega}}^{(u)} - \mathrm{tr}\,\boldsymbol{\Omega})] \\
&\quad = \alpha^2 b^2 p^2 E[|\boldsymbol{F}|^{-2h}] + 2\alpha^2 bp(n_2 - p - 1)E[|\boldsymbol{F}|^{-h}\,\mathrm{tr}\,\boldsymbol{F}] \\
&\qquad - 2\alpha bp(\alpha n_1 p + \mathrm{tr}\,\boldsymbol{\Omega})E[|\boldsymbol{F}|^{-h}] \\
&\quad = pE[L_3(\alpha\widehat{\boldsymbol{\omega}}^{(*)}, \boldsymbol{\omega}) - L_3(\alpha\widehat{\boldsymbol{\omega}}^{(u)}, \boldsymbol{\omega})] \\
&\quad < 0. \qquad \square
\end{aligned}$$

Apart from many practical cases in which we only obtain the eigenvalues of $\boldsymbol{F}$, we may consider purely theoretically formulated estimation problem of $\boldsymbol{\Omega}$ when $\boldsymbol{F}$ is distributed as in (1.1) and all parts of $\boldsymbol{F}$ are observable. We can use our new estimator for this problem. Following the idea in Leung and Muirhead (1987) conversely, we can make a new orthogonally invariant estimator of $\boldsymbol{\Omega}$ (say $\widehat{\boldsymbol{\Omega}}^{(*)}$) from our eigenvalues estimator $\widehat{\boldsymbol{\omega}}^{(*)}$ with

$$\psi_i(\boldsymbol{L}) = \widehat{\omega}_i^{(*)}, \qquad 1 \le i \le p$$

as in (1.5). Consequently our new estimator is given by

$$\widehat{\boldsymbol{\Omega}}^{(*)} = \widehat{\boldsymbol{\Omega}}^{(u)} + b|\boldsymbol{F}|^{-h}\boldsymbol{I}_p.$$

On the risk of this new estimator with respect to the loss function $L_1(\widehat{\boldsymbol{\Omega}}, \boldsymbol{\Omega})$, (1.7), we have the following result.

THEOREM 2.3. *Suppose all the conditions in Theorem* 2.1 *are satisfied, and* α *is given by* (2.14). *Then* $\alpha\widehat{\boldsymbol{\Omega}}^{(*)}$ *dominates* $\alpha\widehat{\boldsymbol{\Omega}}^{(u)}$ *with respect to the* $L_1(\widehat{\boldsymbol{\Omega}}, \boldsymbol{\Omega})$.

PROOF. We have

$$\begin{aligned}
&E[\mathrm{tr}(\alpha\widehat{\boldsymbol{\Omega}}^{(*)} - \boldsymbol{\Omega})^2] - E[\mathrm{tr}(\alpha\widehat{\boldsymbol{\Omega}}^{(u)} - \boldsymbol{\Omega})^2] \\
&\quad = E[\mathrm{tr}\{(\alpha\widehat{\boldsymbol{\Omega}}^{(*)} - \alpha\widehat{\boldsymbol{\Omega}}^{(u)}) + (\alpha\widehat{\boldsymbol{\Omega}}^{(u)} - \boldsymbol{\Omega})\}^2] - E[\mathrm{tr}(\alpha\widehat{\boldsymbol{\Omega}}^{(u)} - \boldsymbol{\Omega})^2] \\
&\quad = E[\mathrm{tr}(\alpha b|\boldsymbol{F}|^{-h}\boldsymbol{I}_p)^2] \\
&\qquad + 2E[\alpha b|\boldsymbol{F}|^{-h}\,\mathrm{tr}(\alpha(n_2 - p - 1)\boldsymbol{F} - \alpha n_1 \boldsymbol{I}_p - \boldsymbol{\Omega})] \\
&\quad = \alpha^2 b^2 pE[|\boldsymbol{F}|^{-2h}] + 2\alpha^2 b(n_2 - p - 1)E[|\boldsymbol{F}|^{-h}\,\mathrm{tr}\,\boldsymbol{F}] \\
&\qquad - 2\alpha b(\alpha n_1 p + \mathrm{tr}\,\boldsymbol{\Omega})E[|\boldsymbol{F}|^{-h}] \\
&\quad = E[L_3(\alpha\widehat{\boldsymbol{\omega}}^{(*)}, \boldsymbol{\omega}) - L_3(\alpha\widehat{\boldsymbol{\omega}}^{(u)}, \boldsymbol{\omega})] \\
&\quad < 0. \qquad \square
\end{aligned}$$

3. Second estimator

In this section we consider empirical Bayes estimators. We postulate that the conditional distribution of $\boldsymbol{F}$ given $\boldsymbol{\Omega}$ is $F_p(n_1, n_2, \boldsymbol{I}_p, \boldsymbol{\Omega})$. We use $W_p(n_1, \boldsymbol{C})$ with some p.d. matrix $\boldsymbol{C}$ as the prior distribution of $\boldsymbol{\Omega}$. That is

$$\boldsymbol{F} \mid \boldsymbol{\Omega} \sim F(n_1, n_2, \boldsymbol{I}_p, \boldsymbol{\Omega}), \qquad \boldsymbol{\Omega} \sim W_p(n_1, \boldsymbol{C}). \tag{3.1}$$

Then from (1.2) the marginal density of $\boldsymbol{F}$ is obtained as

$$\begin{aligned}
f(\boldsymbol{F}) &= K_1|\boldsymbol{C}|^{-n_1/2}|\boldsymbol{F}|^{(n_1-p-1)/2}|\boldsymbol{I}_p+\boldsymbol{F}|^{-(n_1+n_2)/2}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{\left(\frac{n_1+n_2}{2}\right)_{\kappa}}{\left(\frac{n_1}{2}\right)_{\kappa}} \\
&\quad \times \int_{\boldsymbol{\Omega}>0}\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p+\boldsymbol{C}^{-1})\right)|\boldsymbol{\Omega}|^{(n_1-p-1)/2}C_{\kappa}\left(\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p+\boldsymbol{F})^{-1}\right)d\boldsymbol{\Omega} \\
&\quad (K_1 = K_0\{\Gamma_p(n_1/2)\}^{-1}2^{-n_1p/2}) \\
&= K_1 2^{n_1p/2}\Gamma_p\left(\frac{n_1}{2}\right)|\boldsymbol{C}|^{-n_1/2}|\boldsymbol{I}_p+\boldsymbol{C}|^{-n_1/2}|\boldsymbol{F}|^{(n_1-p-1)/2}|\boldsymbol{I}_p+\boldsymbol{F}|^{-(n_1+n_2)/2} \\
&\quad \times {}_1F_0\left(\frac{n_1+n_2}{2};\boldsymbol{F}(\boldsymbol{I}_p+\boldsymbol{F})^{-1}(\boldsymbol{I}_p+\boldsymbol{C}^{-1})^{-1}\right) \\
&\quad (\text{see e.g. Lemma 1.5.2. of Gupta and Nagar (1999)}) \\
&= K_1 2^{n_1p/2}\Gamma_p\left(\frac{n_1}{2}\right)|\boldsymbol{C}|^{-n_1/2}|\boldsymbol{I}_p+\boldsymbol{C}|^{-n_1/2}|\boldsymbol{F}|^{(n_1-p-1)/2}|\boldsymbol{I}_p+\boldsymbol{F}|^{-(n_1+n_2)/2} \\
&\quad \times |\boldsymbol{I}_p-\boldsymbol{F}(\boldsymbol{I}_p+\boldsymbol{F})^{-1}(\boldsymbol{I}_p+\boldsymbol{C}^{-1})^{-1}|^{-(n_1+n_2)/2} \\
&\quad (\text{see e.g. Corollary 1.6.2.1 of Gupta and Nagar (1999)}) \\
&= K_0|\boldsymbol{I}_p+\boldsymbol{C}|^{-n_1/2}|\boldsymbol{F}|^{(n_1-p-1)/2}|\boldsymbol{I}_p+\boldsymbol{F}(\boldsymbol{I}_p+\boldsymbol{C})^{-1}|^{-(n_1+n_2)/2}.
\end{aligned}$$

Therefore the marginal distribution is (central) F distribution, i.e.

$$\boldsymbol{F} \sim F_p(n_1, n_2, \boldsymbol{I}_p+\boldsymbol{C}). \tag{3.2}$$

We also notice that the posterior density function of $\boldsymbol{\Omega}$ is given by

$$\begin{aligned}
f(\boldsymbol{\Omega}\mid\boldsymbol{F}) &= K_2|\boldsymbol{I}_p+\boldsymbol{C}^{-1}|^{n_1/2}|\boldsymbol{I}_p+\boldsymbol{F}\boldsymbol{C}^{-1}(\boldsymbol{I}_p+\boldsymbol{C}^{-1})^{-1}|^{(n_1+n_2)/2} \\
&\quad \times |\boldsymbol{I}_p+\boldsymbol{F}|^{-(n_1+n_2)/2} \\
&\quad \times \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{I}_p+\boldsymbol{C}^{-1})\right)|\boldsymbol{\Omega}|^{(n_1-p-1)/2} \\
&\quad \times {}_1F_1\left(\frac{n_1+n_2}{2},\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p+\boldsymbol{F})^{-1}\right) \\
&\quad \left(K_2^{-1} = 2^{n_1p/2}\Gamma_p\left(\frac{n_1}{2}\right)\right) \\
&= K_2|\boldsymbol{A}|^{-n_2/2}|\boldsymbol{A}-(\boldsymbol{I}_p+\boldsymbol{F})^{-1}\boldsymbol{F}|^{(n_1+n_2)/2} \\
&\quad \times \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{A}\right)|\boldsymbol{\Omega}|^{(n_1-p-1)/2} \\
&\quad \times {}_1F_1\left(\frac{n_1+n_2}{2},\frac{n_1}{2};\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{I}_p+\boldsymbol{F})^{-1}\right) \\
&\quad (\boldsymbol{A} = \boldsymbol{I}_p+\boldsymbol{C}^{-1}).
\end{aligned} \tag{3.3}$$

We calculate the posterior mean of $\boldsymbol{\Omega}$. Since

$$\int_{\Omega>0} f(\boldsymbol{\Omega} \mid \boldsymbol{F})d\boldsymbol{\Omega} = 1,$$

we have for $1 \leq \forall i, \forall j \leq p$

$$\frac{\partial}{\partial a_{ij}} \int_{\Omega>0} f(\boldsymbol{\Omega} \mid \boldsymbol{F})d\boldsymbol{\Omega} = 0, \qquad a_{ij} = (A)_{ij}.$$

This means

$$\begin{aligned}
& K_2 \frac{\partial}{\partial a_{ij}} (|\boldsymbol{A}|^{-n_2/2} |\boldsymbol{A} - (\boldsymbol{I_p} + \boldsymbol{F})^{-1}\boldsymbol{F}|^{(n_1+n_2)/2}) \\
& \quad \times \int_{\Omega>0} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega A}\right) |\boldsymbol{\Omega}|^{(n_1-p-1)/2} \\
& \qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}, \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega F}(\boldsymbol{I_p}+\boldsymbol{F})^{-1}\right) d\boldsymbol{\Omega} \\
& = K_2 |\boldsymbol{A}|^{-n_2/2} |\boldsymbol{A} - (\boldsymbol{I_p} + \boldsymbol{F})^{-1}\boldsymbol{F}|^{(n_1+n_2)/2} \\
& \quad \times (1+\delta_{ij})^{-1} \int_{\Omega>0} \omega_{ij} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega A}\right) |\boldsymbol{\Omega}|^{(n_1-p-1)/2} \\
& \qquad \times {}_1F_1\left(\frac{n_1+n_2}{2}, \frac{n_1}{2}; \frac{1}{2}\boldsymbol{\Omega F}(\boldsymbol{I_p}+\boldsymbol{F})^{-1}\right) d\boldsymbol{\Omega} \\
& \quad (\delta_{ij} : \text{ Kronecker's delta}) \\
& = (1+\delta_{ij})^{-1} E[\omega_{ij} \mid \boldsymbol{F}].
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{\partial}{\partial a_{ij}} \{|\boldsymbol{A}|^{-n_2/2} |\boldsymbol{A} - (\boldsymbol{I_p} + \boldsymbol{F})^{-1}\boldsymbol{F}|^{(n_1+n_2)/2}\} \\
& \quad = \left\{-\frac{n_2}{1+\delta_{ij}}(\boldsymbol{A})^{ij} + \frac{n_1+n_2}{1+\delta_{ij}}(\boldsymbol{A} - (\boldsymbol{I_p}+\boldsymbol{F})^{-1}\boldsymbol{F})^{ij}\right\} \\
& \qquad \times \{|\boldsymbol{A}|^{-n_2/2} |\boldsymbol{A} - (\boldsymbol{I_p} + \boldsymbol{F})^{-1}\boldsymbol{F}|^{(n_1+n_2)/2}\}
\end{aligned}$$

with $(\boldsymbol{X})^{ij} = ((\boldsymbol{X})^{-1})_{ij}$, we have

$$E[\omega_{ij} \mid \boldsymbol{F}] = -n_2(\boldsymbol{A})^{ij} + (n_1+n_2)(\boldsymbol{A} - (\boldsymbol{I_p}+\boldsymbol{F})^{-1}\boldsymbol{F})^{ij}.$$

Consequently

$$\begin{aligned}
E[\boldsymbol{\Omega} \mid \boldsymbol{F}] &= -n_2\boldsymbol{A}^{-1} + (n_1+n_2)(\boldsymbol{A} - (\boldsymbol{I_p}+\boldsymbol{F})^{-1}\boldsymbol{F})^{-1} \\
&= -n_2(\boldsymbol{I_p} + \boldsymbol{C}^{-1})^{-1} \\
&\quad + (n_1+n_2)(\boldsymbol{I_p} + \boldsymbol{C}^{-1} - (\boldsymbol{I_p}+\boldsymbol{F})^{-1}\boldsymbol{F})^{-1}.
\end{aligned} \tag{3.4}$$

In order to use this posterior mean as an estimator of $\boldsymbol{\Omega}$, we need to estimate $\boldsymbol{C}$. We can estimate $\boldsymbol{I_p}+\boldsymbol{C}$ based on $\boldsymbol{F}$ through the relationship (3.2) and use the estimator $\widehat{\boldsymbol{I_p}+\boldsymbol{C}}$ for the estimation of $\boldsymbol{C}$ itself as

$$\widehat{\boldsymbol{C}} = \widehat{\boldsymbol{I_p}+\boldsymbol{C}} - \boldsymbol{I_p}. \tag{3.5}$$

The estimation of $\boldsymbol{I}_p + \boldsymbol{C}$ from $\boldsymbol{F}$ is the problem that was considered in many papers. Gupta (1990) and Konno (1992a) are useful as a review. Leung and Chan (1998) also deal with this problem. Though we have many options in choosing $\widehat{\boldsymbol{I}_p + \boldsymbol{C}}$, for the purpose of the estimation of $\boldsymbol{\omega}$, an orthogonally invariant estimator is preferable since it gives rise to a simple estimator $\widehat{\boldsymbol{\omega}}$. Let the estimator of $\boldsymbol{I}_p + \boldsymbol{C}$ be given by

$$(3.6)\qquad \widehat{\boldsymbol{I}_p + \boldsymbol{C}} = \boldsymbol{H}\boldsymbol{\Psi}(\boldsymbol{L})\boldsymbol{H}', \qquad \boldsymbol{\Psi}(\boldsymbol{L}) = \mathrm{diag}(\psi_1(\boldsymbol{L}), \ldots, \psi_p(\boldsymbol{L}))$$

with the spectral decomposition of $\boldsymbol{F}$ in (1.6). This leads to the estimator

$$(3.7)\qquad \widehat{\boldsymbol{C}} = \boldsymbol{H}(\boldsymbol{\Psi}(\boldsymbol{L}) - \boldsymbol{I}_p)\boldsymbol{H}'.$$

If we substitute this estimator with $\boldsymbol{C}$ in the right side of (3.4), we have the estimator

$$\begin{aligned}\widehat{\boldsymbol{\Omega}} &= -n_2(\boldsymbol{H}(\boldsymbol{I}_p + (\boldsymbol{\Psi} - \boldsymbol{I}_p)^{-1})\boldsymbol{H}')^{-1} \\ &\quad + (n_1 + n_2)(\boldsymbol{H}(\boldsymbol{I}_p + (\boldsymbol{\Psi} - \boldsymbol{I}_p)^{-1})\boldsymbol{H}' - \boldsymbol{H}(\boldsymbol{I}_p + \boldsymbol{L})^{-1}\boldsymbol{L}\boldsymbol{H}')^{-1} \\ &= \boldsymbol{H}\boldsymbol{D}\boldsymbol{H}',\end{aligned}$$

where

$$\boldsymbol{D} = -n_2(\boldsymbol{I}_p + (\boldsymbol{\Psi} - \boldsymbol{I}_p)^{-1})^{-1} + (n_1 + n_2)(\boldsymbol{I}_p + (\boldsymbol{\Psi} - \boldsymbol{I}_p)^{-1} - (\boldsymbol{I}_p + \boldsymbol{L})^{-1}\boldsymbol{L})^{-1}.$$

Especially when

$$(3.8)\qquad \boldsymbol{\Psi}(\boldsymbol{L}) = \boldsymbol{L}\boldsymbol{\Delta}$$

with

$$(3.9)\qquad \boldsymbol{\Delta} = \mathrm{diag}(\delta_1, \ldots, \delta_p),$$

it leads to

$$(3.10)\qquad \boldsymbol{D} = -\boldsymbol{B}_0 + \boldsymbol{B}_1\boldsymbol{L}^{-1} + \boldsymbol{B}_2\boldsymbol{L},$$

where

$$(3.11)\qquad \begin{cases} \boldsymbol{B}_0 = n_2\boldsymbol{I}_p + (n_1 + n_2)(\boldsymbol{I}_p + \boldsymbol{\Delta})^{-1}(\boldsymbol{I}_p - \boldsymbol{\Delta}) \\ \boldsymbol{B}_1 = n_2\boldsymbol{\Delta}^{-1} - (n_1 + n_2)(\boldsymbol{I}_p + \boldsymbol{\Delta})^{-1} \\ \boldsymbol{B}_2 = (n_1 + n_2)(\boldsymbol{I}_p + \boldsymbol{\Delta})^{-1}\boldsymbol{\Delta}. \end{cases}$$

We consider two choices for $\widehat{\boldsymbol{I}_p + \boldsymbol{C}}$. First we consider the case when the M.L.E. estimator,

$$(3.12)\qquad \widehat{\boldsymbol{I}_p + \boldsymbol{C}} = \frac{n_2}{n_1}\boldsymbol{F}$$

is used for the estimation of $\boldsymbol{I}_p + \boldsymbol{C}$. For this estimator, (3.9) was given by

$$\boldsymbol{\Delta} = \frac{n_2}{n_1}\boldsymbol{I}_p.$$

This gives rise to

$$D^{(m)} = n_2 L - n_1 I_p \tag{3.13}$$

for D in (3.10) and the corresponding estimator

$$\widehat{\Omega}^{(m)} = n_2 F - n_1 I_p. \tag{3.14}$$

However there are some orthogonally invariant estimators of $I_p + C$ which are superior to the M.L.E., (3.12) w.r.t. some loss functions. We use the estimator proposed in Gupta and Krishnamoorthy (1990) as the second choice. It is given by (3.8) and (3.9) with

$$\delta_i = \frac{(n_2 - p - 1 + i)(n_2 - p - 2 + i)}{(n_1 + 1 - i)(n_2 - p - 1 + i) + (p - i)(n_1 + n_2 - p - 1)} \tag{3.15}$$

for $1 \leq i \leq p$. This estimator has a simple form and Monte Carlo simulation in the paper shows that this estimator performs better w.r.t. an entropy type loss than some other reasonable estimators including the unbiased estimator of $I_p + C$. The unbiased estimator is known to dominate the M.L.E. with an entropy loss (Stein's loss). The conjecture that the estimator of Gupta and Krishnamoorthy (1990) is minimax was proved in the case $p = 2$ in Konno (1992*b*). Let

$$\widehat{\Omega}^{(g)} = H D^{(g)} H' \tag{3.16}$$

denote another new estimator, where $D^{(g)}$ is given by (3.10) and (3.11) with δ's in (3.15).

Now we have two new estimators, $\widehat{\Omega}^{(m)}$ and $\widehat{\Omega}^{(g)}$, but the next theorem indicates there is still room for improvement.

THEOREM 3.1. *Suppose $n_1 \geq p$, $n_2 > p + 3$. If*

$$\max\left(0, 1 - \frac{2p+6}{n_2}\right) \leq \alpha < 1,$$

then $\alpha\widehat{\Omega}^{(m)}$ dominates $\widehat{\Omega}^{(m)}$ w.r.t. $L_1(\widehat{\Omega}, \Omega)$.

PROOF. The proof is similar to that of Theorem 3.1. in Leung and Muirhead (1987). We only sketch the proof. Using Lemma 3.3 in Leung and Muirhead (1987), we have

$$\begin{aligned} &E[\mathrm{tr}(\widehat{\Omega}^{(m)} - \Omega)^2] - E[\mathrm{tr}(\alpha\widehat{\Omega}^{(m)} - \Omega)^2] \\ &\quad = x_1(\mathrm{tr}\,\Omega)^2 + x_2 \,\mathrm{tr}\,\Omega^2 + x_3 \,\mathrm{tr}\,\Omega + x_4 \\ &\quad > (x_1 + x_2)\,\mathrm{tr}\,\Omega^2 + x_3\,\mathrm{tr}\,\Omega + x_4, \end{aligned}$$

where

$$\begin{aligned} x_1 &= (1-\alpha^2) n_2^2 \beta_0, \\ x_2 &= (1-\alpha^2) n_2^2 \beta_0 \beta_1 - (1-\alpha) 2 n_2 \beta_1^{-1}, \\ x_3 &= (1-\alpha^2) n_2^2 \beta_0 \beta_2 - (1-\alpha^2) 2 n_1 n_2 \beta_1^{-1} - (1-\alpha) 2 n_1 n_2 \beta_1^{-1} + (1-\alpha) 2 n_1, \\ x_4 &= (1-\alpha^2) n_2^2 \beta_0 \beta_3 + (1-\alpha^2) n_1^2 p - (1-\alpha^2) 2 n_1^2 n_2 p \beta_1^{-1} \end{aligned}$$

with β_j, $j=0,\dots,3$ in (3.7) of Leung and Muirhead (1987). Simple calculation leads to

$$\begin{aligned}
x_3 &= (1-\alpha)\beta_1^{-1}(\alpha a+b),\\
x_4 &= (1-\alpha^2)\beta_1^{-1}c,\\
x_1+x_2 &= (1-\alpha)\beta_1^{-1}d,
\end{aligned}$$

where

$$\begin{aligned}
a &= n_2^2\beta_0\beta_1\beta_2 - 2n_1n_2,\\
b &= n_2^2\beta_0\beta_1\beta_2 - 4n_1n_2 + 2n_1\beta_1,\\
c &= n_2^2\beta_0\beta_1\beta_3 + n_1^2p\beta_1 - 2n_1^2n_2p,\\
d &= (1+\alpha)n_2^2\beta_0\beta_1 + (1+\alpha)n_2^2\beta_0\beta_1^2 - 2n_2.
\end{aligned}$$

From straightforward but long calculations, we notice, using $n_1 \geq p$ and $n_2 > p+3$, that $a,b,c>0$ and

$$d = \frac{n_2^2}{n_2-p-3}\left(\alpha - \left(1-\frac{2p+6}{n_2}\right)\right).$$

Apparently if $1-(2p+6)/n_2 \leq \alpha$, then $d \geq 0$. □

As for the estimator $\widehat{\boldsymbol{\Omega}}^{(g)}$, the improvement by the multiplication with α can not be proved as easily as the above theorem. However from an analogy between $\widehat{\boldsymbol{\Omega}}^{(m)}$ and $\widehat{\boldsymbol{\Omega}}^{(g)}$ as empirical Bayes estimators, we conjecture that $\alpha\widehat{\boldsymbol{\Omega}}^{(g)}$ dominates the original estimator $\widehat{\boldsymbol{\Omega}}^{(g)}$. It is also natural to conjecture that $\alpha\widehat{\boldsymbol{\Omega}}^{(g)}$ dominates $\alpha\widehat{\boldsymbol{\Omega}}^{(m)}$ since the former uses better estimator of $\boldsymbol{I}_p+\boldsymbol{C}$. (The Monte Carlo simulation, which was carried out under the same condition as the one in Section 4, indicates these conjectures are right. We omit the details.)

Now we derive the estimator $\widehat{\boldsymbol{\omega}}^{(g)} = (\omega_1^{(g)},\dots,\omega_p^{(g)})$ of $\boldsymbol{\omega}$, using the eigenvalues of $\widehat{\boldsymbol{\Omega}}^{(g)}$, as

$$\omega_i^{(g)} = -b_{0i} + b_{1i}l_i^{-1} + b_{2i}l_i, \qquad i=1,\dots,p, \tag{3.17}$$

where

$$\begin{cases}
b_{0i} = n_2 - (n_1+n_2)\dfrac{\delta_i-1}{\delta_i+1},\\
b_{1i} = n_2\dfrac{1}{\delta_i} - (n_1+n_2)\dfrac{1}{\delta_i+1},\\
b_{2i} = (n_1+n_2)\dfrac{\delta_i}{\delta_i+1},
\end{cases} \tag{3.18}$$

with δ's in (3.15). Note that the estimator $\widehat{\boldsymbol{\omega}}^{(g)}$ is not order-preserving. In order to preserve the order,

$$\widehat{\omega}_1(\boldsymbol{L}) \geq \cdots \geq \widehat{\omega}_p(\boldsymbol{L}), \qquad \forall \boldsymbol{L},$$

we use the order statistics of $\widehat{\boldsymbol{\omega}}^{(g)}$. This modification apparently makes an better estimator w.r.t. the quadratic loss (1.19). The modified estimator $\widehat{\boldsymbol{\omega}}^{(go)} = (\widehat{\omega}_1^{(go)},\dots,\widehat{\omega}_p^{(go)})$ is given by

$$\widehat{\omega}_i^{(go)} = \widehat{\omega}_{(i)}^{(g)}, \qquad i=1,\dots,p, \tag{3.19}$$

where $\widehat{\omega}_{(i)}^{(g)}$ is the i-th largest element of $\widehat{\omega}_1^{(g)}, \ldots, \widehat{\omega}_p^{(g)}$. Consequently we propose another new estimator given by

$$\alpha\widehat{\boldsymbol{\omega}}^{(go)} = (\alpha\widehat{\omega}_1^{(go)}, \ldots, \alpha\widehat{\omega}_p^{(go)}).$$

4. Monte Carlo simulation

We carried out Monte Carlo simulation to compare the estimators proposed in the previous sections with those in the past literature. The following three groups of estimators were the subject for comparison; I: the estimators in the past literature, II: the estimators derived in Section 2, III: the estimators derived in Section 3. As for the group II, III, the specific choice of α in the estimator of the form $\alpha\widehat{\boldsymbol{\omega}}^{(\cdot)}$ is needed. We can't find any information on the "optimal" choice of α in the proofs of the above theorems. Hence we used α_1 in (1.17) for the simulation. Judging from the similarity between L_1 and L_3 (see the proof of Theorem 2.3), α_1 looks more reasonable than α_2. Consequently the list of estimators is given by
I: $\widehat{\boldsymbol{\omega}}^{(u)}$, $\alpha_1\widehat{\boldsymbol{\omega}}^{(u)}$, $\widehat{\boldsymbol{\omega}}^{(l1)}$, $\alpha_2\widehat{\boldsymbol{\omega}}^{(u)}$, $\widehat{\boldsymbol{\omega}}^{(l2)}$
II: $\alpha_1\widehat{\boldsymbol{\omega}}^{(*1)}$, $\alpha_1\widehat{\boldsymbol{\omega}}^{(*2)}$
III: $\alpha_1\widehat{\boldsymbol{\omega}}^{(go)}$.
Unfortunately all the estimators above can be negative. It seems important to find a good estimator that is always strictly positive. Here in this paper we compromise by taking the positive part of these estimators, i.e. for $j = u, l1, l2, *1, *2, go$

$$\widehat{\boldsymbol{\omega}}^{(jp)} = (\widehat{\omega}_1^{(jp)}, \ldots, \widehat{\omega}_p^{(jp)}),$$

where

$$\widehat{\omega}_i^{(jp)} = \begin{cases} \widehat{\omega}_i^{(j)}, & \text{if } \ \widehat{\omega}_i^{(j)} > 0, \\ 0, & \text{if } \ \widehat{\omega}_i^{(j)} \le 0. \end{cases}$$

It is obvious that these positive part estimators dominate the original estimators w.r.t. the loss function (1.19). Therefore we confine the simulation to these modified estimators.

We generated 10000 noncentral Wishart random data when $p = 4$ with specific values of n_1, n_2; $(n_1, n_2) = (10, 20)$, $(10, 50)$, $(10, 100)$, $(20, 50)$, $(20, 100)$, $(50, 100)$. Note that in the MANOVA or discriminant analysis, where this estimation problem arises, we have n_2 greater than n_1 in most of the practical cases. Because of similarity between the results, we only tabulated the three cases as follows.
Table 1: $n_1 = 20$, $n_2 = 50$
Table 2: $n_1 = 20$, $n_2 = 100$
Table 3: $n_1 = 50$, $n_2 = 100$.

Each table contains several values of $\boldsymbol{\omega}$. Every $\boldsymbol{\omega}$ is indicated by Tr (Trace) and Dp (Dispersion Pattern) whose definitions are as follows.
Tr: Trace of $\boldsymbol{\Omega}$, i.e. $\sum_{i=1}^{p} \omega_i$.
Dp: Three patterns A, B and C of dispersion are defined by the value

$$r = \omega_{i+1}/\omega_i, \qquad 1 \le \forall i \le p-1$$

as $r = 1$, 0.5 and 0.1, respectively.
Consequently each ω_i is given by

$$\omega_i = \frac{\mathrm{Tr} \times r^{i-1}}{1 + r + \cdots + r^{p-1}}, \qquad 1 \le i \le p.$$

Table 1. $(n_1 = 20,\ n_2 = 50)$.

Tr = 0.1	Dp = A		Dp = B		Dp = C	
	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.91E + 02	0	3.92E + 02	0	3.84E + 02	0
alpha1(up)	3.57E + 02	9	3.58E + 02	9	3.51E + 02	9
(l1p)	3.78E + 02	3	3.78E + 02	3	3.71E + 02	3
alpha2(up)	3.15E + 02	20	3.15E + 02	20	3.09E + 02	20
(l2p)	3.26E + 02	17	3.26E + 02	17	3.20E + 02	17
alpha1(*1p)	3.57E + 02	9	3.58E + 02	9	3.51E + 02	9
alpha1(gop)	2.05E + 02	48	2.05E + 02	48	2.00E + 02	48
Tr = 1	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.93E + 02	0	3.92E + 02	0	3.78E + 02	0
alpha1(up)	3.58E + 02	9	3.57E + 02	9	3.44E + 02	9
(l1p)	3.78E + 02	4	3.77E + 02	4	3.64E + 02	4
alpha2(up)	3.15E + 02	20	3.13E + 02	20	3.02E + 02	20
(l2p)	3.26E + 02	17	3.24E + 02	17	3.12E + 02	17
alpha1(*1p)	3.58E + 02	9	3.57E + 02	9	3.44E + 02	9
alpha1(gop)	2.05E + 02	48	2.04E + 02	48	1.96E + 02	48
Tr = 10	Risk	Prial	Risk	Prial	Risk	Prial
(up)	4.99E + 02	0	3.94E + 02	0	3.39E + 02	0
alpha1(up)	4.52E + 02	9	3.52E + 02	11	2.99E + 02	12
(l1p)	4.72E + 02	5	3.69E + 02	6	3.14E + 02	7
alpha2(up)	3.94E + 02	21	3.01E + 02	24	2.52E + 02	26
(l2p)	4.04E + 02	19	3.10E + 02	21	2.59E + 02	23
alpha1(*1p)	4.52E + 02	9	3.52E + 02	11	2.99E + 02	12
alpha1(gop)	2.74E + 02	45	2.06E + 02	48	1.79E + 02	47
Tr = 10^2	Risk	Prial	Risk	Prial	Risk	Prial
(up)	2.24E + 03	0	9.84E + 02	0	1.22E + 03	0
alpha1(up)	2.04E + 03	9	8.47E + 02	14	1.08E + 03	11
(l1p)	2.05E + 03	8	8.49E + 02	14	1.08E + 03	11
alpha2(up)	1.82E + 03	19	7.05E + 02	28	9.63E + 02	21
(l2p)	1.82E + 03	19	7.04E + 02	28	9.62E + 02	21
alpha1(*1p)	2.04E + 03	9	8.47E + 02	14	1.08E + 03	11
alpha1(gop)	1.53E + 03	32	7.40E + 02	25	1.11E + 03	9
Tr = 10^3	Risk	Prial	Risk	Prial	Risk	Prial
(up)	4.45E + 04	0	2.36E + 04	0	4.48E + 04	0
alpha1(up)	4.11E + 04	8	2.13E + 04	10	4.20E + 04	6
(l1p)	4.11E + 04	8	2.13E + 04	10	4.20E + 04	6
alpha2(up)	3.84E + 04	14	2.08E + 04	12	4.36E + 04	3
(l2p)	3.84E + 04	14	2.08E + 04	12	4.36E + 04	3
alpha1(*1p)	4.11E + 04	8	2.13E + 04	10	4.20E + 04	6
alpha1(gop)	3.39E + 04	24	2.16E + 04	9	4.29E + 04	4
Tr = 10^4	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.03E + 06	0	1.84E + 06	0	3.89E + 06	0
alpha1(up)	2.82E + 06	7	1.69E + 06	8	3.70E + 06	5
(l1p)	2.82E + 06	7	1.69E + 06	8	3.70E + 06	5
alpha2(up)	2.71E + 06	11	1.73E + 06	6	3.96E + 06	−2
(l2p)	2.71E + 06	11	1.73E + 06	6	3.96E + 06	−2
alpha1(*1p)	2.82E + 06	7	1.69E + 06	8	3.70E + 06	5
alpha1(gop)	2.40E + 06	21	1.71E + 06	7	3.72E + 06	4
Tr = 10^5	Risk	Prial	Risk	Prial	Risk	Prial
(up)	2.89E + 08	0	1.76E + 08	0	3.77E + 08	0
alpha1(up)	2.68E + 08	7	1.62E + 08	8	3.61E + 08	4
(l1p)	2.68E + 08	7	1.62E + 08	8	3.61E + 08	4
alpha2(up)	2.59E + 08	10	1.69E + 08	4	3.90E + 08	−3
(l2p)	2.59E + 08	10	1.69E + 08	4	3.90E + 08	−3
alpha1(*1p)	2.68E + 08	7	1.62E + 08	8	3.61E + 08	4
alpha1(gop)	2.29E + 08	21	1.64E + 08	7	3.62E + 08	4

Table 2. ($n_1 = 20$, $n_2 = 100$).

Tr = 0.1	Dp = A		Dp = B		Dp = C	
	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.11E + 02	0	3.04E + 02	0	3.08E + 02	0
alpha1(up)	2.98E + 02	4	2.91E + 02	4	2.96E + 02	4
(l1p)	3.16E + 02	−2	3.09E + 02	−2	3.14E + 02	−2
alpha2(up)	2.80E + 02	10	2.74E + 02	10	2.78E + 02	10
(l2p)	2.92E + 02	6	2.85E + 02	6	2.90E + 02	6
alpha1(*1p)	2.98E + 02	4	2.91E + 02	4	2.96E + 02	4
alpha1(gop)	1.75E + 02	44	1.69E + 02	44	1.72E + 02	44
Tr = 1	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.15E + 02	0	3.07E + 02	0	3.03E + 02	0
alpha1(up)	3.02E + 02	4	2.94E + 02	4	2.90E + 02	4
(l1p)	3.20E + 02	−1	3.12E + 02	−1	3.07E + 02	−1
alpha2(up)	2.84E + 02	10	2.76E + 02	10	2.72E + 02	10
(l2p)	2.96E + 02	6	2.88E + 02	6	2.83E + 02	6
alpha1(*1p)	3.02E + 02	4	2.94E + 02	4	2.90E + 02	4
alpha1(gop)	1.77E + 02	44	1.71E + 02	44	1.69E + 02	44
Tr = 10	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.94E + 02	0	3.11E + 02	0	2.52E + 02	0
alpha1(up)	3.76E + 02	5	2.95E + 02	5	2.37E + 02	6
(l1p)	3.94E + 02	0	3.10E + 02	0	2.50E + 02	1
alpha2(up)	3.52E + 02	11	2.73E + 02	12	2.17E + 02	14
(l2p)	3.63E + 02	8	2.83E + 02	9	2.26E + 02	10
alpha1(*1p)	3.76E + 02	5	2.95E + 02	5	2.37E + 02	6
alpha1(gop)	2.33E + 02	41	1.77E + 02	43	1.44E + 02	43
Tr = 10^2	Risk	Prial	Risk	Prial	Risk	Prial
(up)	1.77E + 03	0	7.00E + 02	0	8.23E + 02	0
alpha1(up)	1.70E + 03	4	6.49E + 02	7	7.73E + 02	6
(l1p)	1.70E + 03	4	6.52E + 02	7	7.75E + 02	6
alpha2(up)	1.60E + 03	10	5.86E + 02	16	7.17E + 02	13
(l2p)	1.60E + 03	10	5.87E + 02	16	7.18E + 02	13
alpha1(*1p)	1.70E + 03	4	6.49E + 02	7	7.73E + 02	6
alpha1(gop)	1.27E + 03	28	5.63E + 02	20	7.74E + 02	6
Tr = 10^3	Risk	Prial	Risk	Prial	Risk	Prial
(up)	2.60E + 04	0	1.33E + 04	0	2.34E + 04	0
alpha1(up)	2.50E + 04	4	1.26E + 04	5	2.26E + 04	4
(l1p)	2.50E + 04	4	1.26E + 04	5	2.26E + 04	4
alpha2(up)	2.40E + 04	7	1.22E + 04	8	2.26E + 04	4
(l2p)	2.40E + 04	7	1.22E + 04	8	2.26E + 04	4
alpha1(*1p)	2.50E + 04	4	1.26E + 04	5	2.26E + 04	4
alpha1(gop)	2.11E + 04	19	1.28E + 04	3	2.31E + 04	2
Tr = 10^4	Risk	Prial	Risk	Prial	Risk	Prial
(up)	1.47E + 06	0	8.60E + 05	0	1.84E + 06	0
alpha1(up)	1.42E + 06	3	8.25E + 05	4	1.79E + 06	3
(l1p)	1.42E + 06	3	8.25E + 05	4	1.79E + 06	3
alpha2(up)	1.39E + 06	6	8.36E + 05	3	1.85E + 06	0
(l2p)	1.39E + 06	6	8.36E + 05	3	1.85E + 06	0
alpha1(*1p)	1.42E + 06	3	8.25E + 05	4	1.79E + 06	3
alpha1(gop)	1.26E + 06	14	8.44E + 05	2	1.81E + 06	2
Tr = 10^5	Risk	Prial	Risk	Prial	Risk	Prial
(up)	1.36E + 08	0	8.19E + 07	0	1.80E + 08	0
alpha1(up)	1.32E + 08	3	7.87E + 07	4	1.76E + 08	2
(l1p)	1.32E + 08	3	7.87E + 07	4	1.76E + 08	2
alpha2(up)	1.29E + 08	5	8.01E + 07	2	1.82E + 08	−1
(l2p)	1.29E + 08	5	8.01E + 07	2	1.82E + 08	−1
alpha1(*1p)	1.32E + 08	3	7.87E + 07	4	1.76E + 08	2
alpha1(gop)	1.18E + 08	13	8.05E + 07	2	1.77E + 08	2

Table 3. ($n_1 = 50$, $n_2 = 100$).

Tr = 0.1	Dp = A		Dp = B		Dp = C	
	Risk	Prial	Risk	Prial	Risk	Prial
(up)	9.20E + 02	0	9.26E + 02	0	9.43E + 02	0
alpha1(up)	8.81E + 02	4	8.87E + 02	4	9.04E + 02	4
(l1p)	9.23E + 02	0	9.29E + 02	0	9.46E + 02	0
alpha2(up)	8.29E + 02	10	8.34E + 02	10	8.50E + 02	10
(l2p)	8.53E + 02	7	8.59E + 02	7	8.74E + 02	7
alpha1(*1p)	8.81E + 02	4	8.87E + 02	4	9.04E + 02	4
alpha1(gop)	6.05E + 02	34	6.09E + 02	34	6.23E + 02	34
Tr = 1	Risk	Prial	Risk	Prial	Risk	Prial
(up)	9.44E + 02	0	9.22E + 02	0	9.19E + 02	0
alpha1(up)	9.05E + 02	4	8.83E + 02	4	8.80E + 02	4
(l1p)	9.47E + 02	0	9.24E + 02	0	9.22E + 02	0
alpha2(up)	8.50E + 02	10	8.29E + 02	10	8.26E + 02	10
(l2p)	8.75E + 02	7	8.54E + 02	7	8.50E + 02	7
alpha1(*1p)	9.05E + 02	4	8.83E + 02	4	8.80E + 02	4
alpha1(gop)	6.22E + 02	34	6.06E + 02	34	6.04E + 02	34
Tr = 10	Risk	Prial	Risk	Prial	Risk	Prial
(up)	1.04E + 03	0	9.02E + 02	0	7.73E + 02	0
alpha1(up)	9.94E + 02	4	8.58E + 02	5	7.33E + 02	5
(l1p)	1.04E + 03	0	8.96E + 02	1	7.69E + 02	1
alpha2(up)	9.31E + 02	11	7.99E + 02	11	6.78E + 02	12
(l2p)	9.55E + 02	8	8.21E + 02	9	6.99E + 02	10
alpha1(*1p)	9.94E + 02	4	8.58E + 02	5	7.33E + 02	5
alpha1(gop)	6.93E + 02	33	5.89E + 02	35	4.98E + 02	36
Tr = 10^2	Risk	Prial	Risk	Prial	Risk	Prial
(up)	2.77E + 03	0	1.16E + 03	0	1.29E + 03	0
alpha1(up)	2.65E + 03	4	1.07E + 03	7	1.20E + 03	7
(l1p)	2.67E + 03	4	1.08E + 03	6	1.22E + 03	5
alpha2(up)	2.48E + 03	10	9.58E + 02	17	1.10E + 03	14
(l2p)	2.50E + 03	10	9.65E + 02	17	1.11E + 03	14
alpha1(*1p)	2.65E + 03	4	1.07E + 03	7	1.20E + 03	7
alpha1(gop)	2.10E + 03	24	8.71E + 02	25	1.13E + 03	12
Tr = 10^3	Risk	Prial	Risk	Prial	Risk	Prial
(up)	3.03E + 04	0	1.41E + 04	0	2.44E + 04	0
alpha1(up)	2.91E + 04	4	1.34E + 04	5	2.34E + 04	4
(l1p)	2.91E + 04	4	1.34E + 04	5	2.34E + 04	4
alpha2(up)	2.79E + 04	8	1.29E + 04	9	2.33E + 04	4
(l2p)	2.79E + 04	8	1.29E + 04	9	2.33E + 04	4
alpha1(*1p)	2.91E + 04	4	1.34E + 04	5	2.34E + 04	4
alpha1(gop)	2.43E + 04	20	1.34E + 04	5	2.38E + 04	3
Tr = 10^4	Risk	Prial	Risk	Prial	Risk	Prial
(up)	1.48E + 06	0	8.72E + 05	0	1.83E + 06	0
alpha1(up)	1.43E + 06	3	8.36E + 05	4	1.78E + 06	2
(l1p)	1.43E + 06	3	8.36E + 05	4	1.78E + 06	2
alpha2(up)	1.40E + 06	5	8.45E + 05	3	1.85E + 06	−1
(l2p)	1.40E + 06	5	8.45E + 05	3	1.85E + 06	−1
alpha1(*1p)	1.43E + 06	3	8.36E + 05	4	1.78E + 06	2
alpha1(gop)	1.24E + 06	16	8.41E + 05	3	1.79E + 06	2
Tr = 10^5	Risk	Prial	Risk	Prial	Risk	Prial
(up)	1.36E + 08	0	8.16E + 07	0	1.79E + 08	0
alpha1(up)	1.31E + 08	3	7.83E + 07	4	1.75E + 08	2
(l1p)	1.31E + 08	3	7.83E + 07	4	1.75E + 08	2
alpha2(up)	1.29E + 08	5	7.97E + 07	2	1.83E + 08	−2
(l2p)	1.29E + 08	5	7.97E + 07	2	1.83E + 08	−2
alpha1(*1p)	1.31E + 08	3	7.83E + 07	4	1.75E + 08	2
alpha1(gop)	1.15E + 08	15	7.86E + 07	4	1.75E + 08	2

PRIAL (percentage reduction in average loss) of each estimator $\widehat{\omega}$ given by

$$\text{PRIAL} = \frac{\text{Risk of } \widehat{\omega}^{up} - \text{Risk of } \widehat{\omega}}{\text{Risk of } \widehat{\omega}^{up}} \times 100,$$

shows each estimator's superiority over $\widehat{\omega}^{up}$. Note that both 'Risk' and 'Prial' in the following tables are rounded into the nearest integers.

We can summarize the results as follows.

- Within the group I, the estimator $\alpha_j \widehat{\omega}^{(up)}$ shows almost the same or slightly better performance than $\widehat{\omega}^{(ljp)}$ for each $j = 1, 2$. Although original estimators $\alpha_2 \widehat{\omega}^{(u)}$ and $\widehat{\omega}^{(lj)}$ $j = 1, 2$ dominate $\widehat{\omega}^{(u)}$, their positive part estimators show negative PRIAL in several cases.
- Within the group II, the estimators $\alpha_1 \widehat{\omega}^{(*1p)}$ and $\alpha_1 \widehat{\omega}^{(*2p)}$ perform completely alike. (For simplicity, we omit the result on $\alpha_1 \widehat{\omega}^{(*2p)}$ from the tables.)
- In overall comparison, $\alpha_1 \widehat{\omega}^{(up)}$, $\alpha_1 \widehat{\omega}^{(*1p)}$ and $\alpha_1 \widehat{\omega}^{(gop)}$ always outperform $\widehat{\omega}^{(up)}$. Furthermore $\alpha_1 \widehat{\omega}^{(up)}$ and $\alpha_1 \widehat{\omega}^{(*1p)}$ perform equally, while PRIAL of $\alpha_1 \widehat{\omega}^{(gop)}$ is substantially larger than that of these two estimators when $\operatorname{tr} \boldsymbol{\Omega}$ is small and/or the elements of $\boldsymbol{\omega}$ are close to each other.

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References

Fujikoshi, Y. and Satoh, K. (1997). Modified AIC and C_p statistic in multivariate regression models, *Biometrika*, **84**, 707–716.

Gupta, A. K. (1990). Estimation of MANOVA eigenvalues, *Proc. Fifth Vilnius Conference on Prob. Theo. and Math. Statist.* (eds. B. Grigelionis, Y. V. Prohorov, V. V. Sazonov and V. Statulevicius), **1**, 463–469.

Gupta, A. K. and Krishnamoorthy, K. (1990). Improved estimators of the eigenvalues of $\Sigma_1 \Sigma_2^{-1}$, *Statist. Decisions*, **8**, 247–263.

Gupta, A. K. and Nagar, D. K. (1999). *Matrix Variate Distributions*, Chapman & Hall/CRC, Boca Raton.

Gupta, A. K., Sheena, Y. and Fujikoshi, Y. (2002). Estimation of the eigenvalues of noncentrality parameter matrix in noncentral Wishart distribution, Tech. Report, 02-03, Department of Mathematics and Statistics, Bowling Green State University, Ohio.

Konno, Y. (1992*a*). Improved estimation of matrix of normal mean and eigenvalues in the multivariate F-distribution, Ph.D. dissertation, Institute of Mathematics, University of Tsukuba (unpublished).

Konno, Y. (1992*b*). On estimating eigenvalues of the scale matrix of the mulitvariate F distribution, *Sankhyā Ser. A*, **54**, 241–251.

Leung, P. L. and Chan, W. Y. (1998). Estimation of the scale matrix and its eigenvalues in the Wishart and the multivariate F distributions, *Ann. Inst. Statist. Math.*, **50**, 523–530.

Leung, P. L. and Lo, M. (1996). An identity for the noncentral multivariate F distribution with application, *Statist. Sinica*, **6**, 419–431.

Leung, P. L. and Muirhead, R. J. (1987). Estimation of parameter matrices and eigenvalues in MANOVA and canonical correlation analysis, *Ann. Statist.*, **15**, 1651–1666.

Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory*, Wiley, New York.