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BOOTSTRAP BANDWIDTH SELECTION IN KERNEL DENSITY ESTIMATION FROM A CONTAMINATED SAMPLE*

A. DELAIGLE AND I. GIJBELS

Institut de Statistique, Université catholique de Louvain, 20 Voie du Roman Pays, Louvain-la-Neuve, Belgium

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Abstract. In this paper we consider kernel estimation of a density when the data are contaminated by random noise. More specifically we deal with the problem of how to choose the bandwidth parameter in practice. A theoretical optimal bandwidth is defined as the minimizer of the mean integrated squared error. We propose a bootstrap procedure to estimate this optimal bandwidth, and show its consistency. These results remain valid for the case of no measurement error, and hence also summarize part of the theory of bootstrap bandwidth selection in ordinary kernel density estimation. The finite sample performance of the proposed bootstrap selection procedure is demonstrated with a simulation study. An application to a real data example illustrates the use of the method.

Key words and phrases: Bandwidth selection, bootstrap, consistency, deconvolution, errors-in-variables, kernel density estimation.

1. Introduction

In this paper we consider the problem of estimating a density from a sample of size n that has been contaminated by random noise. This problem is usually referred to as a *deconvolution problem*, and has applications in many different fields such as biostatistics, chemistry and public health. See for example Stefanski and Carroll (1990) or Carroll *et al.* (1995) who deal with analyzing such data. Here we consider the so-called *deconvolving kernel estimator* introduced by Carroll and Hall (1988) and investigated by Stefanski and Carroll (1990), among others.

This method of estimation has already received considerable attention in the literature. Recent related works include Wand (1998), Hesse (1999), Rachdi and Sabre (2000) and Zhang and Karunamuni (2000). See also Devroye (1989), Stefanski and Carroll (1990), Fan (1991*a*, *b*, *c*, 1992) and Wand and Jones (1995), among others, for earlier contributions. Most of those papers however deal only with theoretical aspects of the estimation, and very few focus attention on the yet important issue of choosing the bandwidth in practice. Stefanski and Carroll (1990) and Hesse (1999) investigate a cross-validation choice of the bandwidth. However in practice this method suffers from the same drawbacks as those known in the non-contaminated case, such as large variability or multiplicity of the solution. See Delaigle and Gijbels (2004). The latter paper

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proposes plug-in type of bandwidth selectors, based on asymptotic approximations of the mean integrated squared error. These plug-in procedures already improve upon the cross-validation technique. In the present paper we propose a bandwidth selection procedure based on bootstrap techniques, which can be considered as a competitor of the asymptotic techniques. We establish the asymptotic theory which justifies the use of this bootstrap bandwidth selector.

A detailed comparison, via simulation studies, of the three practical bandwidth selection procedures (plug-in, bootstrap and cross-validation) can be found in Delaigle and Gijbels (2004). Their finite sample comparisons revealed that the plug-in method and the bootstrap bandwidth selection method outperform the cross-validation method in all examples considered, and that they are competitive procedures in the sense that none of the two can be claimed to be best in all cases. A very interesting point for the bootstrap procedure is that it does not require the generation of bootstrap samples. Due to its special structure, the bootstrap estimate of the mean integrated squared error can be computed directly from the original sample. In fact, it simplifies a lot further since the only quantity that needs to be computed from the original sample is an empirical characteristic function.

Bootstrap bandwidth selection in case of non-contaminated data has been studied in several papers, including Faraway and Jhun (1990), Marron (1992), Grund and Polzehl (1997) and Hazelton (1999). In the present paper we consider the case of data that are contaminated by random noise, and develop a bootstrap procedure to choose the bandwidth parameter. This bootstrap-based bandwidth selection method requires the choice of a pilot bandwidth. We will see that a good choice for this pilot bandwidth is a bandwidth which is optimal for estimating the integrated squared second derivative of the target density. Such a bandwidth has been proposed by Delaigle and Gijbels (2002), and is easy to use in practice.

We prove the consistency of the proposed bootstrap bandwidth selector, defined as the minimizer of the bootstrap estimator of the mean integrated squared error. We first prove that the bootstrap estimator of the mean integrated squared error of the density estimator converges (in L^2) to the exact mean integrated squared error. We then show that the ratio of the bootstrap bandwidth selector and the minimizer of the mean integrated squared error converges to one in probability. The finite sample performance of the bandwidth selection procedure is illustrated via a simulation study from which we can conclude that the method performs quite well.

The paper is organized as follows. In Section 2 we recall the definition of the deconvolving kernel estimator and some of its theoretical properties. In Section 3 we define the bootstrap estimator of the bandwidth, and discuss how to choose the pilot bandwidth. In Section 4 we establish theoretical properties of the bootstrap bandwidth selector. Finally in Section 5 we illustrate our method on a few simulated examples, and also apply it to some real data. All proofs are deferred to the Appendix.

2. Deconvolving kernel estimator

Let $Y_1, \ldots, Y_n$ be an i.i.d. sample of size n from a random variable Y of unknown density f_Y, satisfying $Y = X + Z$, where X is a random variable distributed according to f_X, and Z is a random variable representing the measurement error, distributed according to f_Z. Suppose that X is independent of Z and f_X and f_Z are continuous. We assume here that the distribution of the error Z is fully known, which is the usual assumption in this context. This assumption might seem to be quite restrictive, but reflects the reality that one often has not sufficient information to estimate the distribution of Z and hence one needs to assume full knowledge of Z. In the case where f_Z is known up to some parameters, one may estimate these parameters through repeated measurements on several individuals. See the real data example. The case where f_Z is totally unknown may also be considered. Such a problem necessitates further observations such as for example a sample from f_Z itself, and will not be studied here. See Barry and Diggle (1995) and Neumann (1997).

Let K be a kernel function, supported on $I\!R$, real-valued and such that $\int K(x)dx = 1$. Denote the Fourier transform of K (or its characteristic function if K is a density) by $\varphi_K(t) = \int e^{itx}K(x)dx$. The *deconvolving kernel estimator* of f_X at $x \in I\!R$ is defined by

$$\widehat{f}_X(x;h) = \frac{1}{nh}\sum_{j=1}^{n} K^Z\left(\frac{x-Y_j}{h};h\right), \tag{2.1}$$

where

$$K^Z(u;h) = (2\pi)^{-1}\int e^{-itu}\varphi_K(t)/\varphi_Z(t/h)dt, \tag{2.2}$$

with $h > 0$ a smoothing parameter depending on n, called the bandwidth and where φ_Z is the characteristic function of Z. Throughout this paper we suppose that for all $t \in I\!R$, $\varphi_Z(t) \neq 0$. See for example Carroll and Hall (1988) and Stefanski and Carroll (1990).

Furthermore we make the assumption that $K^Z(\cdot;h)$ is supported on $I\!R$, is real-valued, and $|K_Z(\cdot;h)|$ is integrable.

Throughout the paper we will use the notation $K_h(x) = h^{-1}K(x/h)$, for any function K.

From studies about the deconvolving kernel density estimator it is already known that the kernel should rather be chosen among densities whose characteristic function has a compact and symmetric support. See for example Stefanski (1990), Stefanski and Carroll (1990), Fan (1992), Wand and Jones (1995), Wand (1998) and Hesse (1999). Although this assumption can be relaxed in the case of ordinary smooth error densities (see Definition 1 below), we will in this paper work exclusively with compactly supported φ_K. The use of such kernels guarantees the existence of the density estimator in (2.1). An example of such a kernel is given by

$$K(x) = \frac{48\cos x}{\pi x^4}\left(1-\frac{15}{x^2}\right) - \frac{144\sin x}{\pi x^5}\left(2-\frac{5}{x^2}\right), \tag{2.3}$$

with $\varphi_K(t) = (1-t^2)^3 1_{[-1,1]}(t)$.

A common way to measure the closeness of the density estimator $\widehat{f}_X(\cdot;h)$ to its target $f_X(\cdot)$ is to compute the Mean Integrated Squared Error (MISE) of $\widehat{f}_X(\cdot;h)$, defined by

$$\begin{aligned}\text{MISE}\{\widehat{f}_X(\cdot;h)\} &= \text{E}\int\{\widehat{f}_X(x;h) - f_X(x)\}^2dx \\ &= \int[\text{Bias}\{\widehat{f}_X(x;h)\}]^2dx + \int \text{Var}\{\widehat{f}_X(x;h)\}dx,\end{aligned}$$

which, after some algebra, can be written as

$$\begin{aligned}
\text{(2.4)} \qquad \mathrm{MISE}\{\widehat{f}_X(\cdot;h)\} &= (2\pi nh)^{-1}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt \\
&\quad + (1-n^{-1})(2\pi)^{-1}\int |\varphi_X(t)|^2|\varphi_K(ht)|^2dt \\
&\quad + R(f_X) - \pi^{-1}\int |\varphi_X(t)|^2\varphi_K(ht)dt,
\end{aligned}$$

where we introduced the notation $R(g) = \int g^2(x)dx$, for any square integrable function g. See for example Wand and Jones (1995) and Wand (1998). The optimal bandwidth h_n for the estimation of f_X is then the bandwidth which minimizes $\mathrm{MISE}\{\widehat{f}_X(\cdot;h)\}$ with respect to h. For simplicity of notation and to highlight the dependency in h, we will rather write $\mathrm{MISE}\{\widehat{f}_X(\cdot;h)\}$ as $\mathrm{MISE}(h)$ in what follows.

The asymptotic properties of the deconvolving kernel estimator have been studied in several papers, among which Stefanski and Carroll (1990), Fan (1991*a*, *b*, *c*, 1992). These properties depend strongly on the error distribution. As in Fan (1991*a*) we classify the errors in two categories, the ordinary smooth distributions and the supersmooth distributions.

DEFINITION 1. The distribution of a random variable Z is said to be
(i) supersmooth of order β if its characteristic function $\varphi_Z(t)$ satisfies:

$$d_0|t|^{\beta_0}\exp(-|t|^\beta/\gamma) \le |\varphi_Z(t)| \le d_1|t|^{\beta_1}\exp(-|t|^\beta/\gamma) \quad \text{as} \quad t \to \infty,$$

for some positive constants d_0, d_1, β, γ and constants β_0 and β_1;
(ii) ordinary smooth of order β if its characteristic function $\varphi_Z(t)$ satisfies:

$$d_0|t|^{-\beta} \le |\varphi_Z(t)| \le d_1|t|^{-\beta} \quad \text{as} \quad t \to \infty,$$

for some positive constants d_0, d_1 and β.

It has been proved in Fan (1991*c*) that for supersmooth error densities (e.g. normal and Cauchy densities) the optimal rate of convergence of the density estimator to f_X is logarithmic, and hence very slow. By contrast this rate is much faster (algebraic) for ordinary smooth error densities (e.g. gamma and Laplace densities). Recall that this rate is $n^{-2/5}$ in the error free case. This difference in convergence rate between the two classes of error densities also shows up in the simulation results in Section 5.

3. Bootstrap selection of the bandwidth

3.1 *Bootstrap estimator of the mean integrated squared error*

The optimal bandwidth h_n defined as the minimizer of $\mathrm{MISE}(h)$ cannot be found in practice since the MISE involves unknown f_X related quantities. In this section we define an estimator of the optimal bandwidth, based on smoothed bootstrap techniques.

Bootstrap procedures for selection of the bandwidth in kernel density estimation from non-contaminated data have been studied in previous papers including Taylor (1989), Faraway and Jhun (1990), Jones *et al.* (1991), Falk (1992), Hall *et al.* (1992), Marron (1992), Grund and Polzehl (1997) and Hazelton (1999).

We now propose a bootstrap bandwidth selection method for contaminated data. Let $\widehat{f}_X(\cdot; g)$ be the deconvolving kernel estimator of f_X obtained from $Y_1, \ldots, Y_n$, with kernel L and pilot bandwidth g. Draw a bootstrap sample $X_1^*, \ldots, X_n^*$ from $\widehat{f}_X(\cdot; g)$, and after having added noise Z, use the contaminated bootstrap sample to construct a deconvolving estimator $\widehat{f}_X^*(\cdot; h)$ of $\widehat{f}_X(\cdot; g)$, with kernel K and bandwidth h. Then the bootstrap estimator of the MISE is given by

$$\begin{aligned}\mathrm{MISE}^*(h) &= \mathrm{E}^* \int \{\widehat{f}_X^*(x; h) - \widehat{f}_X(x; g)\}^2 dx \\ &= \int [\mathrm{Bias}^*\{\widehat{f}_X^*(x; h)\}]^2 dx + \int \mathrm{Var}^*\{\widehat{f}_X^*(x; h)\} dx,\end{aligned}$$

where E^*, Bias^* and Var^* all involve expectations conditionally upon $Y_1, \ldots, Y_n$, and are taken with respect to the pseudo density $\widehat{f}_X(\cdot; g)$, and the estimator of the optimal bandwidth is defined as the minimizer of $\mathrm{MISE}^*(h)$. From (2.4), it is immediate to see that the bootstrap MISE may also be written as

$$\begin{aligned}\mathrm{MISE}^*(h) &= (2\pi nh)^{-1} \int |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt \\ &\quad + (1 - n^{-1})(2\pi)^{-1} \int |\widehat{\varphi}_{X,g}(t)|^2 |\varphi_K(ht)|^2 dt \\ &\quad + R(\widehat{f}_X(\cdot; g)) - \pi^{-1} \int |\widehat{\varphi}_{X,g}(t)|^2 \varphi_K(ht) dt,\end{aligned} \tag{3.1}$$

where $\widehat{\varphi}_{X,g}(t)$ is the Fourier transform of $\widehat{f}_X(\cdot; g)$.

Note that (3.1) is nothing but an approximation of the exact MISE, where the unknown f_X has been replaced by a deconvolving kernel estimator $\widehat{f}_X(\cdot; g)$. Although it may then seem at first that g should be set equal to h, the bandwidth needed to estimate f_X, a closer look at expression (3.1) reveals that g needs not be optimal for the estimation of f_X, but rather for the estimation of quantities involving f_X. See Subsection 3.2. Similarly, the kernel L needs not be equal to K, but this choice is less important and will not be discussed here.

In the case of non-contaminated data Taylor (1989) and Marron (1992), among others, remarked that this bootstrap bandwidth selection procedure, unlike many other bootstrap estimation procedures, does not require the generation of any bootstrap sample in practice. This also holds in case of contaminated data since expression (3.1) can be computed entirely from the original sample. Note that minimizing $\mathrm{MISE}^*(h)$ with respect to h is equivalent to minimizing

$$\begin{aligned}\mathrm{MISE}_2^*(h) &= (2\pi nh)^{-1} \int |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt \\ &\quad + (1 - n^{-1})(2\pi)^{-1} \int |\widehat{\varphi}_{X,g}(t)|^2 |\varphi_K(ht)|^2 dt \\ &\quad - \pi^{-1} \int |\widehat{\varphi}_{X,g}(t)|^2 \varphi_K(ht) dt,\end{aligned} \tag{3.2}$$

with respect to h, and this only requires calculation of $\widehat{\varphi}_{X,g}(\cdot)$ from the original sample. Furthermore we have that $\widehat{\varphi}_{X,g}(t) = \widehat{\varphi}_{Y,n}(t) \cdot \varphi_L(gt)/\varphi_Z(t)$, with $\widehat{\varphi}_{Y,n}$ the empirical characteristic function of Y. See Remark 2 below. So, in practice the only quantity that needs to be calculated in $\mathrm{MISE}_2^*(h)$ is this empirical characteristic function. This simplifies considerably the computations involved in this bootstrap bandwidth selection procedure.

Remark 1. In this method we used a smoothed estimator of f_X since resampling with replacement from the data would lead to a bootstrap estimator of the bias equal to zero, resulting in a bootstrap bandwidth chosen on the basis of the variance only (instead of the whole MISE). See Faraway and Jhun (1990) for the non-contaminated case and Delaigle (1999) for the contaminated case. An alternative to the smoothed bootstrap could be to use a classical non-smooth bootstrap, but with bootstrap samples of size $m < n$, as proposed by Hall (1990) in the non-contaminated case. See Delaigle (2003) for more details.

Remark 2. A useful expression for calculation of $\widehat{\varphi}_{X,g}(t)$ in (3.1) can be derived as follows. Let $\widehat{f}_Y(\cdot;g)$ be the 'usual kernel estimator' of f_Y based on the sample $Y_1,\ldots,Y_n$, and using the kernel L, i.e. $\widehat{f}_Y(x;g) = n^{-1}\sum_{1\leq i\leq n} L_g(x-Y_i)$. By Lemma A.6 of the Appendix, we see that $\widehat{\varphi}_{X,g}(t) = \widehat{\varphi}_{Y,g}(t)/\varphi_Z(t)$, where $\widehat{\varphi}_{Y,g}(t)$ denotes the characteristic function of $\widehat{f}_Y(x;g)$, for which it is easily proved that it equals $\widehat{\varphi}_{Y,n}(t)\cdot\varphi_L(gt)$, with $\widehat{\varphi}_{Y,n}$ the empirical characteristic function of Y. We conclude that $\widehat{\varphi}_{X,g}(t) = \widehat{\varphi}_{Y,n}(t)\cdot\varphi_L(gt)/\varphi_Z(t)$.

3.2 *Choice of the pilot bandwidth*

The quality of the estimator of the density f_X depends strongly on the choice of the bandwidth h, which on its turn depends on the choice of the pilot bandwidth g. In case of error free data, it is already known that, to give good estimates, the bandwidth g should preferably be of an order of magnitude larger than h, at least when using second order kernels. See Faraway and Jhun (1990), Falk (1992), Hall *et al.* (1992), Marron (1992), Hazelton (1999) and Jones (2000). In this context, efficient procedures for choosing g have been proposed. See for example Jones *et al.* (1991), Hall *et al.* (1992) or Jones *et al.* (1996) for a survey. Throughout this paper we assume that K is a second order kernel.

The simple choice of g that we propose here is based on the fact that, under sufficient smoothness conditions, an asymptotic representation of the MISE is

$$\mathrm{AMISE}(h) = \frac{h^4}{4}\mu_2^2(K)R(f_X'') + (2\pi nh)^{-1}\int|\varphi_K(t)|^2\cdot|\varphi_Z(t/h)|^{-2}dt,$$

where $\mu_2(K) = \int u^2K(u)du$ denotes the second moment of the kernel K. See Proposition 4.1 in Section 4. The bootstrap estimator of this quantity is

$$\mathrm{AMISE}^*(h) = \frac{h^4}{4}\mu_2^2(K)R(\widehat{f}_X''(\cdot;g)) + (2\pi nh)^{-1}\int|\varphi_K(t)|^2\cdot|\varphi_Z(t/h)|^{-2}dt. \tag{3.3}$$

From (3.3) we see that asymptotically, the only g related quantity is $R(\widehat{f}_X''(\cdot;g))$, which is best estimated by choosing g as the optimal bandwidth for estimating $R(f_X'')$. See Marron (1992) for a similar remark in the error free case. Delaigle and Gijbels (2002) provide a practical optimal bandwidth for estimating $R(f_X'')$, which is based on minimization of the mean squared error of $R(\widehat{f}_X''(\cdot;g))$. They prove that asymptotically, the MSE-optimal bandwidth g_r for estimating $R(f_X^{(r)})$, for any integer $r \geq 0$, is the bandwidth which minimizes the absolute value of the asymptotic bias of the estimator $R(\widehat{f}_X^{(r)}(\cdot; g_r))$, given by

$$\text{ABias}(R(\widehat{f}_X^{(r)}(\cdot; g_r))) = -g_r^2 \mu_2(K) R(f_X^{(r+1)}) + (2\pi n g_r^{2r+1})^{-1} \int t^{2r} |\varphi_K(t)|^2 \cdot |\varphi_Z(t/g_r)|^{-2} dt. \tag{3.4}$$

The proposed two-stage procedure for selecting g_2 reads as follows:

Step 0: Estimate $R(f_X^{(4)})$ via the normal reference method (assuming a parametric normal model for f_X), i.e.

$$\widehat{R(f_X^{(4)})} = \frac{8! \widehat{\sigma}_X^{-9}}{2^9 4! \sqrt{\pi}}$$

where, for example, $\widehat{\sigma}_X^2 = \widehat{\sigma}_Y^2 - \text{Var}(Z)$, with $\widehat{\sigma}_Y^2$ the empirical variance of the Y-observations, is a consistent estimator for $\sigma_X^2 = \text{Var}(X)$.

Step 1: Substitute $\widehat{R(f_X^{(4)})}$ for $R(f_X^{(4)})$ in (3.4) and select a bandwidth g_3, an optimal bandwidth for estimating $R(f_X^{(3)})$ by minimizing the absolute value of the resulting asymptotic bias. Obtain $R(\widehat{f}_X^{(3)}(\cdot; g_3))$.

Step 2: Substitute $R(\widehat{f}_X^{(3)}(\cdot; g_3))$ for $R(f_X^{(3)})$ in (3.4) and select the bandwidth $g_2 = g$ for estimating $R(f_X'')$.

In our simulation study this choice of initial bandwidth g proved to work well, and hence we propose to use this pilot bandwidth in practice. Of course, any bandwidth g which satisfies the conditions of the theorems of Section 4 would lead to a consistent bootstrap procedure, and therefore any appropriate choice of the initial g could be used in practice.

4. Consistency of the bootstrap method

Throughout this paper we will assume that h and g tend to zero such that $nh \to \infty$ and $ng \to \infty$ as n tends to infinity. These are classical conditions necessary to ensure the convergence of a density estimator to the target density. We also assume that L_Z is absolutely integrable. The following conditions will be necessary.

Condition A.
(A1) $\sup_{x \in \mathbb{R}} |f_X^{(j)}(x)| < \infty$ for $j = 1, \ldots, 4$;
(A2) $\int |f_X''(x)| dx < \infty$ and $\int |f_X^{(3)}(x)| dx < \infty$;
(A3) f_X is square integrable.

Condition B.
(B1) K and L are symmetric and bounded continuous functions such that $\int K(x)dx = 1$, $\int L(x)dx = 1$, $\int y^8 |K(y)| dy < \infty$ and $\int y^2 |L(y)| dy < \infty$;
(B2) $\varphi_K(t)$ and $\varphi_L(t)$ are supported on respectively $[-B_K, B_K]$ and $[-B_L, B_L]$, with $0 < B_K, B_L < \infty$.

Condition C.

(C1) $\{\int |L^Z * L^Z(u;g)|du\}\{\int |t|^j |\varphi_L(t)|^2 |\varphi_Z(t/g)|^{-2} dt\} = o(n^2 g^{j+2})$ for $j = 0,\ldots,8$;

(C2) $\int |L * L^Z(u;g)|du = o(\sqrt{n})$;

(C3) $\lim_{n\to\infty} h^{-1} \int |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt = \infty$ and $\int |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt = o(nh)$;

(C4) $\int t^4 |\varphi_L(t)|^2 |\varphi_Z(t/g)|^{-2} dt = o(ng^5)$.

The above set of conditions might look quite overwhelming at first sight, especially Condition C. However, for a given density function f_Z and given kernels K and L, this condition can be translated into conditions on the bandwidths h and g. In Subsection 4.2 we will discuss to which conditions on h and g it leads in case of a certain class of ordinary smooth error densities. Moreover, we will explain in that section that the practical pilot bandwidth g as proposed in the previous section, and the optimal bandwidth h of for example Fan (1991*b*) satisfy the above assumptions.

Concerning Condition B, an example of a kernel with 8 finite moments is given by

$$K(x) = c\left(\frac{\sin(x/10)}{x/10}\right)^{10},$$

with c a normalizing constant. Its characteristic function has a compact support that is included in $[-1,1]$. See Delaigle (2003).

4.1 *Consistency results*

In what follows we prove that the bootstrap MISE estimator is a consistent estimator of the MISE. Therefore we first establish the asymptotic orders of the bias and variance of the bootstrap integrated squared bias and the bootstrap integrated variance separately. Then we gather those properties in a theorem concerning the whole bootstrap estimated MISE. To simplify the proofs, we will suppose that φ_Z is symmetric. Note however that with a little more effort, the proofs can be adapted to a non symmetric φ_Z.

Remark 3. Note that the symmetry of φ_Z together with the symmetry of a kernel K implies that the function $K^Z(\cdot;h)$ as defined in (2.2) is symmetric. This fact will be used in the proofs.

Proposition 4.1 below, established by Stefanski and Carroll (1990), describes in detail the behaviour of the integrated squared bias, the integrated variance, and mean integrated squared error of the deconvolving kernel estimator. In the following, the integrated squared bias and its bootstrap counterpart will be denoted by respectively $\mathrm{ISB}(h)$ and $\mathrm{ISB}^*(h)$.

PROPOSITION 4.1. (Stefanski and Carroll (1990)) (i) *Under Conditions* (A1), (A2), (B2), *and if* K *is a second order kernel with* $\int |u^3 K(u)|du < \infty$, *we have*

$$\mathrm{ISB}(h) = \frac{h^4}{4}\mu_2^2(K)R(f_X'') + o(h^4).$$

(ii) *Under Conditions* (A3), (B2) *and* (C3) *we have*

$$\int \mathrm{Var}\{\widehat{f}_X(t;h)\}dt = (2\pi nh)^{-1}\int |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt + O(n^{-1}).$$

(iii) *Under Conditions* (A1) *to* (A3), *Conditions* (B2) *and* (C3) *and if* K *is a second order kernel with* $\int |u^3 K(u)| du < \infty$, *we have*

$$\mathrm{MISE}(h) = \frac{h^4}{4}\mu_2^2(K)R(f_X'') + (2\pi nh)^{-1}\int |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt + O(n^{-1}) + o(h^4).$$

In the next two propositions we provide the asymptotic order of the bias and the variance of the bootstrap integrated variance (Proposition 4.2) and of the bootstrap integrated squared bias (Proposition 4.3).

PROPOSITION 4.2. *Under Conditions* (A3), B *and* (C3), *we have*
(i) $\mathrm{Bias}[\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt] = O(n^{-1})$.
(ii) $\mathrm{Var}[\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt] = o(n^{-2})$.

PROPOSITION 4.3. *Under Conditions* (A1), (A2) *and* B
(i) *If* (C4) *is satisfied, we have* $\mathrm{Bias}[\mathrm{ISB}^*(h)] = o(h^4)$.
(ii) *If* (C1) *and* (C2) *are satisfied, we have* $\mathrm{Var}[\mathrm{ISB}^*(h)] = o(h^8)$.

Since $\mathrm{MISE}^*(h) = \mathrm{ISB}^*(h) + \int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt$, and hence

$$\mathrm{E}[\mathrm{MISE}^*(h)] = \mathrm{E}[\mathrm{ISB}^*(h)] + \mathrm{E}\left[\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt\right]$$

and

$$\mathrm{Var}[\mathrm{MISE}^*(h)] \leq \mathrm{Var}[\mathrm{ISB}^*(h)] + \mathrm{Var}\left[\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt\right] + 2\sqrt{\mathrm{Var}[\mathrm{ISB}^*(h)] \cdot \mathrm{Var}\left[\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt\right]},$$

combining Propositions 4.2 and 4.3 leads to the next theorem, which establishes the convergence of the bootstrap MISE estimator to the exact MISE.

THEOREM 4.1. *Under Conditions* A, B *and* C, *we have*

$$\mathrm{E}[\mathrm{MISE}^*(h)] = \mathrm{MISE}(h) + o(h^4) + O(n^{-1}) = \mathrm{MISE}(h) + o(\mathrm{MISE}(h)),$$
$$\mathrm{Var}[\mathrm{MISE}^*(h)] = o(n^{-2}) + o\left(h^8\right) + o(h^4 n^{-1}) = o(\mathrm{MISE}^2(h)),$$

i.e. $\frac{\mathrm{MISE}^*(h)}{\mathrm{MISE}(h)} \xrightarrow{L_2} 1$, *as* $n \to \infty$.

We conclude that under certain conditions, the MISE and its bootstrap counterpart are asymptotically equivalent. From Theorem 4.1, we deduce the following theorem, which shows that if we restrict our search of the bootstrap bandwidth to an interval around the real optimal bandwidth, the MISE and the bootstrap MISE used either with the real optimal bandwidth or with its bootstrap estimator are also asymptotically equivalent.

THEOREM 4.2. *Let h_n (resp. h_n^*) denote the single global minimiser of* MISE(h) (*resp. of* MISE$^*(h)$), *for* $h \in [A_1 h_n, A_2 h_n]$, *with constants* $0 < A_1 < 1$ *and* $A_2 > 1$. *Then under Conditions* A, B *and* C, *we have*

$$\frac{\mathrm{MISE}^*(h_n^*)}{\mathrm{MISE}(h_n^*)} \xrightarrow{P} 1 \quad \textit{and} \quad \frac{\mathrm{MISE}^*(h_n)}{\mathrm{MISE}(h_n)} \xrightarrow{P} 1, \qquad \textit{as} \quad n \to \infty.$$

Finally the next theorem establishes the consistency of our bootstrap bandwidth selector. For simplicity, we only prove, in the Appendix, the theorem for the following subclass of error densities:

$$\{\varphi_Z(t)\}^{-1} = \sum_{j=0}^{p} c_j t^j, \text{ where } c_0 = 1,\, c_1, c_2, \ldots, c_p \text{ are constants with } c_p \neq 0. \tag{4.1}$$

THEOREM 4.3. *Let h_n and h_n^* be as in Theorem* 4.2, *and f_Z such that* (4.1) *holds. Then, under Conditions* A, B *and* C, *we have*

$$\frac{h_n^*}{h_n} \xrightarrow{P} 1, \qquad \textit{as} \quad n \to \infty.$$

From this theorem we learn that under sufficient conditions, as the sample size increases, the bootstrap bandwidth tends to the real optimal bandwidth.

It would be of interest to investigate, in further research, the rate of convergence of the bootstrap bandwidth selector relative to the optimal bandwidth. This is outside the scope of the current paper.

4.2 *Verifying Condition C*

The aim of this subsection is to get a closer look at Condition C, in order to demonstrate that for a given error density and kernels K and L, this set of conditions can be translated into conditions on the bandwidths h and g.

Consider the class of error densities for which $\{\varphi_Z(t)\}^{-1} = \sum_{j=0}^{p} c_{2j} t^{2j}$, where $c_0 = 1$, $c_2, c_4, \ldots, c_{2p-2}$ are constants and c_{2p} is a constant different from zero. Note that f_Z is an ordinary smooth density of order $\beta = 2p$. An example of such an error density is a Laplace density for which $\{\varphi_Z(t)\}^{-1} = 1 + \sigma_Z^2 t^2$, corresponding to a case with $p = 1$ in the above class of densities.

Suppose now that the kernel function K satisfies (B2) and the following condition:

CONDITION D. K is $2p$ times differentiable and $\int |K^{(2j)}(x)dx| < \infty$, for $j = 0, \ldots, p$.

In order to check Condition C, we investigate first the behaviour of the quantities appearing in this condition. The following results can be shown:

Result 1. Under Condition D, we have

$$\int |t|^j |\varphi_K(t)|^2 |\varphi_Z(t/h)|^{-2} dt = O(h^{-4p}), \qquad \text{for} \quad j = 0, 1, 2, \ldots.$$

Result 2. Under Condition D, $\int |K * K^Z(x;h)|dx = O(h^{-2p})$.

Result 3. Under Condition D, $\int |K^Z * K^Z(x;h)|dx = O(h^{-4p})$.

The proofs of these results are reasonably straightforward and are omitted. Details can be obtained from the authors upon request.

The above results will allow us to verify Condition C for this class of error densities and kernels. Assuming that the kernels K and L satisfy Condition D, it is easily seen that the quantities involved in Condition C behave as follows:

$$\left\{\int |L^Z * L^Z(u;g)|du\right\}\left\{\int |t|^j|\varphi_L(t)|^2|\varphi_Z(t/g)|^{-2}dt\right\} = O(g^{-8p}) = o(n^2g^{j+2}) \quad \text{for} \quad j = 0,\ldots,8; \tag{4.2}$$

$$\int |L * L^Z(u;g)|du = O(g^{-2p}) = o(\sqrt{n}); \tag{4.3}$$

$$h^{-1}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt = O(h^{-1-4p}) \to \infty \quad \text{as} \quad n \to \infty; \tag{4.4}$$

$$\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt = O(h^{-4p}) = o(nh); \tag{4.5}$$

$$\int t^4|\varphi_L(t)|^2|\varphi_Z(t/g)|^{-2}dt = O(g^{-4p}) = o(ng^5), \tag{4.6}$$

where the expression following the second equality sign or the arrow indicates what is required by Condition C.

Putting $h = cn^{-\alpha}$ and $g = dn^{-\gamma}$, with c and d positive constants, we can translate the requirements in (4.2)–(4.6) into the following very simple requirements on h and g:

$$0 < \alpha < \frac{1}{4p+1} \quad \text{and} \quad 0 < \gamma < \frac{1}{4p+5}.$$

The optimal bandwidths h and g satisfy these requirements. Indeed, for ordinary smooth error densities of order $\beta = 2p$, the MISE-optimal bandwidth h for estimating f_X in case of contaminated data is of order

$$h \sim n^{-1/(2k+4p+1)},$$

where $k > 0$ denotes the order of the kernel function. This result can be found in Fan (1991*c*). The MSE-optimal bandwidth g for estimating $R(f_X'')$ is, for ordinary smooth error densities order of $\beta = 2p$, of order

$$g \sim n^{-1/(k+4p+5)},$$

as has been shown by Delaigle and Gijbels (2002).

One can also easily check that when h and g satisfy Condition C, then also do $c \cdot h$ and $d \cdot g$ where c and d are positive constants. As a consequence, all bandwidths h and g which behave (in rate) as the optimal bandwidths also satisfy the conditions.

5. Simulations and real data example

5.1 *Simulation study*

We now investigate the finite sample performance of the bootstrap bandwidth selection procedure via a simulation study. Our study involves four f_X densities, chosen because they show some typical features that can be encountered in practice. These densities, in increasing order of estimation difficulty, are:

1. Density #1: $X \sim N(0, 1)$
2. Density #2: $X \sim \chi^2(8)$
3. Density #3: $X \sim 0.5N(-3, 1) + 0.5N(2, 1)$
4. Density #4: $X \sim 0.4\ \text{Gamma}(5) + 0.6\ \text{Gamma}(13)$.

Figure 1 shows the four target densities. From each of these densities, 500 samples of size $n = 50$, 100 and 250 were generated, each of which was then contaminated by a sample from either a $N(0; \sigma_Z^2)$ or a Laplace(σ_Z) error density. For each configuration, the parameter σ_Z was chosen such that the ratio $\operatorname{Var} Z / \operatorname{Var} X$ equals 0.25, except for density #4 where we took $\operatorname{Var} Z / \operatorname{Var} X = 0.1$ (the latter density is more difficult to recover). The 8 moments condition on K imposed by Condition (B1) is needed to prove the consistency results but is mainly technical. From our simulation results it seems that in practice we can use kernels with for example 2 finite moments. In our simulation study we use the kernels $K = L$ as defined in (2.3).

To apply our method in practice, we need to

1. Choose an initial bandwidth g, following the two-stage plug-in procedure as discussed at the end of Subsection 3.2;

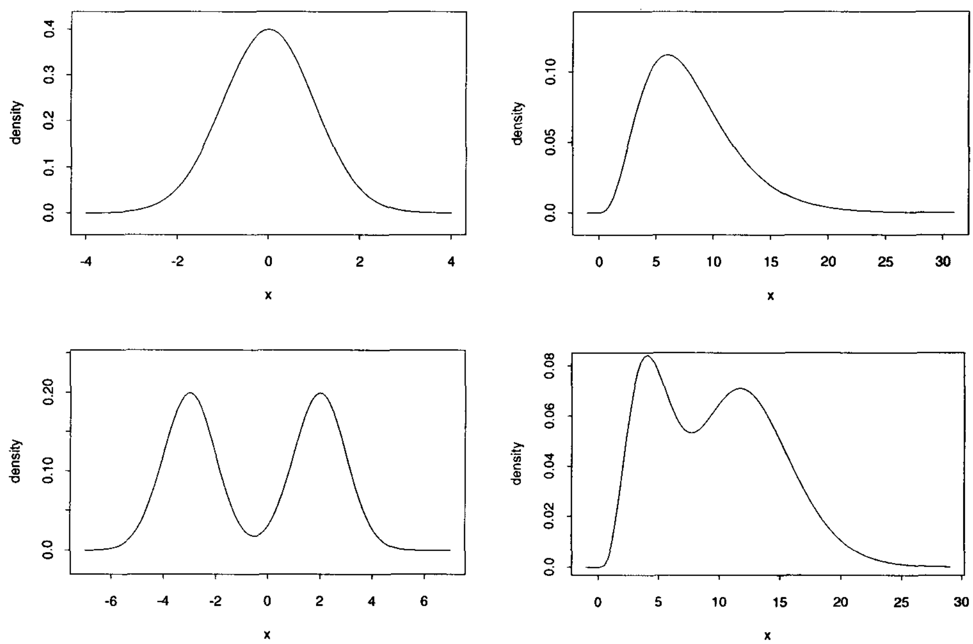

Fig. 1. The four target densities: a normal density (top left panel), a chi-squared density (top right panel), a mixture of two normal densities (bottom left panel), and a mixture of two gamma densities (bottom right panel).

2. On a grid of h, search the value which minimizes $\mathrm{MISE}_2^*(h)$ in (3.2) and obtain $\widehat{h}_{n,\mathrm{boot}}$;

3. Estimate $f_X(x)$ by $\widehat{f}_X(x;\widehat{h}_{n,\mathrm{boot}})$.

In order to evaluate the performance of the estimation procedure, we compute the Integrated Squared Error (ISE) of $\widehat{f}_X(\cdot;\widehat{h}_{n,\mathrm{boot}})$ for each calculated estimate. This allows us to classify the 500 estimates from the best one (i.e. the one with the smallest ISE) till the worst one (i.e. the one with the largest ISE). In all figures reported below, the estimates represented correspond to the first (1st quart), second (median) and third quartile (3rd quart) of these 500 ordered ISE's. The target density is always represented as a solid curve.

Figure 2 shows the results of the estimation of a $\chi^2(8)$ density (density #2) with different error densities (Laplace error and Gaussian error) and sample sizes. We see that even for a sample of size as small as 50, the method performs quite well. We observed similar results when trying to recover a standard normal density, and hence we do not report them here.

In Fig. 3 we compare the results of the bootstrap method with the cross-validation method of Stefanski and Carroll (1990) for the mixed normal density (density #3), with samples of size 50 or 250 contaminated by a Laplace error. The estimation task was a bit more difficult for this mixed density: the estimates managed to recover the two modes, but the peaks were underestimated. In this case, since the density presents quite

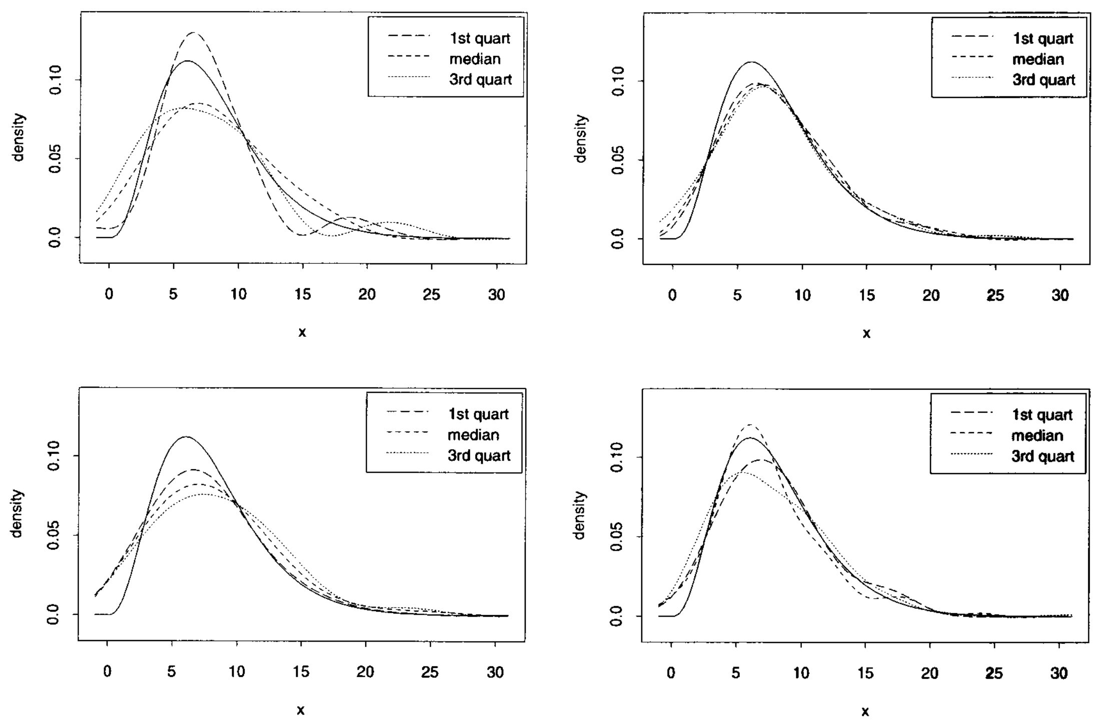

Fig. 2. Estimation of the $\chi^2(8)$ density for a Laplace error with $n = 50$ (top left panel), and $n = 250$ (top right panel), or for a Gaussian error with $n = 50$ (bottom left panel), and $n = 250$ (bottom right panel).

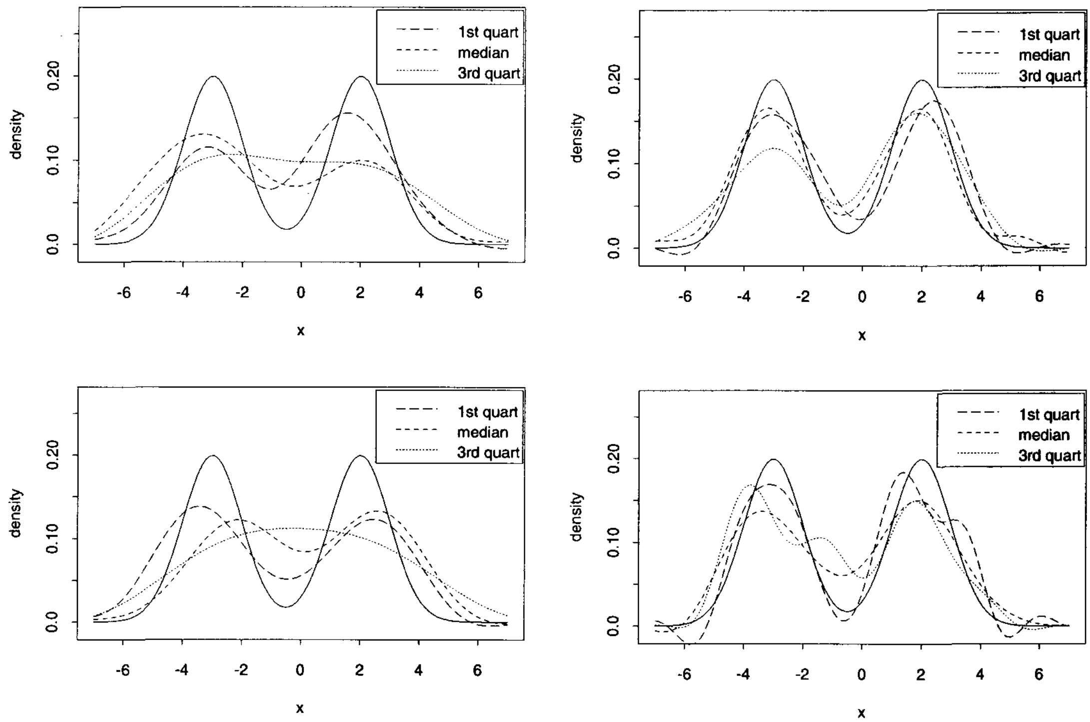

Fig. 3. Estimation of the mixed normal density for a Laplace error with the bootstrap method and $n = 50$ (top left panel), and $n = 250$ (top right panel), or with the CV method with $n = 50$ (bottom left panel), and $n = 250$ (bottom right panel).

different features at different places, the estimator would rather require use of a local bandwidth (but this would require more computations). Despite this difficulty, we see that the bootstrap selection method performs quite well in recovering the density and it outperforms the cross-validation method which gave too variable results. The plug-in method of Delaigle and Gijbels (2002, 2004) gave results similar to the bootstrap case. See Delaigle and Gijbels (2004) for a more complete comparison of the three methods.

Recovering the mixed gamma density (density #4) turned out to be more difficult. Figure 4 shows the results obtained with a Laplace noise for sample sizes 50 and 250. Although the 500 ISE-values were of about the same magnitude as for the other f_X densities, we see that the estimate had difficulties to recover the two modes. Even when, for a larger sample size, the two modes were detected, the estimate then had difficulties to recover their shapes. However as in the case of a mixed normal density, this problem is inherent to the choice of a global parameter, and is not really due to a failure of the method itself. It would be possible to adapt our technique to the search of a local bandwidth, but to the extent of a complication of the method, which would not necessarily guarantee better results.

The conclusions from the simulations remain unchanged if we replace the Laplace error by any ordinary smooth density, or if we replace the Gaussian error by any supersmooth density. We also believe that for most regular densities f_X (i.e. densities without any strong feature) the estimation method with bootstrap bandwidth selector will perform quite well, at least for a reasonable value of the ratio $\operatorname{Var} Z/\operatorname{Var} X$. As a

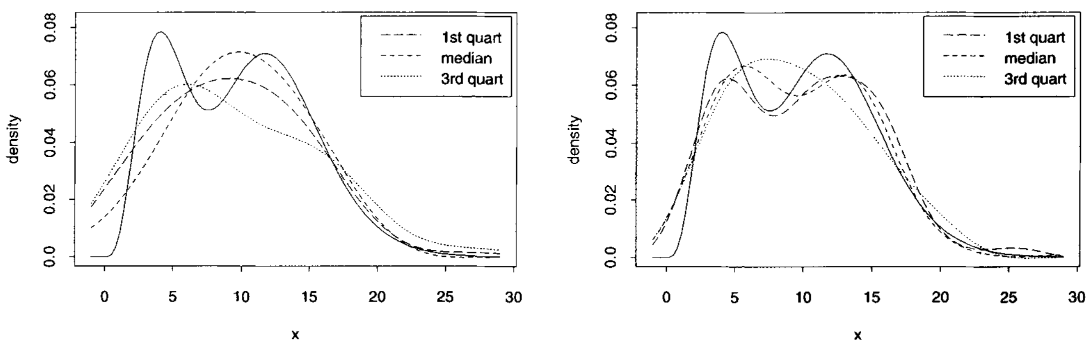

Fig. 4. Estimation of the mixed gamma density for a Laplace error with $n = 50$ (left panel) and $n = 250$ (right panel).

matter of fact one should not expect to get a very good estimate of f_X if the sample is too contaminated by noise.

5.2 *Real data application*

The data come from a pilot study on coronary heart disease, reported by Morris *et al.* (1977) and analyzed in, for example, Clayton (1992) and Cordy and Thomas (1997).

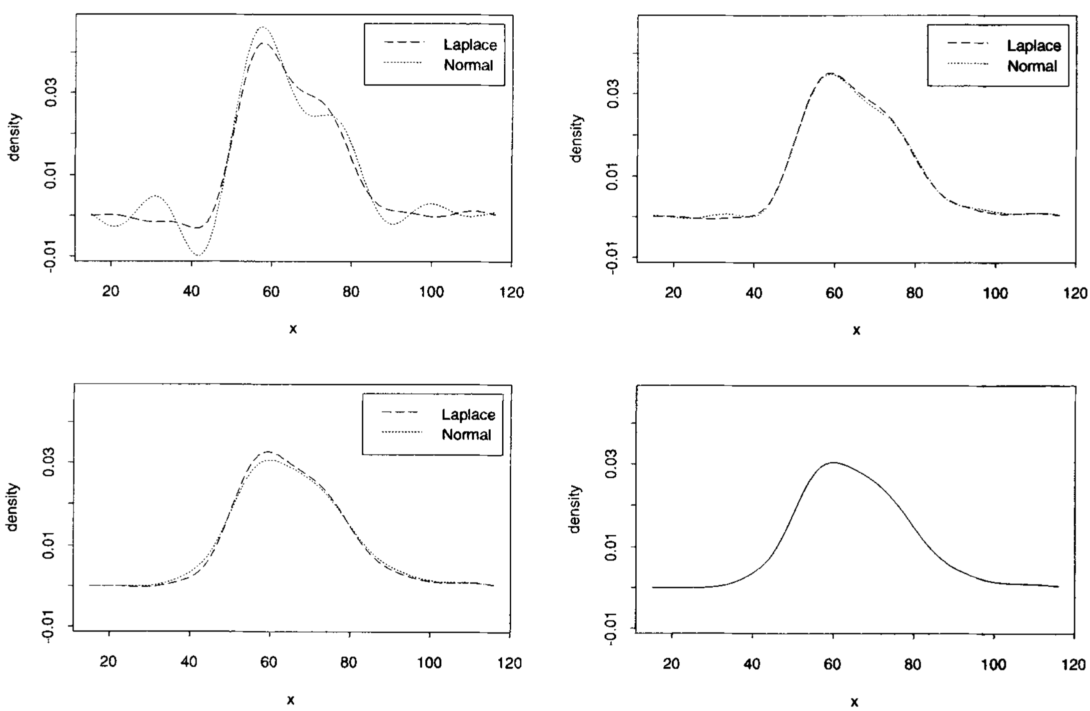

Fig. 5. Estimation of the ratio of poly-unsaturated fat to saturated fat intake for a Laplace error or a normal error with $\mathrm{Var}\, Z = 4/3\, \mathrm{Var}\, X$ (top left panel), $\mathrm{Var}\, Z = (2/3)\, \mathrm{Var}\, X$ (top right panel), $\mathrm{Var}\, Z = (1/3)\, \mathrm{Var}\, X$ (bottom left panel), and no error (bottom right panel).

They consist of measurements of the ratio of poly-unsaturated fat to saturated fat intake for 336 men in a one-week full weighted dietary survey. Among them, 61 individuals completed two such surveys, reporting on their diet during two weeks, separated by 6 months in time. We suppose that the actual ratio is invariant within an individual (see also Clayton (1992)). Since the error distribution is unknown, we will compute the density estimator of the actual ratio in case of a normal or a Laplace error distribution.

The error variance may be estimated through the two repeated measurements on 61 individuals. More precisely we will estimate $\operatorname{Var} Z$ by half the empirical variance of the 61 differences between the repeated measurements on each individual, which corresponds approximately to the situation that $\operatorname{Var} Z = (2/3) \operatorname{Var} X$. This estimation of the error variance may be not very accurate but can at least give us some insight into it. Since we cannot guarantee that this error variance is close to the exact one, we also consider $\operatorname{Var} Z = (4/3) \operatorname{Var} X$, $\operatorname{Var} Z = (1/3) \operatorname{Var} X$ and even $\operatorname{Var} Z = 0$.

The results are depicted in Fig. 5. The figure suggests that the actual density is skewed to the right. The estimators assuming different error distributions do not differ much. Note that the variance estimated through repeated measurements seems to be plausible: the other variances result into estimates that look less smooth or a little bit too smooth. In other words, these variances seem to be either too large or too small.

Acknowledgements

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Appendix

In this section we provide the proofs of the main results as stated in Section 4. These proofs rely on some useful facts about various functionals involving the kernels K and L and the 'kernels' K^Z and L^Z, defined in (2.2), and on some facts about the empirical characteristic function. These facts are summarized in the next section.

Denote by $f * g(\cdot) = \int f(\cdot - y)g(y)dy$ the convolution of the functions f and g. The following notations will be used throughout the appendix:

$$
\begin{aligned}
&D(x) = K(x) - \delta_0(x) && \text{with } \delta_0 \text{ the Dirac delta function}\\
&a(x) = D * D(x) && \\
&b(x) = L^Z * L^Z(x; g) && \\
&B(x) = L * L^Z(x; g) && \\
&\mu_i(K) = \int u^i K(u)du && \text{the } i\text{-th moment of the kernel } K\\
&\mu_i(K * K) = \int u^i (K * K)(u)du && \text{the } i\text{-th moment of the convolution } K * K,
\end{aligned}
\tag{A.1}
$$

and similar moments for the kernel L and the convolution $L * L$.

From Condition B, we know that the functions $\varphi_K(\cdot)$ and $\varphi_L(\cdot)$ are bounded by a finite constant. Without loss of generality we will suppose that this bound is equal to 1.

A.1 *Preliminary lemmas*

We start by recalling a form of Taylor's theorem with integral form of the remainder term.

THEOREM A.1. (Integral form of the remainder term in a Taylor expansion) *Let $m \in I\!N$ and f be a real-valued function which is $(m+1)$ times differentiable on an interval I. Then $\forall a \in I$ and $\forall h \neq 0 \in I$ such that $a+h \in I$, we can write*

$$f(a+h) = \sum_{j=0}^{m} f^{(j)}(a)\frac{h^j}{j!} + \frac{h^{m+1}}{m!}\int_0^1 f^{(m+1)}(a+hu)(1-u)^m du. \tag{A.2}$$

The next lemma is essentially due to Stefanski and Carroll (1990). See also Delaigle and Gijbels (2002) for a proof.

LEMMA A.1. *For a kernel K satisfying Condition* (B2) *we have*

$$\mathrm{E}\left[K^Z\left(\frac{x-Y}{h};h\right)\right] = \mathrm{E}\left[K\left(\frac{x-X}{h}\right)\right]. \tag{A.3}$$

LEMMA A.2. *Under Condition* (B1), *we have*

$$\int |D*D(x)||x|^j dx < \infty \quad \text{for} \quad j = 0, \ldots, 12. \tag{A.4}$$

PROOF OF LEMMA A.2. Under Condition (B1), we have

$$D*D(x) = \int K(x-y)K(y)dy - 2K(x) + \delta_0(x),$$

and thus $|D*D(x)| \leq \int |K(x-y)K(y)|dy + 2|K(x)| + \delta_0(x)$, and hence

$$\begin{aligned}\int |D*D(x)||x|^j dx &\leq \int_{\mathcal{R}^2} |x|^j|K(x-y)K(y)|dydx + 2\int |x|^j|K(x)|dx + \int |x|^j\delta_0(x)dx \\ &= \int_{\mathcal{R}^2} |x|^j|K(x-y)K(y)|dydx + 2\int |x|^j|K(x)|dx.\end{aligned}$$

Using Condition (B1) and the change of variable $u = x-y$, we find

$$\int_{\mathcal{R}^2} |x|^j|K(x-y)K(y)|dydx \leq \sum_{k=0}^{j} C_j^k \int_{\mathcal{R}^2} |u|^k|y|^{j-k}|K(u)||K(y)|dydu < \infty,$$

where C_j^k denotes the binomial coefficient $\binom{j}{k}$. This proves the lemma. □

LEMMA A.3. *Under Condition* (B1), *we have*

$$\mu_i(K) = \mu_j(D*D) = 0 \quad \text{for} \quad i = 1,3,5 \text{ and } j = 0,1,2,3. \tag{A.5}$$

$$\begin{aligned}&\mu_0(L*L) = 1, \qquad \mu_1(L*L) = 0, \\ &\mu_2(|L*L|) < \infty, \qquad \mu_4(D*D) = 6\mu_2^2(K).\end{aligned} \tag{A.6}$$

PROOF OF LEMMA A.3. The proof is straightforward, but rather technical and will be omitted here. See Delaigle (2003). □

LEMMA A.4. *Under Condition* (B2), *for all* $x \in \mathbb{R}$ *and* $j \in \mathbb{N}$, *we have*

$$|b^{(j)}(x)| \leq (2\pi)^{-1} \int |t|^j |\varphi_L(t)|^2 |\varphi_Z(t/g)|^{-2} dt.$$

PROOF OF LEMMA A.4. Under (B2), we have

$$\begin{aligned} b^{(j)}(x) &= \int \{L^Z\}^{(j)}(x-y;g) L^Z(y;g)dy \\ &= (2\pi)^{-2} \int \left(\int (-it)^j e^{-it(x-y)} \varphi_L(t)/\varphi_Z(t/g)dt \right) \\ &\quad \times \left(\int e^{-isy} \varphi_L(s)/\varphi_Z(s/g)ds \right) dy \\ &= (2\pi)^{-2} \int \varphi_{g_1}(y) \bar{\varphi}_{g_2}(y)dy \\ &= (2\pi)^{-1} \int (-it)^j e^{-itx} \varphi_L^2(t) \varphi_Z^{-2}(t/g)dt, \end{aligned}$$

where $g_1(t) = (-it)^j e^{-itx} \varphi_L(t)/\varphi_Z(t/g)$, $\bar{g}_2(s) = \varphi_L(s)/\varphi_Z(s/g)$, and where we used Parseval's identity. This proves the result. □

LEMMA A.5. *Under Conditions* (B2) *and* (C1) *we have*

$$\int |b(u)b^{(j)}(u+x)|du = o(n^2 g^{j+2}) \quad \forall x \in \mathbb{R}, \quad \textit{for} \quad j = 0, \ldots, 8. \tag{A.7}$$

PROOF OF LEMMA A.5. Under Condition (B2), we have

$$\begin{aligned} \int |b(u)b^{(j)}(u+x)|du &\leq \sup_u |b^{(j)}(u)| \int |b(u)|du \\ &\leq (2\pi)^{-1} \int |t|^j |\varphi_L(t)|^2 |\varphi_Z(t/g)|^{-2} dt \int |b(u)|du \\ &= (2\pi)^{-1} \left\{ \int |t|^j |\varphi_L(t)|^2 |\varphi_Z(t/g)|^{-2} dt \right\} \left\{ \int |L^Z * L^Z(u;g)|du \right\} \\ &= o(n^2 g^{j+2}), \end{aligned}$$

where we used Lemma A.4 and Condition (C1). □

Let $\widehat{f}_Y(\cdot;h)$ denote the usual kernel density estimator of f_Y based on the kernel K, and let $\widehat{f}_X(\cdot;h)$ be the deconvolving kernel density estimator of f_X based on the kernel K, and defined in (2.1).

LEMMA A.6. *Under Condition* B, *we have*

$$\widehat{f}_Y(x;h) = (\widehat{f}_X(\cdot;h) * f_Z)(x).$$

PROOF OF LEMMA A.6. Straightforward by (2.1), (2.2) and the Fourier inversion Theorem. □

Let $\widehat{\varphi}_{Y,n}(t) = n^{-1}\sum_{j=1}^{n} e^{itY_j}$, denote the empirical characteristic function of Y. The following lemma summarizes some useful facts about this quantity.

LEMMA A.7. *For the empirical characteristic function we have*

$$|\widehat{\varphi}_{Y,n}(t)|^2 = \frac{1}{n} + \frac{1}{n^2}\sum_{\substack{j,k=1\\ j\neq k}}^{n} e^{it(Y_j-Y_k)} \qquad \textit{and}$$
$$E|\widehat{\varphi}_{Y,n}(t)|^2 = |\varphi_Y(t)|^2 + n^{-1}(1-|\varphi_Y(t)|^2) = |\varphi_Y(t)|^2 + O(n^{-1}).$$

PROOF OF LEMMA A.7. The proof is rather straightforward and omitted. See Delaigle (2003) for details. □

A.2 *Proofs of Propositions* 4.2 *and* 4.3

A.2.1 *Proof of Proposition* 4.2

To deal with the integrated variance, first note that under (A3) and (B2) and after basic manipulations, the integrated variance of the deconvolving kernel density estimator may be written as follows (see for example Stefanski and Carroll (1990))

$$\int \mathrm{Var}\{\widehat{f}_X(t;h)\}dt = \frac{1}{2\pi nh}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt + \frac{1}{2\pi n}\int |\varphi_K(ht)|^2|\varphi_X(t)|^2dt, \tag{A.8}$$

and its bootstrap estimator as

$$\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt = \frac{1}{2\pi nh}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt + \frac{1}{2\pi n}\int |\varphi_K(ht)|^2|\widehat{\varphi}_{X,g}(t)|^2dt. \tag{A.9}$$

In what follows we will calculate the expectation and the variance of this bootstrap estimator. For simplicity we introduce the shortcut notation $\mathrm{Var}^*(t;h)$ for $\mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}$.

PROOF OF STATEMENT (i). From (A.9) we find

$$\mathrm{E}\left[\int \mathrm{Var}^*(t;h)dt\right] = \frac{1}{2\pi nh}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt + \frac{1}{2\pi n}\int |\varphi_K(ht)|^2\,\mathrm{E}\{|\widehat{\varphi}_{X,g}(t)|^2\}dt.$$

Recall from Remark 2 that $\widehat{\varphi}_{X,g}(t) = \widehat{\varphi}_{Y,n}(t)\cdot\varphi_L(gt)/\varphi_Z(t)$, and hence

$$\begin{aligned}\mathrm{E}[|\widehat{\varphi}_{X,g}(t)|^2] &= \mathrm{E}[|\widehat{\varphi}_{Y,n}(t)|^2]\cdot|\varphi_L(gt)|^2|\varphi_Z(t)|^{-2}\\ &= |\varphi_X(t)|^2|\varphi_L(gt)|^2 + |\varphi_L(gt)|^2|\varphi_Z(t)|^{-2}O(n^{-1}),\end{aligned}$$

where we used Lemma A.7 and the fact that $\varphi_Y(t) = \varphi_X(t) \cdot \varphi_Z(t)$. Therefore we can write

$$\begin{aligned} \mathrm{E}\left[\int \mathrm{Var}^*(t;h)dt\right] &= \frac{1}{2\pi nh}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt \\ &\quad + \frac{1}{2\pi nh}\int |\varphi_K(t)|^2|\varphi_X(t/h)|^2|\varphi_L(gt/h)|^2dt \\ &\quad + O(n^{-2}h^{-1})\int |\varphi_K(t)|^2|\varphi_L(gt/h)|^2|\varphi_Z(t/h)|^{-2}dt, \end{aligned}$$

where, by (A3), the behaviour of the second term is provided by

$$n^{-1}\int |\varphi_K(ht)|^2|\varphi_X(t)|^2|\varphi_L(gt)|^2dt \le n^{-1}\int |\varphi_X(t)|^2dt = O(n^{-1}),$$

and, by (C3), the behaviour of the third term is described by

$$\begin{aligned} n^{-2}h^{-1}\int |\varphi_K(t)|^2|\varphi_L(gt/h)|^2|\varphi_Z(t/h)|^{-2}dt &\le (n^2h)^{-1}\int |\varphi_K(t)|^2|\varphi_Z(t/h)|^{-2}dt \\ &= o(n^{-1}). \end{aligned}$$

Using (A.8) completes the proof of statement (i). □

PROOF OF STATEMENT (ii). Note first that from (A.9), we get $\mathrm{Var}[\int \mathrm{Var}^*\{\widehat{f}_X^*(t;h)\}dt] = (4\pi^2n^2)^{-1}\,\mathrm{Var}[\int |\varphi_K(ht)|^2|\widehat{\varphi}_{X,g}(t)|^2dt]$, and hence the task is to compute the latter variance term.

Let $\varphi_K^Z(t)$ denote $|\varphi_L(gt)|^2|\varphi_Z(t)|^{-2}|\varphi_K(ht)|^2$, and note that this is an even function. Then by Remark 2 and Lemma A.7, we have

$$\begin{aligned} \text{(A.10)}\quad & \mathrm{Var}\left[\int |\varphi_K(ht)|^2|\widehat{\varphi}_{X,g}(t)|^2|dt\right] \\ &= \mathrm{Var}\left[n^{-2}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\int e^{it(Y_j-Y_k)}\varphi_K^Z(t)dt\right] \\ &= n^{-4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{\substack{j',k'=1\\ j'\neq k'}}^{n}\mathrm{Cov}\left[\int e^{it(Y_j-Y_k)}\varphi_K^Z(t)dt, \int e^{it(Y_{j'}-Y_{k'})}\varphi_K^Z(t)dt\right] \\ &= 2n^{-3}(n-1)\,\mathrm{Var}\left[\int e^{it(Y_1-Y_2)}\varphi_K^Z(t)dt\right] \\ &\quad + 4n^{-3}(n-1)(n-2) \\ &\quad \times \mathrm{Cov}\left[\int e^{it(Y_1-Y_2)}\varphi_K^Z(t)dt, \int e^{it(Y_1-Y_3)}\varphi_K^Z(t)dt\right]. \end{aligned}$$

We will now compute separately the two terms appearing on the right-hand side of (A.10).

For the first term, let $T = \int e^{it(Y_1-Y_2)}\varphi_K^Z(t)dt$. Then $\mathrm{Var}(T) \leq \mathrm{E}(T^2)$, where under (C3)

$$
\begin{aligned}
\mathrm{E}(T^2) &= \mathrm{E}\int_{\mathcal{R}^2} e^{i(t+u)Y_1}e^{-i(t+u)Y_2}|\varphi_L(gt)|^2|\varphi_Z(t)|^{-2}|\varphi_K(ht)|^2 \\
&\qquad \times |\varphi_L(gu)|^2|\varphi_Z(u)|^{-2}|\varphi_K(hu)|^2 dtdu \\
&= \int_{\mathcal{R}^2} |\varphi_Y(t+u)|^2|\varphi_Z(t)|^{-2}|\varphi_Z(u)|^{-2}| \\
&\qquad \times \varphi_L(gt)|^2|\varphi_K(ht)|^2|\varphi_L(gu)|^2|\varphi_K(hu)|^2 dtdu \\
&\leq \int_{\mathcal{R}^2} |\varphi_Z(t)|^{-2}|\varphi_Z(u)|^{-2}|\varphi_K(ht)|^2|\varphi_K(hu)|^2 dtdu \\
&= o(n^2).
\end{aligned}
\tag{A.11}
$$

For the second term, let T be defined as above and put $U = \int e^{iu(Y_1-Y_3)}\varphi_K^Z(u)du$. We then have

$$
\begin{aligned}
|\,\mathrm{E}(TU)| &= \left|\int_{\mathcal{R}^2} \varphi_Y(t+u)\bar{\varphi}_Y(t)\bar{\varphi}_Y(u)\varphi_K^Z(t)\varphi_K^Z(u)dtdu\right| \\
&\leq \int_{\mathcal{R}^2} |\varphi_Y(t+u)||\varphi_Y(t)||\varphi_Y(u)||\varphi_Z(t)|^{-2}|\varphi_K(ht)|^2|\varphi_Z(u)|^{-2}|\varphi_K(hu)|^2 dtdu \\
&= \int_{\mathcal{R}^2} |\varphi_Y(t+u)||\varphi_X(t)||\varphi_X(u)||\varphi_Z(u)|^{-1}|\varphi_Z(t)|^{-1}|\varphi_K(ht)|^2|\varphi_K(hu)|^2 dtdu \\
&\leq \int_{\mathcal{R}^2} |\varphi_X(t)||\varphi_X(u)||\varphi_Z(u)|^{-1}|\varphi_Z(t)|^{-1}|\varphi_K(ht)||\varphi_K(hu)|dtdu
\end{aligned}
$$

and this term is of order $o(n)$ since by Cauchy-Schwartz and applying Conditions (A3) and (C3) we have

$$
\begin{aligned}
\int |\varphi_X(t)||\varphi_Z(t)|^{-1}|\varphi_K(ht)|dt &\leq \left[\int |\varphi_X(t)|^2 dt\right]^{1/2}\left[\int |\varphi_Z(t)|^{-2}|\varphi_K(ht)|^2 dt\right]^{1/2} \\
&= o(\sqrt{n}).
\end{aligned}
$$

For the expectation of the random quantity T we get

$$
\begin{aligned}
E(T) &= \int_{\mathcal{R}^3} e^{it(y-z)}\varphi_K^Z(t)f_Y(y)f_Y(z)dtdydz \\
&= \int \varphi_Y(t)\varphi_Y(-t)\varphi_K^Z(t)dt \\
&= \int |\varphi_Y(t)|^2\varphi_K^Z(t)dt \\
&= \int |\varphi_X(t)|^2|\varphi_L(gt)|^2|\varphi_K(ht)|^2 dt \\
&\leq \int |\varphi_X(t)|^2 dt,
\end{aligned}
$$

which is of order $O(1)$ by Condition (A3). Similarly for $E(U)$.

Therefore the covariance term $\mathrm{Cov}(T, U) = E(TU) - E(T)E(U)$ is of order $o(n)$. Substituting this finding and (A.11) into (A.10) we get the result. □

A.2.2 *Proof of Proposition* 4.3

Some arguments of this proof are similar to those used by Jones *et al.* (1991) for proving their results. The main technical difficulties here come from the fact that we have to deal with contaminated data.

Recall the definition of $D(\cdot)$ in (A.1), and denote by $D_h(\cdot) = \frac{1}{h}D(\cdot/h)$ its usual rescaled version. It is easy to verify that $D_h(x)$ is equal to $K_h(x) - \delta_0(x)$.

Using Lemma A.1 and Condition B, we can rewrite the integrated squared bias as follows

$$
\begin{aligned}
\text{(A.12)} \qquad \mathrm{ISB}(h) &= \int \{\mathrm{E}[\widehat{f}_X(x;h)] - f_X(x)\}^2 dx \\
&= \int \{\mathrm{E}[K_h^Z(x-Y;h)] - f_X(x)\}^2 dx \\
&= \int \left\{\int D_h(x-y) f_X(y) dy\right\}^2 dx \\
&= \int_{\mathcal{R}^2} D_h * D_h(z-y) f_X(y) f_X(z) dy dz \\
&= \int (D_h * D_h * f_X)(z) f_X(z) dz.
\end{aligned}
$$

In what follows, we use the shortcut notation $D_h * D_h * f_X(z)$ to denote $(D_h * D_h * f_X)(z)$ and similar expressions. Recall that, under Condition B, $L^Z(\cdot;g)$ is symmetric (see Remark 3). The bootstrap estimator of the integrated squared bias can then be written as

$$
\begin{aligned}
\mathrm{ISB}^*(h) &= \int (D_h * D_h * \widehat{f}_X(\cdot;g))(z) \widehat{f}_X(z;g) dz \\
&= n^{-2} \sum_{i,j=1}^{n} \int D_h * D_h * L_g^Z(z-Y_j;g) L_g^Z(z-Y_i;g) dz \\
&= n^{-2} \sum_{i,j=1}^{n} \int D_h * D_h * L_g^Z(u;g) L_g^Z(u-Y_i+Y_j;g) du \\
&= n^{-2} \sum_{i,j=1}^{n} \int D_h * D_h * L_g^Z(u;g) L_g^Z(Y_i-Y_j-u;g) du \\
&= n^{-1} D_h * D_h * L_g^Z * L_g^Z(0;g) + n^{-2} \sum_{\substack{i,j=1 \\ j\neq i}}^{n} T_{ij},
\end{aligned}
$$

where we used a change of variable $u = z - Y_j$, and introduced the notation

$$
\begin{aligned}
\text{(A.13)} \qquad T_{ij} &= \int D_h * D_h * L_g^Z(z-Y_j;g) L_g^Z(z-Y_i;g) dz \\
&= (D_h * D_h * L_g^Z * L_g^Z)(Y_i - Y_j; g).
\end{aligned}
$$

Statement (i) of Proposition 4.3 is an immediate consequence of the following lemma, the proof of which is given below.

LEMMA A.8. *Under Condition* B,
(i) *if* (A1) *and* (A2) *are satisfied, we have* $\mathrm{E}(T_{12}) = \mathrm{ISB}(h) + O(g^2h^4)$.
(ii) *if* (C4) *is satisfied, we have* $D_h * D_h * L_g^Z * L_g^Z(0; g) = o(nh^4)$.

To compute the variance of the bootstrap ISB, note first that

$$
\begin{aligned}
\text{(A.14)} \qquad \mathrm{Var}[\mathrm{ISB}^*(h)] &= n^{-4}\,\mathrm{Var}\left[\sum_{\substack{i,j=1\\ j\neq i}}^{n} T_{ij}\right] \\
&= n^{-4}\,\mathrm{Cov}\left[\sum_{\substack{i,j=1\\ j\neq i}}^{n} T_{ij}, \sum_{\substack{i',j'=1\\ j'\neq i'}}^{n} T_{i'j'}\right] \\
&= n^{-4}[2n(n-1)\,\mathrm{Var}(T_{12}) + 4n(n-1)(n-2)\,\mathrm{Cov}\,(T_{12}, T_{13})].
\end{aligned}
$$

The behaviour of the variance and covariance term in this expression is established in Lemma A.9 below, and hence the proof of statement (ii) of Proposition 4.3 is completed with the proof of that lemma.

LEMMA A.9. *Under Conditions* (A1) *and* B,
(i) *if* (C1) *is satisfied we have*

$$\mathrm{Var}(T_{12}) \leq \mathrm{E}(T_{12}^2) = o(n^2h^8g)$$

(ii) *if* (A2) *and* (C2) *are satisfied we have*

$$\mathrm{Cov}(T_{12}, T_{13}) = o(nh^8).$$

PROOF OF LEMMA A.8. *Proof of statement* (i): Using Lemma A.1, the symmetry of L, and introducing the notation $(L * L)_g(\cdot)$ for the usual rescaled version of $L * L$ we can write

$$
\begin{aligned}
\text{(A.15)} \qquad \mathrm{E}(T_{12}) &= \mathrm{E}\int D_h * D_h * L_g(z - X_2)L_g(z - X_1)dz \\
&= \int_{\mathcal{R}^2} D_h * D_h * L_g * L_g(x - y)f_X(x)f_X(y)dxdy \\
&= \int_{\mathcal{R}^3} D_h * D_h(x - y - u)L_g * L_g(u)f_X(x)f_X(y)dxdydu \\
&= \int_{\mathcal{R}^3} D_h * D_h(x - v)L_g * L_g(v - y)f_X(x)f_X(y)dxdydv \\
&= \int_{\mathcal{R}^3} D_h * D_h(x - v)f_X(v - gw)L * L(w)f_X(x)dxdwdv \\
&= \int_{\mathcal{R}^2} D_h * D_h(x - v)f_X(v)f_X(x)dxdv \\
&\quad + g^2\int_{\mathcal{R}^3}\int_0^1 D_h * D_h(x - v)L * L(w)w^2(1 - t)
\end{aligned}
$$

$$
\begin{aligned}
&\times f_X''(v-gtw)f_X(x)dtdwdxdv\\
&= \mathrm{ISB}(h) + g^2\int_{\mathcal{R}^3}\int_0^1 (D*D)_h(x-v)L*L(w)w^2(1-t)\\
&\qquad\times f_X''(v-gtw)f_X(x)dtdwdxdv\\
&= \mathrm{ISB}(h) + g^2\cdot(I),
\end{aligned}
$$

where we used a first change of variable $v = y+u$ and a second change of variable $w = (v-y)/g$, followed by a first order Taylor expansion of f_X around v and an application of Lemma A.3.

We will now show that the term (I) in expression (A.15) is of order $O(h^4)$. To see this, use the change of variable $z = (x-v)/h$, a third order Taylor expansion of f_X around v, and get

$$
\begin{aligned}
|(I)| &= \left|\int_{\mathcal{R}^3}\int_0^1 D*D(z)L*L(w)w^2(1-t)f_X''(v-gtw)f_X(v+hz)dtdwdzdv\right|\\
&= \frac{h^4}{3!}\left|\int_{\mathcal{R}^3}\int_0^1\int_0^1 D*D(z)z^4L*L(w)w^2(1-t)\right.\\
&\qquad\left.\times f_X''(v-gtw)(1-s)^3f_X^{(4)}(v+zsh)dsdtdwdzdv\right|\\
&\le cst.h^4\int_{\mathcal{R}^3}\int_0^1\int_0^1 |D*D(z)|z^4|L*L(w)|w^2(1-t)(1-s)^3\\
&\qquad\times|f_X''(v-gtw)|dsdtdwdzdv\\
&= O(h^4),
\end{aligned}
$$

where we used Lemma A.3, Condition (A1), Conditions (A2) and (B1) and Lemma A.2.

Proof of statement (ii): First note that using the notations in (A.1) and applying Condition (B2) we have

$$D_h * D_h * L_g^Z * L_g^Z(0;g) = (D*D)_h * (L^Z * L^Z)_g(0;g) = a_h * b_g(0).$$

Now we can write

$$
\begin{aligned}
a_h * b_g(x) &= \int a_h(x-y)b_g(y)dy\\
&= \int a(u)b_g(x-hu)du\\
&= \frac{h^4}{3!}\iint_0^1 a(u)u^4(1-t)^3b_g^{(4)}(x-thu)dtdu\\
&= \frac{h^4g^{-5}}{3!}\iint_0^1 a(u)u^4(1-t)^3b^{(4)}\left(\frac{x-thu}{g}\right)dtdu,
\end{aligned}
$$

where we used the change of variable $u = (x-y)/h$, a third order Taylor expansion of b_g around x and Lemma A.3.

Finally from Lemmas A.4 and A.2 and using (C4), we obtain

$$|a_h * b_g(0)| \le \frac{h^4g^{-5}}{3!2\pi}\int|a(u)|u^4du\int_0^1|1-t|^3dt\int v^4|\varphi_L(v)|^2|\varphi_Z(v/g)|^{-2}dv = o(nh^4),$$

which proves the statement. □

PROOF OF LEMMA A.9. *Proof of statement* (i): Note first that, by Definition (A.13) and the notations in (A.1), $\mathrm{E}(T_{12}^2) = \mathrm{E}[a_h * b_g(Y_1 - Y_2).a_h * b_g(Y_1 - Y_2)]$. Now, by the changes of variables $s = (x - y - z)/h$, $t = z/g$ and $w = (hs + gt - u)/h$, we have

$$
\begin{aligned}
\text{(A.16)}\quad & \mathrm{E}[a_h * b_g(Y_1 - Y_2).a_h * b_g(Y_1 - Y_2)] \\
&= \int_{\mathcal{R}^4} a_h(x-y-z)b_g(z)a_h(x-y-u)b_g(u)f_Y(x)f_Y(y)dxdydzdu \\
&= \int_{\mathcal{R}^4} a(s)b(t)a_h(hs+gt-u)b_g(u)f_Y(sh+y+gt)f_Y(y)dsdydtdu \\
&= \int_{\mathcal{R}^4} a(s)b(t)a(w)b_g(hs+gt-wh)f_Y(sh+y+gt)f_Y(y)dsdydtdw \\
&= g^{-1}\int_{\mathcal{R}^4} a(s)b(t)a(w)b(t+h(s-w)/g) \\
&\qquad \times f_Y(y+sh+gt)f_Y(y)dsdydtdw.
\end{aligned}
$$

Using a seventh order Taylor expansion of the function $b(\cdot)$ around t, we can write

$$
\begin{aligned}
b(t+h(s-w)/g) &= \sum_{k=0}^{7}\frac{h^k g^{-k}}{k!}(s-w)^k b^{(k)}(t) \\
&\quad + \frac{h^8 g^{-8}}{7!}(s-w)^8\int_0^1 (1-\theta)^7 b^{(8)}(t+h\theta(s-w)/g)d\theta,
\end{aligned}
$$

where by the binomial expansion, $(s-w)^k = \sum_{j=0}^{k} C_k^j(-1)^{k-j}s^j w^{k-j}$. Combining this with expression (A.16) and using Lemma A.3, we find

$$
\begin{aligned}
\text{(A.17)}\quad & \mathrm{E}[a_h * b_g(Y_1 - Y_2).a_h * b_g(Y_1 - Y_2)] \\
&= \sum_{k=4}^{7}\sum_{j=0}^{k-4}\frac{h^k g^{-k-1}}{k!}C_k^j(-1)^{k-j}\mu_{k-j}(a) \\
&\quad \times \int_{\mathcal{R}^3} a(s)b(t)b^{(k)}(t)s^j f_Y(y+sh+gt)f_Y(y)dsdydt \\
&\quad + \frac{h^8 g^{-9}}{7!}\int_{\mathcal{R}^4}\int_0^1 a(s)b(t)a(w)(s-w)^8(1-\theta)^7 b^{(8)}(t+\theta h(s-w)/g) \\
&\qquad \times f_Y(y+sh+gt)f_Y(y)d\theta dsdydtdw \\
&= \sum_{k=5}^{7}\sum_{\substack{1\le j\le k-4\\ 4-j\le \ell\le 3}} h^{k+\ell}g^{-k-1}(I_1) \\
&\quad + \sum_{\substack{4\le k\le 7\\ 0\le j\le k-4}} h^{k+4}g^{-k-1}(I_2) + h^8 g^{-9}(7!)^{-1}(I_3),
\end{aligned}
$$

where

$$
(I_1) = C_k^j(-1)^{k-j}(k!\ell!)^{-1}\mu_{k-j}(a)\mu_{\ell+j}(a)\int_{\mathcal{R}^2} b(t)b^{(k)}(t)f_Y(y)f_Y^{(\ell)}(y+gt)dydt,
$$

$$(I_2) = C_k^j(-1)^{k-j}(k!3!)^{-1}\mu_{k-j}(a)$$
$$\times \int_{\mathcal{R}^3}\int_0^1 a(s)b(t)b^{(k)}(t)s^j s^4(1-\xi)^3 f_Y^{(4)}(y+gt+\xi hs)f_Y(y)d\xi dsdydt,$$
$$(I_3) = \int_{\mathcal{R}^4}\int_0^1 a(s)b(t)a(w)(s-w)^8(1-\theta)^7 b^{(8)}(t+\theta h(s-w)/g)$$
$$\times f_Y(y+sh+gt)f_Y(y)d\theta dwdsdydt,$$

and where we used a third order Taylor expansion of f_Y around $y+gt$ and applied Lemma A.3. By using Lemmas A.5 and A.2 and Condition (A1), we can now conclude that

$$\begin{aligned}
&|\operatorname{E}[a_h * b_g(Y_1-Y_2).a_h * b_g(Y_1-Y_2)]| \\
&\quad\le \sum_{k=5}^{7}\sum_{\substack{1\le j\le k-4\\ 4-j\le \ell\le 3}} o(n^2h^{k+\ell}g^{k+2-k-1}) \\
&\qquad + \sum_{\substack{4\le k\le 7\\ 0\le j\le k-4}} o(n^2h^{k+4}g^{k+2-k-1})\iint_0^1 |a(s)||s|^{j+4}ds \\
&\qquad + o(n^2h^8g)\int_{\mathcal{R}^2}|a(s)||a(w)|(s-w)^8dwds \\
&\quad = o(n^2h^8g),
\end{aligned}$$

which proves the statement.

Proof of statement (ii): In order to evaluate $\operatorname{Cov}(T_{12},T_{13}) = \operatorname{E}(T_{12}T_{13}) - \operatorname{E}(T_{12})\operatorname{E}(T_{13})$ we first investigate the expectation of the product term. It should be understood that all operations below are carried out after having written the expectations as integrals, so the various steps should be understood as such.

Using Condition B, the notation from (A.1), a first change of variables $s' = x+Y_2$, $t' = z+Y_3$ and Lemma A.1, a second set of changes of variables $u = (x-s')/h$, $v = (x-t')/h$, $s = (s'-y)/g$ and $t = (t'-z)/g$, we find

$$\begin{aligned}
\operatorname{E}[T_{12}T_{13}] &= \operatorname{E}[a_h * b_g(Y_1-Y_2).a_h * b_g(Y_1-Y_3)] \\
&= \operatorname{E}\left[\int_{\mathcal{R}^2} a_h(Y_1-Y_2-x)b_g(x)a_h(Y_1-Y_3-z)b_g(z)dxdz\right] \\
&= \int_{\mathcal{R}^2}\operatorname{E}[a_h(Y_1-s')a_h(Y_1-t')]\operatorname{E}[b_g(s'-Y_2)]\operatorname{E}[b_g(t'-Y_3)]ds'dt' \\
&= \int_{\mathcal{R}^2}\operatorname{E}[a_h(Y_1-s')a_h(Y_1-t')]\operatorname{E}[B_g(s'-X_2)]\operatorname{E}[B_g(t'-X_3)]ds'dt' \\
&= \int_{\mathcal{R}^5} a_h(x-s')a_h(x-t')B_g(s'-y)B_g(t'-z)f_Y(x)f_X(y)f_X(z)ds'dt'dxdydz \\
&= \int_{\mathcal{R}^5} a(u)B(s)a(v)B(t)f_Y(x)f_X(x-hu-gs)f_X(x-hv-gt)dsdtdxdudv \\
&= \frac{h^4}{3!}\int_{\mathcal{R}^5}\int_0^1 u^4a(u)B(s)a(v)B(t)f_Y(x)(1-\theta)^3f_X^{(4)}(x-gs-\theta hu) \\
&\qquad\times f_X(x-hv-gt)d\theta dsdtdxdudv
\end{aligned}$$

$$= \frac{h^8}{3!3!} \int_{\mathcal{R}^5} \int_0^1 \int_0^1 u^4 a(u) B(s) a(v) B(t) f_Y(x) v^4 (1-\theta)^3 f_X^{(4)}(x - gs - \theta hu)$$
$$\times (1-\xi)^3 f_X^{(4)}(x - gt - \xi hv) d\xi d\theta ds dt dx du dv,$$

where we applied subsequently a third order Taylor expansion of f_X around $x - gs$, and a third order Taylor expansion of f_X around $x - gt$ and used Condition (A1) and Lemma A.3.

Using Condition (A1) and Lemma A.2 we then find

$$|\,\mathrm{E}[T_{12}T_{13}]| \le M \frac{h^8}{3!3!} \left\{ \int |u^4 a(u)| du \right\}^2 \left\{ \int |B(s)| ds \right\}^2 \int f_Y(x) dx = o(nh^8),$$

where we also used Condition (C2), and where M is a positive constant. From (A.15) and Proposition 4.1 (i) it is clear that $\mathrm{E}(T_{ij}) = \mathrm{E}(T_{ik}) = O(h^4)$. This then proves that $\mathrm{Cov}(T_{12}, T_{13}) = o(nh^8)$. □

A.3 *Proof of Theorem* 4.3

The proof of this result uses arguments similar to those used in the non-contaminated case by Hall (1983) and Scott and Terrell (1987). Note that the error densities characterised by (4.1) are ordinary smooth of order $\beta = p$, such that $(2\pi)^{-1} \int |\varphi_K(t)|^2 \cdot |\varphi_Z(t/h)|^{-2} dt = a_{2p} h^{-2p} + o(h^{-2p})$, with a_{2p} a constant. We deduce that $\mathrm{MISE}(h) = \mathrm{AMISE}(h) + o(\mathrm{AMISE}(h))$, where $\mathrm{AMISE}(h) = h^4 \mu_2^2(K) R(f_X'')/4 + a_{2p} h^{-2p} (nh)^{-1}$. Let $C = h_n^* / h_{\mathrm{AMISE}}$, where

$$\text{(A.18)} \qquad \begin{aligned} h_{\mathrm{AMISE}} &= \mathrm{argmin}_h\, \mathrm{AMISE}(h) \\ &= [(2p+1) a_{2p} / (\mu_2^2(K) R(f_X''))]^{1/(2p+5)} n^{-1/(2p+5)}. \end{aligned}$$

Below, we show that $C \xrightarrow{P} 1$. Since $h_{\mathrm{AMISE}}/h_n \to 1$, the proof of the theorem then follows immediately. By (A.18), we find

$$\begin{aligned} \mathrm{AMISE}(h_{\mathrm{AMISE}}) = {} & a_{2p}^{4/(2p+5)} n^{-4/(2p+5)} [\mu_2^2(K) R(f_X'')/(2p+1)]^{(2p+1)/(2p+5)} \\ & \times [(2p+1)/4 + 1], \end{aligned}$$

and

$$\begin{aligned} \mathrm{AMISE}(C h_{\mathrm{AMISE}}) = {} & a_{2p}^{4/(2p+5)} n^{-4/(2p+5)} [\mu_2^2(K) R(f_X'')/(2p+1)]^{(2p+1)/(2p+5)} \\ & \times [C^4 (2p+1)/4 + C^{-(2p+1)}]. \end{aligned}$$

Taking the ratio of these two expressions, we obtain

$$\text{(A.19)} \qquad \begin{aligned} \mathrm{AMISE}(h_n^*)/\,\mathrm{AMISE}(h_{\mathrm{AMISE}}) &= [C^4(2p+1) + 4C^{-(2p+1)}]/[2p+5] \\ &= f(C), \end{aligned}$$

with $f :]0, +\infty[\to [1, +\infty[$ defined by $f(x) = [x^4(2p+1) + 4x^{-(2p+1)}]/[2p+5]$. Similarly to Theorem 4.2, one can show that $f(C) \xrightarrow{P} 1$, as $n \to \infty$, i.e. $\forall \eta > 0$, $\lim_{n\to\infty} P(|f(C) - 1| > \eta) = 0$. Using this result and the fact that f is strictly convex and minimised by $f(1) = 1$, we deduce that, for all $0 < \epsilon < 1$,

$$\lim_{n\to\infty} P(|C - 1| > \epsilon) = \lim_{n\to\infty} P(\{C > 1 + \epsilon\} \cup \{C < 1 - \epsilon\})$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} P(\{f(C) > f(1+\epsilon)\} \cup \{f(C) > f(1-\epsilon)\}) \\
&\leq \lim_{n\to\infty} P(f(C) > \min(f(1+\epsilon), f(1-\epsilon))) \\
&= \lim_{n\to\infty} P(f(C) - 1 > \min(f(1+\epsilon), f(1-\epsilon)) - 1) \\
&= 0,
\end{aligned}$$

since $\min(f(1+\epsilon), f(1-\epsilon)) - 1 > 0$. Consequently, one also has $\lim_{n\to\infty} P(|C-1| > \epsilon) = 0$ for all $\epsilon \geq 1$. This proves that $C \xrightarrow{P} 1$, as $n \to \infty$. □

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