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A BIVARIATE UNIFORM AUTOREGRESSIVE PROCESS

MIROSLAV M. RISTIĆ AND BILJANA Č. POPOVIĆ

Department of Mathematics, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Yugoslavia, e-mail: miristic@ptt.yu

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Abstract. We define the bivariate first order stationary autoregressive process $\{(X_n, Y_n)\}$ with uniform marginal distribution where $\{X_n\}$ and $\{Y_n\}$ are the two stationary sequences with uniform $\mathcal{U}(0,1)$ marginal distributions. We also estimate the unknown parameters of the model.

Key words and phrases: Uniform autoregressive process, new uniform autoregressive process, first-order, second-order, bivariate uniform autoregressive process, autocovariance and autocorrelation matrix, estimation.

1. Introduction

The uniform autoregressive processes belong to the special class of autoregressive processes. Their marginal distributions are absolutely continuous $\mathcal{U}(0,1)$. Some well defined processes of this type are the uniform autoregressive process with positive autocorrelations (Chernick (1981)), the first order uniform autoregressive process with negative lag one correlations (Chernick and Davis (1982)) and the new uniform first order autoregressive process NUAR(1) (Ristić and Popović (2000a)). Lawrance (1992) discussed the uniform autoregressive process of the first order in general. The unknown parameters of these processes were estimated by Ristić and Popović (2000a, 2000b). We define here a bivariate process from the same class. Section 2 of this paper is devoted to the definition itself. The autocovariance and the autocorrelation structure are expanded in this section also. In Section 3 we prove the existence and the properties of the solution of the difference equation which defines the model. Estimations of parameters is discussed in Section 4.

2. The construction of the process

Following the definition of the bivariate autoregressive process with exponential marginal distribution presented by Dewald *et al.* (1989), we set the definition of the bivariate first order uniformly distributed process.

Let the two stationary processes $\{X_n\}$ and $\{Y_n\}$ be defined as

$$\begin{aligned} X_n &= U_{n1}X_{n-1} + V_{n1}Y_{n-1} + \varepsilon_{n1}, \\ Y_n &= U_{n2}X_{n-1} + V_{n2}Y_{n-1} + \varepsilon_{n2}, \end{aligned} \tag{2.1}$$

where $\{(U_{ni}, V_{ni})\}$, $i = 1, 2$, are independent sequences of independent identically distributed (i.i.d.) random vectors with the discrete probability distribution:

$U_{ni} \backslash V_{ni}$	0	β_i
0	0	$\frac{-\beta_i}{\alpha_i-\beta_i}$
α_i	$\frac{\alpha_i}{\alpha_i-\beta_i}$	0

The sequences $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are independent sequences of i.i.d. random variables with the probability distributions

$$P\{\varepsilon_{ni} = j_i(\alpha_i - \beta_i) - \beta_i\} = \alpha_i - \beta_i; \qquad j_i = 0, 1, \ldots, \frac{1}{\alpha_i - \beta_i} - 1,$$

where $1/(\alpha_i - \beta_i) \in \{2, 3, \ldots\}$, $i = 1, 2$.

The random vectors (U_{ni}, V_{ni}), $i = 1, 2$ and $(\varepsilon_{n1}, \varepsilon_{n2})$ are also independent.

If we let the random variables X_{n-1} and Y_{n-1} be uniformly distributed with $\mathcal{U}(0,1)$ probability distributions, then the random variables X_n and Y_n will be distributed in the same way. So, we have just defined the first order autoregressive time series $\{\boldsymbol{Z}_n\} = \{(X_n, Y_n)'\}$ which we have named BUAR(1) process.

The equation (2.1) can be represented in the vector form:

$$\boldsymbol{Z}_n = \boldsymbol{M}_n \boldsymbol{Z}_{n-1} + \boldsymbol{\varepsilon}_n, \tag{2.2}$$

where $\boldsymbol{\varepsilon}_n = (\varepsilon_{n1}, \varepsilon_{n2})'$ and

$$\boldsymbol{M}_n = \begin{pmatrix} U_{n1} & V_{n1} \\ U_{n2} & V_{n2} \end{pmatrix}.$$

The equations (2.1) enable us to determine the autocovariance and the autocorrelation matrix as follows:

$$\boldsymbol{\Gamma}(k) = \mathrm{Cov}(\boldsymbol{Z}_n, \boldsymbol{Z}_{n-k}) = \begin{pmatrix} \gamma_{XX}(k) & \gamma_{XY}(k) \\ \gamma_{YX}(k) & \gamma_{YY}(k) \end{pmatrix},$$

where $\gamma_{XX}(k) = \mathrm{Cov}(X_n, X_{n-k})$, $\gamma_{XY}(k) = \mathrm{Cov}(X_n, Y_{n-k})$, $\gamma_{YX}(k) = \mathrm{Cov}(Y_n, X_{n-k})$ and $\gamma_{YY}(k) = \mathrm{Cov}(Y_n, Y_{n-k})$. We shall set $u_i = E(U_{ni})$, $v_i = E(V_{ni})$, $i = 1, 2$, and let $\boldsymbol{M}$ be the matrix

$$\boldsymbol{M} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

The simple calculation proves that

$$\boldsymbol{\Gamma}(k) = \boldsymbol{M} \cdot \boldsymbol{\Gamma}(k-1) = \boldsymbol{M}^k \boldsymbol{\Gamma}(0), \tag{2.3}$$

where the autocovariance matrix $\boldsymbol{\Gamma}(0)$ is defined as

$$\boldsymbol{\Gamma}(0) = \frac{1}{12} \begin{pmatrix} 1 & \frac{u_1u_2+v_1v_2}{1-u_1v_2-u_2v_1} \\ \frac{u_1u_2+v_1v_2}{1-u_1v_2-u_2v_1} & 1 \end{pmatrix}.$$

Let us solve the $\boldsymbol{M}^k$. If we use Caley-Hamilton theorem, it follows that $\boldsymbol{M}^2 = (u_1 + v_2)\boldsymbol{M} + (u_2v_1 - u_1v_2)\boldsymbol{I}_2$, where $\boldsymbol{I}_2$ is the 2-by-2 identity matrix. So, by induction, it will be $\boldsymbol{M}^k = a_k\boldsymbol{M} + b_k\boldsymbol{I}_2$, where the coefficients a_k and b_k, $k = 2, 3, \ldots$, are defined according to the recurrent relations $a_{k+1} = a_k a_2 + b_k$ and $b_{k+1} = a_k b_2$ with the initial conditions $a_0 = 0$, $b_0 = 1$, $a_1 = 1$, $b_1 = 0$, $a_2 = u_1 + v_2$, $b_2 = u_2 v_1 - u_1 v_2$. These recurrent relations produce the difference equation $a_{k+2} - a_2 a_{k+1} - b_2 a_k = 0$, where $k \geq 2$. This difference equation can easily be computed using the method described by Brockwell and Davis ((1987), Section 3.6). If the equation $\lambda^2 - a_2\lambda - b_2 = 0$ corresponding to the last difference equation has two different roots, real or complex, λ_1 and λ_2, then the solution of the difference equation is

$$a_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}.$$

The other coefficient b_k is then

$$b_k = -\lambda_1\lambda_2 \cdot \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}.$$

The matrix $\boldsymbol{M}^k$ is then

$$\boldsymbol{M}^k = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} (u_1 - \lambda_2)\lambda_1^k + (\lambda_1 - u_1)\lambda_2^k & v_1(\lambda_1^k - \lambda_2^k) \\ u_2(\lambda_1^k - \lambda_2^k) & (v_2 - \lambda_2)\lambda_1^k + (\lambda_1 - v_2)\lambda_2^k \end{pmatrix}.$$

Two real and equal solutions of the equation imply that

$$\boldsymbol{M}^k = \begin{pmatrix} (k+1)\lambda_1^k - v_2 k\lambda_1^{k-1} & v_1 k\lambda_1^{k-1} \\ u_2 k\lambda_1^{k-1} & (k+1)\lambda_1^k - u_1 k\lambda_1^{k-1} \end{pmatrix}.$$

The eigenvalues of the matrix $\boldsymbol{M}$ are less than 1 in absolute value, so, the eigenvalues of $\boldsymbol{M}^k$ are also less than 1 in absolute value. One consequence will be that the matrix $\boldsymbol{M}^k$ converge to zero matrix when $k \to \infty$. The autocorrelation matrix of BUAR(1) will be

$$\boldsymbol{R}(k) = \mathrm{Corr}(\boldsymbol{X}_n, \boldsymbol{X}_{n-k}) = \boldsymbol{M}^k\boldsymbol{R}(0),$$

where $\boldsymbol{R}(0) = 12\boldsymbol{\Gamma}(0)$.

Example 1. Let parameters' values be $\alpha_1 = 0.4$, $\beta_1 = -0.1$, $\alpha_2 = 0.35$ and $\beta_2 = -0.15$. Then the autocovariance and the autocorrelation matrix will be

$$\boldsymbol{\Gamma}(0) = \begin{pmatrix} 0.0833 & 0.0648 \\ 0.0648 & 0.0833 \end{pmatrix}, \quad \boldsymbol{R}(0) = \begin{pmatrix} 1 & 0.778 \\ 0.778 & 1 \end{pmatrix},$$

while the matrix of expectations $\boldsymbol{M}$ will be

$$\boldsymbol{M} = \begin{pmatrix} 0.32 & -0.02 \\ 0.245 & -0.045 \end{pmatrix}.$$

The scatter diagram of 100 simulated values is plotted in Fig. 1.

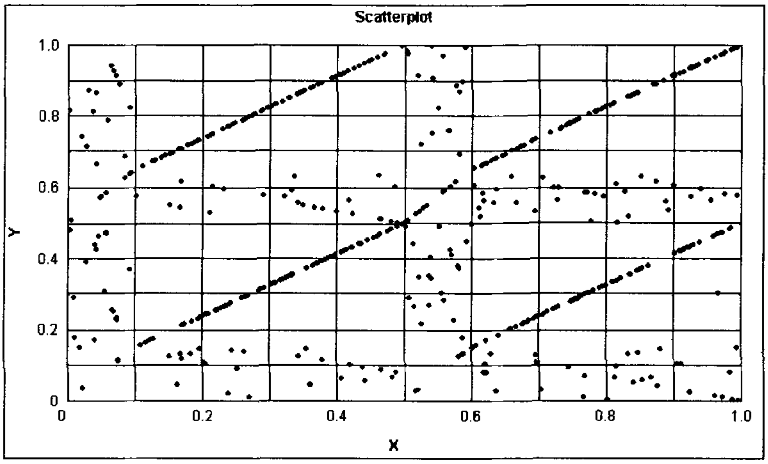

Fig. 1. Scatter plot using a simulated sequence of 100 values.

3. Well defined process

The main result is that the process is well defined meaning that the solution of (2.2) exists. Let us set the equation again:

$$\boldsymbol{Z}_n = \boldsymbol{M}_n \boldsymbol{Z}_{n-1} + \boldsymbol{\varepsilon}_n.$$

We will search for the mean square solution of this equation.

Nicholls and Quinn (1982) have considered the process $\{\boldsymbol{T}_n\}$ defined as

$$\boldsymbol{T}_n = \boldsymbol{M}_n \boldsymbol{T}_{n-1} + \boldsymbol{\eta}_n,$$

where the processes $\{\boldsymbol{M}_n\}$ and $\{\boldsymbol{\eta}_n\}$ are independent and $E(\boldsymbol{M}_n) = \boldsymbol{0}$, $E(\boldsymbol{\eta}_n) = \boldsymbol{0}$ for all n. The process which we have just defined above can be translated by $\boldsymbol{T}_n = \boldsymbol{Z}_n - 1/2(1,1)'$. In this way, the expectations will become zeros, but the independence of the sequences will be disturbed. So, Theorem 2.2 from Nicholls and Quinn ((1982), p. 21) can't be applied directly. The same is with two theorems (3.1 and 3.2) set by Andel (1991).

By applying the backward shift operator to equation (2.2) we have

$$\boldsymbol{Z}_n = \boldsymbol{Q}_k \boldsymbol{Z}_{n-k-1} + \sum_{j=0}^{k} \boldsymbol{Q}_{j-1} \boldsymbol{\varepsilon}_{n-j},$$

where $\boldsymbol{Q}_j = \prod_{r=0}^{j} \boldsymbol{M}_{n-r}$ and $\boldsymbol{Q}_{-1} = \boldsymbol{I}_2$. We will prove immediately that the right side of the equation

$$\boldsymbol{Z}_n - \sum_{j=0}^{k} \boldsymbol{Q}_{j-1} \boldsymbol{\varepsilon}_{n-j} = \boldsymbol{Q}_k \boldsymbol{Z}_{n-k-1}$$

converges in mean square to the zero matrix.

As

$$\operatorname{vec} E \left(\boldsymbol{Z}_n - \sum_{j=0}^{k} \boldsymbol{Q}_{j-1} \boldsymbol{\varepsilon}_{n-j} \right)^2 = \Lambda^{k+1} \operatorname{vec} \Gamma(0),$$

where $\Lambda = E(\boldsymbol{M}_n \otimes \boldsymbol{M}_n)$ and the eigenvalues of the matrix Λ are less than one in modulus, it will be (2.2) has the mean square solution

$$\boldsymbol{W}_n = \sum_{j=0}^{\infty} \boldsymbol{Q}_{j-1}\boldsymbol{\varepsilon}_{n-j}. \tag{3.1}$$

The solution is strictly stationary for, it is of the same form for each n. If we set $\mathcal{F}_n$ be σ-field generated by the set $\{(\boldsymbol{M}_m, \boldsymbol{\varepsilon}_m), m \leq n\}$, then it can be seen that the solution (3.1) is $\mathcal{F}_n$-measurable. As the solution is $\mathcal{F}_n$-measurable on the ergodic set $\{(\boldsymbol{M}_m, \boldsymbol{\varepsilon}_m), m \leq n\}$, it will be ergodic also (Doob (1953), p. 458). Finally, we shall prove that the solution is a wide sense stationary process.

As the eigenvalues of the matrices Λ, $\boldsymbol{M} \otimes \boldsymbol{I}_2$ and $\boldsymbol{I}_2 \otimes \boldsymbol{M}$ are less than one in modulus, it follows that

$$\begin{aligned} \operatorname{vec}\operatorname{Var}(\boldsymbol{W}_n) &= (\boldsymbol{I}_4 - \Lambda)^{-1}[(\boldsymbol{I}_4 - \boldsymbol{M} \otimes \boldsymbol{I}_2)^{-1} - \boldsymbol{I}_4]\operatorname{vec}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') \\ &\quad + (\boldsymbol{I}_4 - \Lambda)^{-1}[(\boldsymbol{I}_4 - \boldsymbol{I}_2 \otimes \boldsymbol{M})^{-1} - \boldsymbol{I}_4]\operatorname{vec}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') \\ &\quad + (\boldsymbol{I}_4 - \Lambda)^{-1}\operatorname{vec}\operatorname{Var}(\boldsymbol{\varepsilon}_n), \end{aligned}$$

where $\boldsymbol{\varepsilon} = E(\boldsymbol{\varepsilon}_n)$. The components of the vector $\operatorname{vec}\operatorname{Var}(\boldsymbol{W}_n)$ are all finite, so that the elements of the vector $\operatorname{Var}(\boldsymbol{W}_n)$ are finite also. This proves the stationarity of the solution.

This solution is also unique almost surely.

We have just proved the following theorem:

THEOREM 3.1. *The mean square solution of the equation* (2.2) *is unique*, $\mathcal{F}_n$-*measurable*, *strictly*, *ergodic and also wide sense stationary*, *ergodic and its explicit form is*

$$\boldsymbol{W}_n = \sum_{j=0}^{\infty} \boldsymbol{Q}_{j-1}\boldsymbol{\varepsilon}_{n-j},$$

where $\boldsymbol{Q}_j = \prod_{r=0}^{j} \boldsymbol{M}_{n-r}$.

4. Estimation of the unknown parameters

Consistent estimates of the parameters of the model are obtained from the definition of the model itself. If we have the sample $(\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_N)$ from the only one realization of the process, we can use the fact

$$\begin{aligned} \beta_1(Y_{n-1} - 1) \leq X_n \leq \alpha_1(X_{n-1} - 1) + 1, \\ \beta_2(Y_{n-1} - 1) \leq Y_n \leq \alpha_2(X_{n-1} - 1) + 1, \end{aligned}$$

and conclude that

$$\hat{\alpha}_{1N} = \min_{1\leq n\leq N}\left\{\frac{X_n - 1}{X_{n-1} - 1}\right\}, \qquad \hat{\beta}_{1N} = \max_{1\leq n\leq N}\left\{\frac{X_n}{Y_{n-1} - 1}\right\}$$

$$\hat{\alpha}_{2N} = \min_{1\leq n\leq N}\left\{\frac{Y_n - 1}{X_{n-1} - 1}\right\}, \qquad \hat{\beta}_{2N} = \max_{1\leq n\leq N}\left\{\frac{Y_n}{Y_{n-1} - 1}\right\}.$$

Now, we shall prove that the obtained estimates are consistent. Let $G_{1N}(x) = P\{\alpha_{1N} \geq x\}$ be the survival function of the estimate α_{1N}. After some simple calculations, we obtain that

$$G_{1N}(x) = 1, \qquad \text{for} \quad x \leq \alpha_1,$$

and

$$G_{1N}(x) \leq (1-\alpha_1)^{N-1} \to 0, \qquad N \to \infty, \qquad \text{for} \quad x > \alpha_1.$$

This implies that

$$G_{1N}(x) \to G_1(x) = \begin{cases} 1, & x \leq \alpha_1, \\ 0, & x > \alpha_1, \end{cases}$$

and it means that α_{1N} converges in probability to α_1. In the quite same way we can prove the consistency of all three other estimates. This completes the proof for the consistency of the proposed estimates.

This method gives the exact values of the parameters for relatively small number of observations ($N = 17$ for Example 1). In fact, the exact values of the parameters can be obtained after a random number of observations for the model. The same results were obtained by Gaver and Lewis (1980) when the first-order autoregressive gamma process had been discussed.

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References

Andel, J. (1991). On stationarity of a multiple doubly stochastic model, *Kybernetika*, **27**, 114–119.

Brockwell, P. and Davis, R. (1987). *Time Series: Theory and Methods*, Springer-Verlag, New York.

Chernick, M. R. (1981). A limit theorem for the maximum of autoregressive processes with uniform marginal distributions, *Ann. Probab.*, **9**, 145–149.

Chernick, M. R. and Davis, R. A. (1982). Extremes in autoregressive processes with uniform marginal distributions, *Statist. Probab. Lett.*, **1**, 85–88.

Dewald, L. S., Lewis, P. A. W. and McKenzie, E. (1989). A bivariate first-order autoregressive time series model in exponential variables ($BEAR(1)$), *Management Science*, **35**(10), 1236–1246.

Doob, J. L. (1953). *Stochastic Processes*, Wiley, New York.

Gaver, D. P. and Lewis, P. A. W. (1980). First-order autoregressive gamma sequences and point processes, *Adv. Appl. Prob.*, **12**, 727–745.

Lawrance, A. J. (1992). Uniformly distributed first-order autoregressive time series models and multiplicative congruential random number generators, *J. Appl. Probab.*, **29**, 896–903.

Nicholls, D. F. and Quinn, B. G. (1982). *Random Coefficient Autoregressive Models: An Introduction*, Springer-Verlag, Heidelberg.

Ristić, M. M. and Popović, B. Č. (2000*a*). A new uniform AR(1) time series model ($NUAR(1)$), *Publ. de L' Inst. Math.-Beograd*, **68**(82), 145–152.

Ristić, M. M. and Popović, B. Č. (2000*b*). Parameter estimation for uniform autoregressive processes, *Novi Sad J. Math.*, **30**(1), 89–95.