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EDGEWORTH EXPANSION IN CENSORED LINEAR REGRESSION MODEL

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Abstract. For the censored simple linear regression model, we establish a one-term Edgeworth expansion for the Koul, Susarla and Van Ryzin type estimator of the regression coefficient. Our approach is to represent the estimator of the regression coefficient as an asymptotic U-statistic plus some ignorable terms and hence apply the known results on the Edgeworth expansions for asymptotic U-statistic. The counting process and martingale techniques are used to provide the proof of the main results.

Key words and phrases: Censored data, regression, martingale, asymptotic U-statistic, Edgeworth expansion.

1. Introduction

Suppose that in the simple linear regression model

$$Y_i = X_i\beta + e_i,$$

Y_i are not completely observable and the observations are (X_i, Z_i, δ_i), $i = 1, \dots, n$, where β is one-dimensional unknown parameter, X_i's are observable covariates, $Z_i = \min(Y_i, C_i)$, $\delta_i = I(Y_i \le C_i)$, the residual e_i's are i.i.d. r.v.'s with mean zero and finite variance σ_0^2 and the (C_i, X_i)'s are i.i.d. random vectors that are independent of e_i's.

The study on censored linear regression model has received considerable attention in the statistical literature. There are two main trends in this body of literature: one trend is to extend the least squares method (LSE) in the complete data case to the incomplete data case. For instances, Buckley and James (1979) introduced an adaptive approach to estimate β, their method has become a very influential method in biostatistics, astronomy, and econometrics. Koul, Susarla and Van Ryzin (hereafter abbreviated KSV) (1981) suggested another method. They defined a transformed data

$$Y_{iG} = \frac{\delta_i Z_i}{1 - G(Z_i)}, \qquad i = 1, \dots, n,$$

where G denotes the distribution of censored variable C, and noticed that $E(Y_{iG} \mid X_i) = X_i\beta$. Hence, if G is known, then one can use the ordinary least square estimator to obtain

$$\beta_G = \left(\sum_{i=1}^{n} X_i^2\right)^{-1} \sum_{i=1}^{n} X_i Y_{iG}.$$

When G is unknown, replacing G by Kaplan-Meier estimator G_n yields

$$\widehat{\beta}_n = \left(\sum_{i=1}^{n} X_i^2\right)^{-1} \sum_{i=1}^{n} X_i Y_{iG_n}, \quad \text{where} \quad Y_{iG_n} = \frac{\delta_i Z_i}{1 - G_n(Z_i)}.$$

In their 1981 paper, KSV used a U-statistic representation to derive the asymptotic distribution of the estimator. This approach to estimating β was subsequently refined and extended by Leurgans (1987) and Zheng (1984). Srinivasan and Zhou (1991) and Zhou (1992) used the martingale structure of the counting processes associated with the underlying problem and developed a martingale representation for the estimator and derived the asymptotic normality of the estimator. Lai *et al.* (1995) developed a general asymptotic distribution theory for the estimators defined by estimating equations. This general theory was used to establish asymptotic normality of synthetic LSE in censored regression models. Recently Qin and Jing (2001) developed an empirical likelihood procedure for the construction of confidence interval for the parameter β in censored regression models. The other trend has been to extend robust estimators to incomplete data settings. For example, Tsiatis (1990), Lai and Ying (1992), Zhou (1992), Ying (1993), Ritov (1990), Lai and Ying (1994) among others.

It is well known that the convergence rate of the distribution $n^{1/2}(\widehat{\beta}_n - \beta)$ to normality is of order $n^{-1/2}$. In this paper, we shall study some higher-order approximations to the distribution of the standardized version of $\widehat{\beta}_n$. In particular, we shall establish a one-term Edgeworth expansion for the KSV type estimator $\widehat{\beta}_n$ of β when β is one dimensional and (Y_i, X_i, C_i)'s are i.i.d. random vectors. Our approach is to approximate β_n by a U-statistic plus some negligible term and then apply the known result of the Edgeworth expansion for asymptotic U-statistic (e.g., Lai and Wang (1993)) to obtain the desired expansion. In the proof, we shall use the counting process and martingales techniques.

Another, perhaps the most popular, approach to regression problem is the Cox model and partial likelihood analysis. Gu (1992) established the one-term Edgeworth expansion for the parameter estimator in Cox model. For linear regression model in complete data case, the Edgeworth expansion for the parametric estimator can be found in monograph by Hall (1992). Finally, it should be pointed that the method in our paper can be used to derive Edgeworth expansion for each component estimator of the parameter vector in censored multiple regression model under similar censoring settings.

The layout of the paper is as follows. Section 2 gives the main results of the paper. Section 3 contains the proof of the main result. In Section 4, we give the proofs of some technical lemmas needed in Section 3.

2. Main results

2.1 *Notations and preliminaries*

Let

$$\begin{aligned} F(t) &= P(Y_i < t), & \tau_F &= \inf\{t : F(t) = 1\}, \\ G(t) &= P(C_i < t), & \tau_G &= \inf\{t : G(t) = 1\}, \\ p(t) &= P(Z_i \geq t), & \tau &= \inf\{t : p(t) = 0\}. \end{aligned}$$

Throughout this paper, we suppose that F and G are continuous and $\tau_F < \tau_G$. For any $K(t)$, we write $\overline{K}(t) = 1 - K(t)$. Clearly, since $p(t) = \overline{F}(t)\overline{G}(t)$, it follows that $\tau = \tau_F$.

Furthermore, let

$$w_i(s) = I(Z_i \ge s) - p(s), \qquad L_i(s) = I(Z_i \ge s),$$
$$L(s) = \sum_{i=1}^{n} L_i(s) = \sum_{i=1}^{n} w_i(s) + np(s),$$
$$\Lambda(s) = \int_{-\infty}^{s} (1 - G(t))^{-1} dG(t),$$
$$M_i(s) = I(Z_i \le s, \delta_i = 0) - \int_{-\infty}^{s} I(C_i \ge t, Y_i > t) d\Lambda(t),$$
$$N(s) = \sum_{i=1}^{n} I(Z_i \le s, \delta_i = 0).$$

It is well known that $\{M_i(s), -\infty < s < \infty\}$ is a square integrable martingale with respect to the filtration $\mathcal{F}_s = \sigma\{Z_k I(Z_k \le s), \delta_k I(Z_k \le s), 1 \le k \le n\}$ and its predictable variation process is $\langle M_i \rangle(s) = \int_{-\infty}^{s} I(C_i \ge t, Y_i > t) d\Lambda(t)$. The Kaplan-Meier estimator of G is given by

$$G_n(t) = 1 - \prod_{s \le t} (1 - \Delta N(s))/L(s).$$

Some further notation will be needed before we introduce the main results. Denote $a_1 \wedge a_2 = \min(a_1, a_2)$, $a_1 \vee a_2 = \max(a_1, a_2)$. Also we define

$$W_n = n^{-1} \sum_{i=1}^{n} X_i^2,$$
$$B_i(t) = X_i Y_{iG} I(Z_i > t), \qquad B(t) = EB_i(t),$$
$$B_{i0}(t) = B_i(t) - B(t), \qquad \overline{B}_0(t) = n^{-1} \sum_{i=1}^{n} B_{i0}(t),$$
$$p_1(s) = P(Z_i \le s, \delta_i = 0), \qquad g(s) = \int_{-\infty}^{s} p^{-2}(t) dp_1(t),$$
$$\mu_i(s) = -g(Z_i \wedge s) + p^{-1}(Z_i) I(Z_i \le s, \delta_i = 0),$$
$$\xi_i(s) = (1 - G(s))\mu_i(s), \qquad i = 1, \ldots, n,$$
$$\widehat{\alpha}_i = \int_{-\infty}^{\tau} \frac{B(t)}{p(t)} dM_i(t) + X_i(Y_{iG} - X_i\beta),$$
$$\begin{aligned}\widehat{\beta}_{ij} = &\int_{-\infty}^{\tau} \frac{1}{p(t)} [B_{j0}(t) dM_i(t) + B_{i0}(t) dM_j(t)] \\ &+ \int_{-\infty}^{\tau} \frac{B(t)}{p(t)(G(t) - 1)} [\xi_j(t) dM_i(t) + \xi_i(t) dM_j(t)] \\ &- \int_{-\infty}^{\tau} \frac{B(t)}{p^2(t)} [w_j(t) dM_i(t) + w_i(t) dM_j(t)] \\ &+ 2\int_{-\infty}^{\tau} \int_{-\infty}^{\tau} \frac{B(t \vee s)}{p(t)p(s)} dM_i(t) dM_j(s).\end{aligned}$$

Finally we define

$$\sigma^2 = E[\widehat{\alpha}_1^2], \tag{2.1}$$

$$\kappa_3 = E[\widehat{\alpha}_1^3] + 3E[\widehat{\alpha}_1\widehat{\alpha}_2\widehat{\beta}_{12}],$$
$$b = \sigma^{-1}\int_{-\infty}^{\tau} \frac{B^2(t)}{p(t)} \frac{dG(t)}{1-G(t)},$$

and also define σ_n^2 and κ_{3n} to the corresponding σ^2 and κ_3 except that the upper limit τ in the integrals of $\widehat{\alpha}_i$ and $\widehat{\beta}_{ij}$ are replaced by u_n.

2.2 *Main results*

In this section, we are interested in obtaining Edgeworth expansions for the distribution of the standardized slope estimator $\widehat{\beta}_n$. First problem here is how to choose the normalizing factor. From (2.1), it follows that

$$\sigma^2 = E[X_1(Y_{1G} - X_1\beta)]^2 - \int_{-\infty}^{\tau} \frac{B^2(t)}{p(t)} \frac{dG(t)}{1-G(t)}. \tag{2.2}$$

See Theorem 2 of Lai *et al.* (1995). As in Lai *et al.* (1995), for technical reasons, we need to truncate the domain of the integral in σ^2 (and other quantities involving such integrals) to a fixed sequence u_n with $u_n < \tau$. To be more specific, let us define

$$\sigma_n^2 = E[X_1(Y_{1G} - X_1\beta)]^2 - \int_{-\infty}^{u_n} \frac{B^2(t)}{p(t)} \frac{dG(t)}{1-G(t)}.$$

Then in this paper, we are interested in deriving Edgeworth expansions for the following standardized slope estimator $\widehat{\beta}_n$,

$$T_n = \frac{n^{1/2}W_n(\widehat{\beta}_n - \beta)}{\sigma_n}.$$

Before stating our main results, we shall make the following assumptions:

(A1): $|\kappa_3| < \infty$.

(A2): For every $\epsilon > 0$, there exists a $u_n < \tau$ such that for all large n,

$$\int_{R^1}\int_{u_n}^{\tau} \frac{n|xy|dJ(x,y)}{\overline{F}^{1/2}(y)} < \epsilon,$$

where $J(x,y) = P(X_i < x, Y_i < y)$.

(A3): $\int_{-\infty}^{\tau} \frac{dG(t)}{\overline{F}^8(t)} < \infty$.

Assumption (A2) imposing condition on the distributions J and F near the tail of F is similar to but stronger than the condition (C3) of Lai *et al.* (1995). Assumption (A3) is used to control the behavior of Kaplan-Meier estimator G_n of G near the tail of the distribution, similar conditions (see the conditions (1.8) and (2.2) in Chen and Lo (1997)) have been assumed to obtain the weak and strong convergence of Kaplan-Meier estimator in Gu and Lai (1990), and Chen and Lo (1997). Therefore, it should come at no surprise to us that conditions required for Edgeworth expansions in this paper are stronger than those for the asymptotic normality as was done in Lai *et al.* (1995).

THEOREM 2.1. *Assume that* (A1)–(A3) *hold. Then we have*

$$P(T_{1n} \le z) = \Phi(z) - n^{-1/2}\phi(z)P_1(z) + o(n^{-1/2})$$

uniformly in z, *where* $P_1(z) = \frac{1}{6}\kappa_3\sigma^{-3}(z^2-1) + b$.

Remark 1. Notice that the form of the Edgeworth expansion in Theorem 2.1 looks similar to that in the case of the smooth function of independent sample means. The term κ_3/σ^3 in $P_1(z)$ corrects for the skewness of the distribution. On the other hand, the term b in $P_1(z)$ corrects for the bias in variance σ^2 due to the presence of censoring; when there is no censoring in observations (i.e., $G(t) = 0$ for all $t < \tau$), $b = 0$, this bias effect will disappear.

Remark 2. From Theorem 2.1, we can easily obtain the following Edgeworth expansion:

$$P\left(\frac{n^{1/2}W_n(\widehat{\beta}_n - \beta)}{\sigma} \le z\right) = \Phi(z) - n^{-1/2}\phi(z)P_2(z) + o(n^{-1/2}),$$

where $P_2(z) = P_1(z) + n^{1/2}(\sigma_n - \sigma)/\sigma$.

3. The proof of Theorem 2.1

First we introduce two lemmas that are needed to prove the theorem, whose proof will be deferred to the Appendix. Let

$$\overline{B}(t) = n^{-1}\sum_{i=1}^{n} B_i(t), \qquad M(s) = \sum_{i=1}^{n} M_i(s),$$

$$D(t) = \frac{1 - G_n(t-)}{1 - G(t)}, \qquad \overline{w}(t) = n^{-1}\sum_{i=1}^{n} w_i(t),$$

$$A_{n1}(t) = \overline{B}(t)D(t), \qquad A_{n2}(t,s) = \overline{B}(t \vee s)D(t)D(s).$$

LEMMA 3.1. *Under the conditions of Theorem* 2.1, *we have*

$$n^{1/2}W_n(\widehat{\beta}_n - \beta) = U_n + \gamma_n + R_n,$$

where $\gamma_n = n^{-1/2}\int_{-\infty}^{u_n} \frac{B^2(t)}{p(t)}\frac{dG(t)}{1-G(t)}$, U_n *is a* U-*statistic defined by*

$$\begin{aligned}
U_n = {} & n^{-1/2}\sum_{i=1}^{n}\left(\int_{-\infty}^{u_n}\frac{B(t)}{p(t)}dM_i(t) + X_i(Y_{iG} - X_i\beta)\right) \\
& + n^{-3/2}\sum_{1\le i<j\le n}\left[\int_{-\infty}^{u_n}\left(\frac{B_{j0}(t)}{p(t)}dM_i(t) + \frac{B_{i0}(t)}{p(t)}dM_j(t)\right)\right. \\
& \qquad + \int_{-\infty}^{u_n}\frac{B(t)}{p(t)(G(t)-1)}(\xi_j(t)dM_i(t) + \xi_i(t)dM_j(t)) \\
& \qquad - \int_{-\infty}^{u_n}\frac{B(t)}{p^2(t)}(w_i(t)dM_j(t) + w_j(t)dM_i(t)) \\
& \qquad \left. + 2\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)}{p(t)p(s)}dM_i(t)dM_j(s)\right],
\end{aligned}$$

and R_n *is the error term given by*

$$R_n = R_1 + R_2 + R_3 + R_4 + R_5,$$

$$\begin{aligned}
R_1 &= n^{-1/2} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{u_n}^{Z_i} X_i Y_{iG} \frac{D(t)}{L(t)} dM_j(t) \\
&\quad + n^{-3/2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\int_{u_n}^{Z_i} \int_{u_n}^{Z_i} + 2 \int_{-\infty}^{u_n} \int_{u_n}^{Z_i} \right) A_{n2}(t,s) \frac{dM_j(t) dM_k(s)}{n^{-2} L(t) L(s)} \\
&\quad + n^{-1/2} \sum_{i=1}^{n} X_i Y_{iG} \frac{G_n(Z_i) - G(Z_i)}{1 - G(Z_i)} Q\left(\frac{G_n(Z_i) - G(Z_i)}{1 - G(Z_i)} \right) \\
&\qquad\qquad (\textit{where} \quad Q(x) = x^2/(1-x)), \\
R_2 &= n^{-1/2} \sum_{j=1}^{n} \int_{-\infty}^{u_n} \frac{A_{n1}(t) \overline{w}^2(t)}{(p(t) + \theta_n(t) \overline{w}(t))^3} dM_j(t) \\
&\quad - n^{-3/2} \sum_{j=1}^{n} \int_{-\infty}^{u_n} \frac{A_{n1}(t)}{p^2(t)} w_j(t) dM_j(t) \qquad (\textit{where} \quad 0 \le \theta_n(t) \le 1), \\
R_3 &= n^{-1/2} \sum_{i=1}^{n} \int_{-\infty}^{u_n} \frac{\overline{B}_0(t)[D(t) - 1]}{p(t)} dM_i(t) \\
&\quad - n^{-3/2} \sum_{i \ne j} \int_{-\infty}^{u_n} \frac{A_{n1}(t) - B(t)}{p^2(t)} w_i(t) dM_j(t), \\
R_4 &= n^{-3/2} \sum_{i=1}^{n} \int_{-\infty}^{u_n} \frac{B_{i0}(t)}{p(t)} dM_i(t) + n^{-1/2} \sum_{i=1}^{n} \int_{-\infty}^{u_n} \frac{B(t) r_n(t)}{p(t)(G(t) - 1)} dM_i(t) \\
&\quad + n^{-3/2} \sum_{i=1}^{n} \int_{-\infty}^{u_n} \frac{B(t) \xi_i(t) dM_i(t)}{p(t)(G(t) - 1)} \\
&\qquad\qquad (\textit{where} \quad r_n(t) = G_n(t) - G(t) - \overline{\xi}(t)), \\
R_5 &= n^{-3/2} \sum_{j=1}^{n} \left[\int_{-\infty}^{u_n} \int_{-\infty}^{u_n} \frac{B(t \vee s)}{p(t)p(s)} dM_j(t) dM_j(s) \right. \\
&\qquad \left. - E\left(\int_{-\infty}^{u_n} \int_{-\infty}^{u_n} \frac{B(t \vee s)}{p(t)p(s)} dM_j(t) dM_j(s) \right) \right] \\
&\quad + n^{-3/2} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{u_n} \int_{-\infty}^{u_n} \frac{A_{n2}(t,s) - B(t \vee s)}{p(t)p(s)} dM_j(t) dM_k(s) \\
&\quad + n^{-3/2} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{u_n} \int_{-\infty}^{u_n} \frac{A_{n2}(t,s) r_n(t,s)}{p(t)p(s)} dM_j(t) dM_k(s) \\
&\qquad\qquad (\textit{where} \quad r_n(t,s) = p(t)p(s)/((p(t) + \overline{w}(t))(p(s) + \overline{w}(s))) - 1).
\end{aligned} \tag{3.1}$$

LEMMA 3.2. $P(|R_j| \ge o(n^{-1/2})) = o(n^{-1/2})$ *for* $j = 1, 2, 3, 4, 5$.

PROOF OF THEOREM 2.1. From Lemma 3.2, it follows that

$$P(|R_n| \geq o(n^{-1/2})) = o(n^{-1/2}). \tag{3.2}$$

By Lemma 3.1, $n^{1/2}W_n(\widehat{\beta}_n - \beta)$ can be expressed as a U-statistic U_n plus a constant term γ_n and a negligible remainder term R_n, therefore we can apply the results on Edgeworth expansion for asymptotic U-statistic in Lai and Wang (1993) to derive the Edgeworth expansion for T_{1n}. Now we shall modify the arguments in the proof of Theorem 1 in Lai and Wang (1993), where their condition (C) can be verified by using the arguments similar to those on pp. 523–pp. 524 of that paper to show that

$$P\left(\frac{U_n + \gamma_n}{\sigma_n} \leq z\right) = P\left(\frac{U_n}{\sigma_n} \leq z - \frac{\gamma_n}{\sigma_n}\right) = K_n\left(z - \frac{\gamma_n}{\sigma_n}\right) + o\left(n^{-1/2}\right), \tag{3.3}$$

uniformly for z, where

$$K_n(x) = \Phi(x) - \frac{1}{6}n^{-1/2}\phi(x)\kappa_{3n}\sigma_n^{-3}(x^2 - 1),$$

κ_{3n} is κ_3 defined in Theorem 2.1 except for τ being replaced by u_n. In fact, tracing the proof of Theorem 1 of Lai and Wang (1993) (hereafter abbreviated LW), and taking $T = n^{(r-1)/r}(\log n)^{-1}$ for $4 \geq r > 2$ in (4.1) of LW and $F_n(z) = P(U_n/\sigma_n \leq z - \gamma_n/\sigma_n)$, $G_n(z) = K_n(z - \gamma_n/\sigma_n)$ in (4.2) of LW, it can be shown that the rates of the right side of (4.4) and (4.5) in LW are $o(n^{-1/2})$. Then combining (4.3) and (4.1) of LW imply (3.3) above. Also noticing $u_n = \tau + o(1)$, $\sigma_n \to \sigma$, $\sqrt{n}\gamma_n \to b\sigma$ and $\kappa_{3n} \to \kappa_3$, then from (3.2) above and the following inequlity (see Serfling (1980))

$$\sup_x |P(X + Y \leq x) - K_{n1}(x)| \leq \sup_x |P(X \leq x) - K_{n1}(x)| + ba + P(|Y| > a),$$

where X, Y are r.v.'s, a, b are constants, and $K_{n1}(z) = \Phi(z) - n^{-1/2}\phi(z)P_1(z)$, it follows that

$$\begin{aligned}
&\sup_z \left| P\left(\frac{n^{1/2}W_n(\widehat{\beta}_n - \beta)}{\sigma_n} \leq z\right) - K_{n1}(z)\right| \\
&\leq \sup_z \left| P\left(\frac{U_n + \gamma_n}{\sigma_n} \leq z\right) - K_{n1}(z)\right| + o(n^{-1/2}) \\
&\leq \sup_z \left| K_n\left(z - \frac{\gamma_n}{\sigma_n}\right) - K_{n1}(z)\right| + o(n^{-1/2}) \\
&\leq \sup_z \left| K_n\left(z - \frac{\gamma_n}{\sigma_n}\right) - K_n(z) + n^{-1/2}\phi(z)b\right| \\
&\quad + \sup_z \left| K_n(z) - \left(\Phi(z) - n^{-1/2}\phi(z)\frac{\kappa_3}{6\sigma^3}(z^2 - 1)\right)\right| + o(n^{-1/2}) \\
&= o(n^{-1/2}).
\end{aligned}$$

Hence the proof is completed.

Appendix

In order to prove Lemmas 3.1 and 3.2, we need the following lemma.

LEMMA A.1. *Let* $\bar{\xi}(u) = \frac{1}{n}\sum_{i=1}^{n} \xi_i(u)$, *we have for all* $u < \tau$
(i) $G_n(u) = G(u) + \bar{\xi}(u) + r_n(u)$,
(ii) $E[\mu_i(u)] = 0$, $\mathrm{Cov}(\mu_i(u), \mu_i(v)) = g(u \wedge v)$,
(iii) $E|r_n(u)|^k = O((n^{-1}p^{-3}(u)\log n)^k)$ *for* $k \geq 1$.

PROOF OF LEMMA A.1. (i) and (ii) can be found in Lo *et al.* (1989). We shall only prove (iii) for the case $k = 1$. From the proof of Lemma 2.1 in Lo *et al.* (1989), we have

$$E|r_n(u)| \leq (1 - G(u))E|R_n(u)| + 2E(\bar{\xi}(u))^2 + 2E(R_n(u))^2,$$

where

$$R_n(u) = R_{n1}(u) + R_{n2}(u) + R_{n3}(u),$$

$R_{n1}(u)$, $R_{n2}(u)$, $R_{n3}(u)$ are defined in Lo *et al.* (1989).

Tracing the proof of Theorem 1 in Lo and Singh (1986), it can be shown that for any $b > 0$,

$$P(|R_{n1}(u)| > C_b n^{-1}p^{-2}(u)\log n) = O(n^{-b}),$$
$$P(|R_{n2}(u)| > C_b n^{-1}p^{-3}(u)\log n) = O(n^{-b}),$$

where C_b is a constant depending only on b. An application of the lemma in Burke *et al.* (1988) leads to

$$P(|R_{n3}(u)| > C_b n^{-1}p^{-2}(u)\log n) = O(n^{-b}),$$

so

$$E|R_n(u)| = O(n^{-1}p^{-3}(u)\log n).$$

Similarly

$$E|R_n(u)|^2 = O(n^{-2}p^{-6}(u)\log^2 n) = O(n^{-1}p^{-3}(u)\log n).$$

Now that

$$\begin{aligned}
E\bar{\xi}^2(u) &= n^{-1}\sup_{u \leq u}(1 - G(u))^2 E\mu_1^2(u) \\
&\leq 2n^{-1}[Eg^2(Z_1 \wedge u) + E(p^{-2}(Z_1)I(Z_1 \leq u, \delta_1 = 0))] \\
&= O\left(n^{-1}\left[\left(\int_{-\infty}^{\tau}\frac{G(t)}{\overline{F}^2(t)}\right)^2 + \int_{-\infty}^{\tau}\frac{G(t)}{\overline{F}^2(t)}\right]\right) \\
&= O(n^{-1}),
\end{aligned}$$

hence $E|r_n(u)| = O(n^{-1}p^{-3}(u)\log n)$.

PROOF OF LEMMA 3.1. Let

$$S_{nG_n} = \sum_{i=1}^{n} X_i(Y_{iG_n} - X_i\beta), \qquad S_{nG} = \sum_{i=1}^{n} X_i(Y_{iG} - X_i\beta).$$

Then

$$n^{1/2}W_n(\widehat{\beta}_n - \beta) = n^{-1/2}S_{nG_n} = n^{-1/2}S_{nG} + n^{-1/2}(S_{nG_n} - S_{nG}). \tag{A.1}$$

From Equation (3.2.13) of Gill (1980):

$$\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}=\int_{-\infty}^{Z_i}\frac{D(t)}{L(t)}dM(t)=\int_{-\infty}^{\infty}\frac{D(t)}{L(t)}I(Z_i>t)dM(t).$$

Also we have

$$\begin{aligned}Y_{iG_n}&=\frac{Y_{iG}}{D(t)}=Y_{iG}\left(1-\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right)^{-1}\\&=Y_{iG}\left[1+\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}+Q\left(\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right)\right],\end{aligned}$$

where $Q(x)=x^2/(1-x)$. Therefore,

$$\begin{aligned}&n^{-1/2}(S_{nG_n}-S_{nG})\\&=n^{-1/2}\sum_{i=1}^{n}X_iY_{iG_n}\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\\&=n^{-1/2}\sum_{i=1}^{n}X_iY_{iG}\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\left(1+\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right)\\&\quad+n^{-1/2}\sum_{i=1}^{n}X_iY_{iG}\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}Q\left(\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right)\\&=n^{-1/2}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n}X_iY_{iG}I(Z_i>t)\right)\frac{D(t)}{L(t)}dM_j(t)\\&\quad+n^{-1/2}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n}X_iY_{iG}I(Z_i>t\vee s)\right)\\&\qquad\qquad\times\frac{D(t)D(s)}{L(t)L(s)}dM_j(t)dM_k(s)\\&\quad+n^{-1/2}\sum_{i=1}^{n}X_iY_{iG}\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}Q\left(\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right)\\&=n^{-1/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\frac{A_{n1}(t)}{n^{-1}L(t)}dM_j(t)\\&\quad+n^{-3/2}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}A_{n2}(t,s)\frac{dM_j(t)dM_k(s)}{n^{-2}L(t)L(s)}+R_1\\&\equiv J_{n1}+J_{n2}+R_1,\end{aligned}\tag{A.2}$$

where u_n is a sequence of real numbers chosen as in Assumption (A2).

Note that

$$\begin{aligned}&n^{-1}L(t)=p(t)+\overline{w}(t),\\&A_{n1}(t)=(B(t)+\overline{B}_0(t))(1+(D(t)-1))\\&\qquad=B(t)+\overline{B}_0(t)+B(t)(D(t)-1)+\overline{B}_0(t)(D(t)-1),\end{aligned}\tag{A.3}$$

$$\begin{aligned}
D(t)-1 &= (G(t)-1)^{-1}(\overline{\xi}(u)+r_n(u)), \qquad \text{(from Lemma A.1)}\\
A_{n2}(t,s) &= \overline{B}(t\vee s)D(t)D(s)\\
&\equiv B(t\vee s)+R_A(t,s).
\end{aligned}$$

Then we can apply Taylor expansion to $(p(t)+x)^{-1}$ around 0 to further decompose J_{n1} and J_{n2} as

$$\begin{aligned}
\text{(A.4)}\quad J_{n1} &= n^{-1/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\frac{A_{n1}(t)dM_j(t)}{p(t)+\overline{w}(t)}\\
&= n^{-1/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\frac{A_{n1}(t)dM_j(t)}{p(t)}\\
&\quad - n^{-3/2}\sum_{1\le i<j\le n}\int_{-\infty}^{u_n}\frac{A_{n1}(t)}{p^2(t)}[w_i(t)dM_j(t)+w_j(t)dM_i(t)]+R_2\\
&= n^{-1/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B(t)}{p(t)}dM_i(t)+n^{-1/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)}{p(t)}dM_i(t)\\
&\quad + n^{-1/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B(t)[D(t)-1]}{p(t)}dM_i(t)\\
&\quad - n^{-3/2}\sum_{1\le i<j\le n}\int_{-\infty}^{u_n}\frac{B(t)}{p^2(t)}[w_i(t)dM_j(t)+w_j(t)dM_i(t)]+R_2+R_3\\
&= n^{-1/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B(t)}{p(t)}dM_i(t)\\
&\quad + n^{-3/2}\sum_{1\le i<j\le n}\int_{-\infty}^{u_n}\frac{1}{p(t)}[B_{i0}(t)dM_j(t)+B_{j0}(t)dM_i(t)]\\
&\quad + n^{-3/2}\sum_{1\le i<j\le n}\int_{-\infty}^{u_n}\frac{B(t)}{p(t)(G(t)-1)}(\xi_j(t)dM_i(t)+\xi_i(t)dM_j(t))\\
&\quad - n^{-3/2}\sum_{1\le i<j\le n}\int_{-\infty}^{u_n}\frac{B(t)}{p^2(t)}[w_i(t)dM_j(t)+w_j(t)dM_i(t)]\\
&\quad + R_2+R_3+R_4
\end{aligned}$$

and

$$\begin{aligned}
\text{(A.5)}\quad J_{n2} &= n^{-3/2}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{A_{n2}(t,s)}{(p(t)+\overline{w}(t))(p(s)+\overline{w}(s))}dM_j(t)dM_k(s)\\
&= n^{-3/2}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)+R_A(t,s)}{p(t)p(s)}(1+r_n(t,s))dM_j(t)dM_k(s)\\
&= n^{-1/2}E\left(\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)}{p(t)p(s)}dM_j(t)dM_j(s)\right)\\
&\quad + 2n^{-3/2}\sum_{1\le j<k\le n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)}{p(t)p(s)}dM_j(t)dM_k(s)+R_5
\end{aligned}$$

$$= \gamma_n + 2n^{-3/2} \sum_{1\le j<k\le n} \int_{-\infty}^{u_n}\int_{-\infty}^{u_n} \frac{B(t\vee s)}{p(t)p(s)} dM_j(t)dM_k(s) + R_5.$$

Therefore, Lemma 3.1 follows from (A.1), (A.2), (A.4) and (A.5), with

$$\begin{aligned} R_n &= n^{1/2}W_n(\widehat{\beta}_n - \beta) - U_n - \gamma_n \\ &= R_1 + R_2 + R_3 + R_4 + R_5. \end{aligned} \tag{A.6}$$

Proof of Lemma 3.2. (I) We first give the proof of

$$P(|R_1| \ge o(n^{-1/2})) = o(n^{-1/2}). \tag{A.7}$$

Note that

$$R_1 = R_{11} + R_{12} + 2R_{13} + R_{14},$$

where

$$\begin{aligned}
R_{11} &= n^{-1/2}\sum_{i=1}^n X_iY_{iG}\left(\int_{u_n}^{Z_i} \frac{D(t)}{L(t)} dM(t)\right), \\
R_{12} &= n^{-1/2}\sum_{i=1}^n X_iY_{iG}\left(\int_{u_n}^{Z_i} \frac{D(t)}{L(t)} dM(t)\right)^2, \\
R_{13} &= \left(\int_{-\infty}^{u_n} D(t)\frac{dM(t)}{L(t)}\right)\left(n^{-1/2}\sum_{i=1}^n X_iY_{iG}\int_{u_n}^{Z_i} D(s)\frac{dM(s)}{L(s)}\right) \\
&\equiv R_{13}^{(1)} \times R_{13}^{(2)}, \\
R_{14} &= n^{-1/2}\sum_{i=1}^n X_iY_{iG}\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)} Q\left(\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right).
\end{aligned}$$

Note that $\int_{u_n}^{Z_i} \frac{D(t)}{L(t)} dM(t)$ is a martingale. By Assumption (A2), (2.35) in Lai *et al.* (1995) and martingale inequality, for every $\epsilon > 0$, we have

$$\begin{aligned}
&P(|R_{11}| > \epsilon^{1/2}n^{-1/2}) \\
&\le C_0\epsilon^{-1/2}E\left[\sum_{i=1}^n |X_iY_{iG}|(n^{-1/2}\overline{G}^{-1}(Z_i)\overline{F}^{-1/2}(Z_i)I(Z_i > u_n))\right] \\
&\le C_0\epsilon^{1/2}n^{-1/2},
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
&P(|R_{12}| > \epsilon^{1/2}n^{-1/2}) \\
&\le P\left(\max_i \left|\int_{u_n}^{Z_i} \frac{D(t)}{L(t)} dM(t)\right| n^{-1/2}\sum_{i=1}^n |X_iY_{iG}| \left|\int_{u_n}^{Z_i} \frac{D(t)}{L(t)} dM(t)\right| \right. \\
&\qquad\qquad \left. > \epsilon^{1/2}n^{-1/2}\right) \\
&\le P\left(n^{-1/2}\sum_{i=1}^n |X_iY_{iG}| \left|\int_{u_n}^{Z_i} \frac{D(t)}{L(t)} dM(t)\right| > \epsilon^{1/2}m^{-1}n^{-1/2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+P\left(\max_i\left|\int_{u_n}^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|>m\right)\\
&\le C_0\epsilon^{1/2}n^{-1/2}+C_0m^{-1}n^{-1/2}E(\overline{G}^{-1}(Z_{(n)})\overline{F}^{-1/2}(Z_{(n)})I(Z_{(n)}>u_n)),
\end{aligned}
$$

where $m>0$ and $C_0>0$ are generic constants, and $Z_{(n)}=\max_i\{Z_i\}$. Since

$$
E(\overline{G}^{-1}(Z_{(n)})\overline{F}^{-1/2}(Z_{(n)})I(Z_{(n)}>u_n))=\int_{u_n}^{\tau}\frac{d(\overline{p}(t))^n}{\overline{G}(t)\overline{F}^{1/2}(t)}\longrightarrow 0,\qquad \text{as}\quad n\longrightarrow\infty,
$$

so $P(|R_{12}|>\epsilon^{1/2}n^{-1/2})=o(n^{-1/2})$. For R_{13}, we have

$$
P(|R_{13}|>\epsilon^{1/2}n^{-1/2})\le P(|R_{13}|>\epsilon^{1/2}n^{-1/2},|R_{13}^{(1)}|\le m)+P(|R_{13}^{(1)}|>m)
$$

and

$$
\begin{aligned}
P(|R_{13}^{(1)}|>m)&\le m^{-2}E\left\langle\int_{-\infty}^{u_n}\frac{D(t)}{L(t)}dM(t)\right\rangle\\
&=m^{-2}n^{-1}\int_{-\infty}^{u_n}E\left(\frac{D^2(t)}{n^{-1}L(t)}\right)d\Lambda(t)\\
&\le m^{-2}n^{-1}\int_{-\infty}^{u_n}E(nL^{-1}(t))\frac{d\Lambda(t)}{(1-G(t))^2}\\
&=O\left(n^{-1}\int_{-\infty}^{\tau}p^{-1}(t)dG(t)\right)\\
&=O\left(n^{-1}\int_{-\infty}^{\tau}\overline{F}^{-1}(t)dG(t)\right)=o(n^{-1/2}).
\end{aligned}
$$

Note that $R_{13}^{(2)}=R_{11}$. A similar argument to (A.8) leads to

$$
P(|R_{13}|>\epsilon^{1/2}n^{-1/2},|R_{13}^{(1)}|\le m)\le P(|R_{13}^{(2)}|>\epsilon^{1/2}m^{-1}n^{-1/2})\le C_0\epsilon^{1/2}n^{-1/2}.
$$

Therefore, $P(|R_{13}|>o(n^{-1/2}))=o(n^{-1/2})$.

Finally, let us investigate R_{14}. Fixing a $u<\tau$, we can decompose R_{14} into

$$
\begin{aligned}
R_{14}&=n^{-1/2}\sum_{i=1}^n\left(X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}\right)\left(\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right)^3\\
&=n^{-1/2}\sum_{i=1}^n\left(X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}\right)\left(\int_{-\infty}^{u}\frac{D(t)}{L(t)}dM(t)+\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right)^3\\
&=n^{-1/2}\sum_{i=1}^n\left(X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}\right)\left(\frac{G_n(u)-G(u)}{1-G(u)}+\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right)^3\\
&\equiv R_{14}^{(1)}+n^{-1/2}\sum_{i=1}^n\left(X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}\right)\left(\frac{G_n(u)-G(u)}{1-G(u)}\right)^3\\
&\equiv R_{14}^{(1)}+R_{14}^{(2)}+n^{1/2}E(X_1Y_{1G})\left(\frac{G_n(u)-G(u)}{1-G(u)}\right)^3\\
&\equiv R_{14}^{(1)}+R_{14}^{(2)}+R_{14}^{(3)},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{14}^{(1)} &= n^{-1/2}\sum_{i=1}^{n}\left(X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}\right)\\
&\quad\times\left[\left(\frac{G_n(u)-G(u)}{1-G(u)}\right)^3-\left(\frac{G_n(u)-G(u)}{1-G(u)}+\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right)^3\right]\\
R_{14}^{(2)} &= n^{1/2}\left(\frac{G_n(u)-G(u)}{1-G(u)}\right)^3\left(E(X_1Y_{1G})-n^{-1}\sum_{i=1}^{n}X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}\right)\\
R_{14}^{(3)} &= n^{1/2}E(X_1Y_{1G})\left(\frac{G_n(u)-G(u)}{1-G(u)}\right)^3.
\end{aligned}
$$

Using Lemma A.1 and noticing that $E(|r_n(u)|^k) = O((n^{-1}p^{-3}(u)\log n)^k) = O((n^{-1}\log n)^k)$ for fixed $u<\tau$, we can get $P(|R_{14}^{(3)}|>o(n^{-1/2}))=o(n^{-1/2})$. Since $Z_{(n)}\le\tau_F<\tau_G$ a.s. and

$$
\left|\frac{G_n(Z_i)-G(Z_i)}{1-G(Z_i)}\right|\le\sup_{t\le\tau_F}\left|\frac{G_n(t)-G(t)}{1-G(t)}\right|\le\frac{\sup_{t\le\tau_F}|G_n(t)-G(t)|}{1-G(\tau_F)}\longrightarrow 0\qquad\text{a.s.},
$$

we have

$$
n^{-1}\sum_{i=1}^{n}X_iY_{iG}\frac{1-G(Z_i)}{1-G_n(Z_i)}-E(X_1Y_{1G})\longrightarrow 0\qquad\text{a.s.},
$$

then making use of $E[(\frac{G_n(u)-G(u)}{1-G(u)})^3]=O(n^{-3/2})$, it can be shown that

$$
P(|R_{14}^{(2)}|>o(n^{-1/2}))=o(n^{-1/2}).
$$

For the term $R_{14}^{(1)}$, we have

$$
\begin{aligned}
|R_{14}^{(1)}| &\le C_0n^{-1/2}\sum_{i=1}^{n}|X_iY_{iG}|\left[\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|\right.\\
&\qquad\left.+\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|^2+\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|^3\right]\\
&\le C_0n^{1/2}\left[\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|\right.\\
&\qquad\left.+\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|^2+\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|^3\right]\qquad\text{a.s.}
\end{aligned}
$$

For every $\epsilon>0$ and $k=1,2,3$, when n is large enough, we have

$$
E(n^{1/k}\overline{G}^{-1}(Z_{(n)})\overline{F}^{-1/2}(Z_{(n)})I(Z_{(n)}>u))=n^{1/k}\int_u^{\tau}\frac{d(\overline{p}(t))^n}{\overline{G}(t)\overline{F}^{1/2}(t)}\le\epsilon^2,
$$

then applying martingale inequality and (2.35) in Lai *et al.* (1995), we get

$$\begin{aligned}
&P\left(n^{1/2}\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right|^k > \epsilon n^{-1/2}\right)\\
&= P\left(\max_i\left|\int_u^{Z_i}\frac{D(t)}{L(t)}dM(t)\right| > \epsilon^{1/k}n^{-1/k}\right)\\
&\le C_0\epsilon^{-1/k}n^{-1/2}E(n^{1/k}\overline{G}^{-1}(Z_{(n)})\overline{F}^{-1/2}(Z_{(n)})I(Z_{(n)}>u))\\
&\le C_0\epsilon^{2-1/k}n^{-1/2},
\end{aligned}$$

therefore,

$$P(|R_{14}^{(1)}| > (n\log n)^{-1/2}) = o(n^{-1/2}),$$

and $P(|R_{14}| > o(n^{-1/2})) = o(n^{-1/2})$. (A.7) is thus proved.

(II) In this part we show that

$$P(|R_2| \ge o(n^{-1/2})) = o(n^{-1/2}). \tag{A.9}$$

From (A.3), R_2 can be written as

$$\begin{aligned}
R_2 &= n^{-1/2}\sum_{j=1}^n\int_{-\infty}^{u_n}\frac{A_{n1}(t)\overline{w}^2(t)}{(p(t)+\theta_n(t)\overline{w}(t))^3}dM_j(t) - n^{-3/2}\sum_{j=1}^n\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)}{p^2(t)}w_j(t)dM_j(t)\\
&\quad - n^{-3/2}\sum_{j=1}^n\int_{-\infty}^{u_n}\frac{B(t)}{p^2(t)}w_j(t)dM_j(t) - n^{-3/2}\sum_{j=1}^n\int_{-\infty}^{u_n}\frac{\overline{B}(t)(D(t)-1)}{p^2(t)}w_j(t)dM_j(t)\\
&= R_{22}^{(1)} + R_{22}^{(2)} + R_{22}^{(3)} + R_{22}^{(4)}.
\end{aligned}$$

For the term $R_{22}^{(2)}$, we have

$$\begin{aligned}
P(|R_{22}^{(2)}| > (n\log n)^{-1/2}) &\le n^{-2}(\log n)E\left[\sum_{i=1}^n\int_{-\infty}^{u_n}\frac{\overline{B}_0^2(t)w_i^2(t)}{p^4(t)}L_i(t)d\Lambda(t)\right]\\
&\le n^{-1}(\log n)\int_{-\infty}^{u_n}p^{-4}(t)\sup_{t\le u_n}E(\overline{B}_0(t))^2d\Lambda(t)\\
&= O\left(n^{-2}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-4}(t)dG(t)\right) = o(n^{-1/2}).
\end{aligned}$$

Similarly, we can show that $P(|R_{22}^{(i)}| > (n\log n)^{-1/2}) = o(n^{-1/2})$, $i = 3, 4$.

Now let us investigate the term $R_{22}^{(1)}$. Note that for any $t \le \tau_F$,

$$|A_{n1}(t)| \le \frac{\sum_{i=1}^n |X_iY_{iG}|}{n(1-G(\tau_F))} \longrightarrow \frac{E|X_1Y_{1G}|}{1-G(\tau_F)} < \infty \qquad \text{a.s.}$$

and $\sup_{t\le u_n} E|A_{n1}(t)|^4 < \infty$. So, we have

$$P(|R_{22}^{(1)}| > (n\log n)^{-1/2}) \le (\log n)E\left[\sum_{i=1}^n\int_{-\infty}^{u_n}\frac{A_{n1}^2(t)\overline{w}^4(t)}{(p(t)+\theta_n(t)\overline{w}(t))^6}L_i(t)d\Lambda(t)\right]$$

$$
\begin{aligned}
&\le C_0 n(\log n)\int_{-\infty}^{u_n} p^{-6}(t)E(A_{n1}^2(t)\overline{w}^4(t))d\Lambda(t)\\
&= O\left(n^{-1}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-6}(t)dG(t)\right) = o(n^{-1/2}).
\end{aligned}
$$

(A.9) is thus proved.

(III) In this part we will prove that

$$
P(|R_3| \ge o(n^{-1/2})) = o(n^{-1/2}). \tag{A.10}
$$

From (A.3), we can write

$$
\begin{aligned}
R_3 &= n^{-1/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)[D(t)-1]}{p(t)}dM_i(t) - n^{-3/2}\sum_{i\ne j}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)}{p^2(t)}w_i(t)dM_j(t)\\
&\quad - n^{-3/2}\sum_{i\ne j}\int_{-\infty}^{u_n}\frac{\overline{B}(t)(D(t)-1)}{p^2(t)}w_i(t)dM_j(t)\\
&= R_{31}+R_{32}+R_{33}.
\end{aligned}
$$

First we investigate the term R_{31}. By Lemma A.1, we have

$$
\begin{aligned}
&P(|R_{31}| > (n\log n)^{-1/2})\\
&\quad\le P\left(\left|n^{-1/2}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)\overline{\xi}(t)}{p(t)(1-G(t))}dM(t)\right| > 2^{-1}(n\log n)^{-1/2}\right)\\
&\qquad + P\left(\left|n^{-1/2}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)r_n(t)}{p(t)(1-G(t))}dM(t)\right| > 2^{-1}(n\log n)^{-1/2}\right)\\
&\quad\equiv H_1 + H_2,
\end{aligned}
$$

where, by using the following facts

$$
\sup_{t\le u_n} E(\overline{B}_0(t))^4 = O(n^{-2}), \qquad \sup_{t\le u_n} E(\overline{\xi}(t))^4 = O(n^{-2}), \tag{A.11}
$$

we get

$$
\begin{aligned}
H_1 &\le 4(\log n)E\left\langle\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)\overline{\xi}(t)}{p(t)(1-G(t))}dM(t)\right\rangle\\
&= 4(\log n)\int_{-\infty}^{u_n}E\left[\left(\frac{\overline{B}_0(t)\overline{\xi}(t)}{p(t)(1-G(t))}\right)^2 L(t)\right]d\Lambda(t)\\
&\le 4n(\log n)\int_{-\infty}^{u_n}[E(\overline{B}_0(t))^4]^{1/2}[E(\overline{\xi}(t))^4]^{1/2}\frac{d\Lambda(t)}{(p(t)(1-G(t)))^2}\\
&= O\left(n^{-1}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-2}(t)dG(t)\right) = o(n^{-1/2}),\\
H_2 &\le 4(\log n)E\left\langle\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)r_n(t)}{p(t)(1-G(t))}dM(t)\right\rangle\\
&= 4(\log n)\int_{-\infty}^{u_n}E\left[\left(\frac{\overline{B}_0(t)r_n(t)}{p(t)(1-G(t))}\right)^2 L(t)\right]d\Lambda(t)
\end{aligned}
$$

$$\leq 4n(\log n)\int_{-\infty}^{u_n}[E(\overline{B}_0(t))^4]^{1/2}[E(r_n(t))^4]^{1/2}\frac{d\Lambda(t)}{(p(t)(1-G(t)))^2}$$
$$= O\left(n^{-2}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-8}(t)dG(t)\right) = o(n^{-1/2}), \quad \text{(from Lemma A.1)}.$$

For the term R_{32}, we have

$$P(|R_{32}| > (n\log n)^{-1/2})$$
$$\leq P\left(n^{-3/2}\left|\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)}{p^2(t)}w_i(t)dM_j(t)\right| > \frac{(n\log n)^{-1/2}}{2}\right)$$
$$+ P\left(n^{-3/2}\left|\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\int_{-\infty}^{u_n}\frac{\overline{B}_0(t)}{p^2(t)}w_j(t)dM_i(t)\right| > \frac{(n\log n)^{-1/2}}{2}\right)$$
$$\equiv V_1 + V_2.$$

Using (A.11) and the following fact $\sup_t E(n^{-1/2}\sum_{i=j}^{n} w_i(t))^4 = O(1)$, $(1 \leq j \leq n-1)$, we get

$$V_1 \leq \frac{4\log n}{n^2}\sum_{j=1}^{n-1} E\int_{-\infty}^{u_n}\left(\frac{\overline{B}_0(t)}{p^2(t)}\sum_{i=j+1}^{n} w_i(t)\right)^2 L_j(t)d\Lambda(t)$$
$$\leq \frac{4\log n}{n^2}\sum_{j=1}^{n-1}\int_{-\infty}^{u_n} E^{1/2}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} B_{k0}(t)\right)^4 E^{1/2}\left(\frac{1}{\sqrt{n}}\sum_{i=j+1}^{n} w_i(t)\right)^4 \frac{d\Lambda(t)}{p^4(t)}$$
$$= O\left(n^{-1}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-4}(t)dG(t)\right) = o(n^{-1/2}).$$

Similarly, we can get $V_2 = o(n^{-1/2})$. So $P(|R_{32}| > (n\log n)^{-1/2}) = o(n^{-1/2})$.

Similar to R_{32}, we can show that $P(|R_{33}| > (n\log n)^{-1/2}) = o(n^{-1/2})$. (A.10) is thus proved.

(IV) In this part we will show that

$$P(|R_4| \geq o(n^{-1/2})) = o(n^{-1/2}). \tag{A.12}$$

We can write

$$R_4 = n^{-3/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B_{i0}(t)}{p(t)}dM_i(t) + n^{-1/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B(t)r_n(t)}{p(t)(G(t)-1)}dM_i(t)$$
$$+ n^{-3/2}\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B(t)\xi_i(t)}{p(t)(G(t)-1)}dM_i(t)$$
$$\equiv R_{41} + R_{42} + R_{43}.$$

Since R_{41} is a martingale, we have

$$P(|R_{41}| > (n\log n)^{-1/2})$$

$$\le n^{-2}(\log n)E\left\langle \sum_{i=1}^{n}\int_{-\infty}^{u_n} p^{-1}(t)B_{i0}(t)dM_i(t)\right\rangle$$
$$= n^{-2}(\log n)\sum_{i=1}^{n}E\int_{-\infty}^{u_n} p^{-2}(t)B_{i0}^2(t)I(Z_i\ge t)d\Lambda(t)$$
$$\le n^{-1}(\log n)\int_{-\infty}^{u_n}\left(\sup_t E(B_1(t)-B(t))^2\right)p^{-2}(t)d\Lambda(t)$$
$$= O\left(n^{-1}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-2}(t)dG(t)\right) = o(n^{-1/2})$$

and

$$P(|R_{42}| > (n\log n)^{-1/2})$$
$$\le (\log n)E\left[\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B^2(t)r_n^2(t)L_i(t)}{p^2(t)(G(t)-1)^2}d\Lambda(t)\right]$$
$$\le (n\log n)\int_{-\infty}^{u_n}\frac{B^2(t)E(r_n^2(t))}{p^2(t)(G(t)-1)^2}d\Lambda(t)$$
$$\le C_0 n^{-1}(\log n)\int_{-\infty}^{u_n}\frac{B^2(t)}{p^8(t)(G(t)-1)^2}d\Lambda(t)$$
$$= O\left(n^{-1}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-8}(t)dG(t)\right) = o(n^{-1/2}), \qquad \text{(by Lemma A.1)}$$

and

$$P(|R_{43}| > (n\log n)^{-1/2})$$
$$\le n^{-2}(\log n)E\left\langle\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B(t)\xi_i(t)dM_i(t)}{p(t)(G(t)-1)}\right\rangle$$
$$= n^{-2}(\log n)E\left[\sum_{i=1}^{n}\int_{-\infty}^{u_n}\frac{B^2(t)\xi_i^2(t)L_i(t)}{p^2(t)(G(t)-1)^2}d\Lambda(t)\right]$$
$$= O\left(n^{-1}(\log n)\int_{-\infty}^{\tau}\overline{F}^{-2}(t)dG(t)\right) = o(n^{-1/2}).$$

(A.12) is thus proved.

(V) Finally, we will show that

$$P(|R_5| \ge o(n^{-1/2})) = o(n^{-1/2}). \tag{A.13}$$

Write $R_5 = R_{51} + R_{52} + R_{53}$, where R_{5i} represents the i-th term in (3.1), $i = 1, 2, 3$. It can be shown that

$$E|r_n(t,s)|^m = O(n^{-m/2}p^{-m/2}(t)p^{-m/2}(s)), \qquad (m \ge 1),$$

which in turn implies that $P(|R_{53}| \ge o(n^{-1/2})) = o(n^{-1/2})$. It is also easy to see that $P(|R_{51}| \ge o(n^{-1/2})) = o(n^{-1/2})$. So in the following, we shall try to show that

$$P(|R_{52}| \ge o(n^{-1/2})) = o(n^{-1/2}). \tag{A.14}$$

Note that we can further decompose R_{52} into

$$\begin{aligned} R_{52} &= 2n^{-3/2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{\overline{B}(t\vee s)-B(t\vee s)}{p(t)p(s)}dM_j(t)dM_k(s) \\ &\quad + 2n^{-3/2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{\overline{B}(t\vee s)(D(t)-1)}{p(t)p(s)}dM_j(t)dM_k(s) \\ &\quad + 2n^{-3/2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{\overline{B}(t\vee s)D(t)(D(s)-1)}{p(t)p(s)}dM_j(t)dM_k(s) \\ &\quad + n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{A_{n2}(t,s)-B(t\vee s)}{p(t)p(s)}dM_j(t)dM_j(s) \\ &\equiv 2R_{52}^{(1)}+2R_{52}^{(2)}+2R_{52}^{(3)}+R_{52}^{(4)}. \end{aligned}$$

Denote $M_{nj}(s)=\sum_{k=j+1}^{n}M_k(s)$, then we can write

$$R_{52}^{(1)} = n^{-3/2}\sum_{j=1}^{n-1}\int_{-\infty}^{u_n}\left[\int_{-\infty}^{u_n}\frac{\overline{B}(t\vee s)-B(t\vee s)}{p(t)p(s)}dM_{nj}(s)\right]dM_j(t).$$

Therefore,

$$\begin{aligned} &P(|R_{52}^{(1)}|>(n\log n)^{-1/2}) \\ &\le \frac{\log n}{n^2}\sum_{j=1}^{n-1}\int_{-\infty}^{u_n}E\left(\left(\int_{-\infty}^{u_n}\frac{\overline{B}(t\vee s)-B(t\vee s)}{p(t)p(s)}dM_{nj}(s)\right)^2\right)L_j(t)d\Lambda(t) \\ &= \frac{\log n}{n^2}\sum_{j=1}^{n-1}\int_{-\infty}^{u_n}E\left\langle\int_{-\infty}^{u_n}\frac{\overline{B}(t\vee s)-B(t\vee s)}{p(t)p(s)}L_j(t)dM_{nj}(s)\right\rangle d\Lambda(t) \\ &= \frac{\log n}{n^2}\sum_{j=1}^{n-1}\int_{-\infty}^{u_n}E\left(\int_{-\infty}^{u_n}\frac{(\overline{B}(t\vee s)-B(t\vee s))^2}{p^2(t)p^2(s)}L_j(t)d<M_{nj}(s)>\right)d\Lambda(t) \\ &= \frac{\log n}{n^2}\sum_{j=1}^{n-1}\int_{-\infty}^{u_n}E\left(\int_{-\infty}^{u_n}\frac{(\overline{B}(t\vee s)-B(t\vee s))^2}{p^2(t)p^2(s)}L_j(t)\sum_{k=j+1}^{n}L_k(s)d\Lambda(s)\right)d\Lambda(t) \\ &\le \log n\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}E(\overline{B}(t\vee s)-B(t\vee s))^2\frac{d\Lambda(s)d\Lambda(t)}{p^2(t)p^2(s)} \\ &= O\left(n^{-1}(\log n)\left(\int_{-\infty}^{\tau}\overline{F}^{-2}(t)dG(t)\right)^2\right) = o(n^{-1/2}). \end{aligned}$$

Here we have used the following fact

$$\sup_t E(n^{1/2}(\overline{B}(t)-B(t)))^2 = \sup_t E\left(n^{-1/2}\sum_{i=1}^{n}(B_j(t)-B(t))\right)^2 = O(1).$$

Similarly, we can show, along with Lemma A.1, that

$$P(|R_{52}^{(i)}|>(n\log n)^{-1/2}) = o(n^{-1/2}),\qquad i=2,3.$$

For the term $R_{52}^{(4)}$, we have the following decompositions:

$$\begin{aligned}
R_{52}^{(4)} &= n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}(\overline{B}(t\vee s)-B(t\vee s))\frac{dM_j(t)dM_j(s)}{p(t)p(s)}\\
&\quad+n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}(\overline{B}(t\vee s)-B(t\vee s))(D(t)-1)\frac{dM_j(t)dM_j(s)}{p(t)p(s)}\\
&\quad+2n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}B(t\vee s)(D(t)-1)\frac{dM_j(t)dM_j(s)}{p(t)p(s)}\\
&\quad+n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}(\overline{B}(t\vee s)-B(t\vee s))D(t)(D(s)-1)\frac{dM_j(t)dM_j(s)}{p(t)p(s)}\\
&\quad+n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}B(t\vee s)(D(t)-1)(D(s)-1)\frac{dM_j(t)dM_j(s)}{p(t)p(s)}\\
&\equiv L_1+L_2+2L_3+L_4+L_5.
\end{aligned}$$

We have

$$\begin{aligned}
&P(|L_1|>(n\log n)^{-1/2})\\
&\quad\le n^{-2}\log nE\left(\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}(\overline{B}(t\vee s)-B(t\vee s))\frac{\sum_{j=1}^{n}dM_j(t)dM_j(s)}{p(t)p(s)}\right)^2\\
&\quad=O\left(n^{-1}(\log n)\left(\int_{-\infty}^{\tau}\overline{F}^{-2}(t)dG(t)\right)^2\right)=o(n^{-1/2}).
\end{aligned}$$

Similarly, using Lemma A.1, it can be shown that

$$P(|L_i|>(n\log n)^{-1/2})=o(n^{-1/2})\qquad\text{for}\quad i=2,4,5.$$

Now we further decompose L_3 into

$$\begin{aligned}
L_3 &= n^{-5/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)\xi_j(t)}{(G(t)-1)p(t)p(s)}dM_j(t)dM_j(s)\\
&\quad+n^{-5/2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)\xi_i(t)}{(G(t)-1)p(t)p(s)}dM_j(t)dM_j(s)\\
&\quad+n^{-3/2}\sum_{j=1}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t\vee s)r_n(t)}{(G(t)-1)p(t)p(s)}dM_j(t)dM_j(s)\\
&\equiv L_{31}+L_{32}+L_{33},
\end{aligned}$$

we have

$$\begin{aligned}
&P(|L_{32}|>(n\log n)^{-1/2})\le n(\log n)EL_{32}^2\\
&\quad=\frac{\log n}{n^4}E\Bigg(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t_1\vee s_1)}{(G(t_1)-1)p(t_1)p(s_1)}
\end{aligned}$$

$$\times \frac{B(t_2 \vee s_2)}{(G(t_2)-1)p(t_2)p(s_2)}\xi_i(t_1)\xi_k(t_2)dM_j(t_1)dM_j(s_1)dM_l(t_2)dM_l(s_2)\Bigg)$$

$$= \frac{\log n}{n^4}E\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq i}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t_1 \vee s_1)}{(G(t_1)-1)p(t_1)p(s_1)}\right.$$

$$\left.\times \frac{B(t_2 \vee s_2)}{(G(t_2)-1)p(t_2)p(s_2)}\xi_i(t_1)\xi_i(t_2)dM_j(t_1)dM_j(s_1)dM_l(t_2)dM_l(s_2)\right)$$

$$+ \frac{\log n}{n^4}E\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\int_{-\infty}^{u_n}\frac{B(t_1 \vee s_1)}{(G(t_1)-1)p(t_1)p(s_1)}\right.$$

$$\left.\times \frac{B(t_2 \vee s_2)}{(G(t_2)-1)p(t_2)p(s_2)}\xi_i(t_1)dM_i(t_2)dM_i(s_2)\xi_k(t_2)dM_j(t_1)dM_j(s_1)\right)$$

$$= O\left(n^{-1}(\log n)\left(\int_{-\infty}^{\tau}\overline{F}^{-2}(t)dG(t)\right)^2\left(\int_{-\infty}^{\tau}\overline{F}^{-1}(t)dG(t)\right)^2\right)$$

$$= o(n^{-1/2}).$$

It is easy to prove that $P(|L_{3i}| > (n\log n)^{-1/2}) = o(n^{-1/2})$ for $i = 1, 3$. We proved (A.13), and thus Lemma 3.2.

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