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ON THE ASYMPTOTIC ACCURACY OF THE BOOTSTRAP UNDER ARBITRARY RESAMPLING SIZE

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Abstract. We study the order of convergence of the Kolmogorov-Smirnov distance for the bootstrap of the mean and the bootstrap of quantiles when an arbitrary bootstrap sample size is used. We see that for the bootstrap of the mean, the best order of the bootstrap sample is of the order of n, where n is the sample size. In the case of non-lattice distributions and the bootstrap of the sample mean; the bootstrap removes the effect of the skewness of the distribution only when the bootstrap sample equals the sample size. However, for the bootstrap of quantiles, the preferred order of the bootstrap sample is $n^{2/3}$. For the bootstrap of quantiles, if the bootstrap sample is of order n^2 or bigger, the bootstrap is not consistent.

Key words and phrases: Bootstrap, quantile.

1. Main results

Let $X_1, \ldots, X_n$ be independent identically distributed random variables (i.i.d.r.v.'s) from a cumulative distribution function (c.d.f.) F. The bootstrap (term coined by Efron (1979)) consists in doing Monte-Carlo from the sample, i.e., take i.i.d.r.v.'s $X^*_{n,1}, \ldots, X^*_{n,n}$ from the c.d.f. F_n, where F_n is the empirical distribution function based on $X_1, \ldots, X_n$. This procedure has been shown to be very useful in many statistical situations (see for example Beran and Ducharme (1991); Hall (1992); Efron and Tibshirani (1993); Shao and Tu (1995); Giné (1997); and Politis *et al.* (1999)).

Singh (1981) considered the a.s. behavior of the (conveniently normalized) bootstrap process

$$\{\Pr^*\{m^{1/2}(\bar{X}^*_{n,n} - \bar{X}_n) \le x\} - \Pr\{n^{1/2}(\bar{X}_n - \mu) \le x\} : x \in \mathbb{R}\},$$

where $\Pr^*$ is the conditional bootstrap probability, $\bar{X}^*_{n,m} = m^{-1}\sum_{j=1}^m X^*_{n,j}$, $\bar{X}_n = n^{-1}\sum_{j=1}^n X_j$ and $\mu = E[X]$. Convoluting this process with a normal density Beran ((1984), Theorem 3) obtained the weak convergence of this process in the sup norm.

Several variations of this method have appeared in the literature. One of them is to consider different bootstrap sample size (see for example Bickel and Freedman (1981) and Bickel *et al.* (1997)). In this case, a Monte-Carlo sample $X^*_{n,1}, \ldots, X^*_{n,m}$ from the empirical d.f. F_n is obtained. Here, we propose to study the asymptotic accuracy of the bootstrap of the sample mean and the bootstrap of quantiles under arbitrary bootstrap sample size.

The bootstrap m out of n has been studied by several authors in different situations. Athreya (1987) considered this bootstrap to correct the inconsistency of the unusual bootstrap under data coming from distributions with heavy tails. Politis and Romano (1994) and Politis *et al.* (1999) considered a variation of the bootstrap when samples are taken without replacement with bootstrap sample size smaller than the original sample. In particular, in Corollary 2.3.1 in Politis *et al.* (1999), it is shown that the bootstrap with replacement and bootstrap sample size m_n is consistent if $n^{-1}m_n^2 \to 0$,

We examine the weak convergence of the normalized difference of the distribution of a statistic and its bootstrap version. We study the weak convergence of the stochastic processes indexed by $I\!R$ obtained using the bootstrap. We will use the definition of weak convergence of stochastic processes introduced by Hoffmann-Jørgensen (1991) (see also van der Vaart and Wellner (1996); Dudley (1999)). The considered stochastic processes indexed by $I\!R$ are random elements of $l_\infty(I\!R)$, where $l_\infty(I\!R)$ denotes the Banach space consisting by the bounded functions in $I\!R$ with the norm $\|x\|_\infty = \sup_{x\in I\!R}|x(t)|$, where $x \in l_\infty(I\!R)$. This allows to use the Hoffmann-Jørgensen definition of weak convergence.

In the case of the bootstrap of the sample mean, we have to make two cases, according to whether the distribution is either lattice or nonlattice.

THEOREM 1.1. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.i.d.r.v.'s with a nonlattice distribution. Suppose that $E[X^4] < \infty$.*

(i) *If $\frac{m_n}{n} \to 0$, then*

$$\{m_n^{1/2}(\Pr{}^*\{m^{1/2}(\bar{X}^*_{n,m_n} - \bar{X}_n) \le x\} - \Pr\{n^{1/2}(\bar{X}_n - \mu) \le x\}) : x \in I\!R\}$$

converges weakly in $l_\infty(I\!R)$ to

$$\{6^{-1}\sigma^{-3}E[(X-\mu)^3](1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) : x \in I\!R\},$$

where ϕ is the density of a standard normal density.

(ii) *If $\frac{n}{m_n} \to a$, for some $0 \le a < \infty$, then*

$$\{n^{1/2}(\Pr{}^*\{m^{1/2}(\bar{X}^*_{n,m_n} - \bar{X}_n) \le x\} - \Pr\{n^{1/2}(\bar{X}_n - \mu) \le x\}) : x \in I\!R\}$$

converges weakly in $l_\infty(I\!R)$ to

$$\begin{aligned}&\{2^{-1}\sigma^{-3}(\mathrm{Var}((X-\mu)^2))^{1/2}gx\phi(\sigma^{-1}x)\\ &\quad + (a^{1/2}-1)6^{-1}\sigma^{-3}E[(X-\mu)^3](1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) : x \in I\!R\},\end{aligned}$$

where g is a r.v. with a standard normal distribution.

The conclusion of the previous theorem is that in the case of the sample mean is better to take the bootstrap sample size equal to the sample size. When the bootstrap sample has order smaller than the sample size, the rate of convergence of the bootstrap decreases. By the continuous mapping theorem (see e.g. Theorem 3.6.7 in Dudley (1999); or Theorem 1.11.1 in van der Vaart and Wellner (1996)), in the case (ii) in the previous theorem,

$$\begin{aligned}(1.1)\qquad &\sup_{x\in I\!R} n^{1/2}|\Pr{}^*\{m^{1/2}(\bar{X}^*_{n,m_n} - \bar{X}_n) \le x\} - \Pr\{n^{1/2}(\bar{X}_n - \mu) \le x\}|\\ &\xrightarrow{\mathrm{d}} \sup_{x\in I\!R} |2^{-1}\sigma^{-3}(\mathrm{Var}((X-\mu)^2))^{1/2}gx\phi(\sigma^{-1}x)\\ &\quad + (a^{1/2}-1)6^{-1}\sigma^{-3}E[(X-\mu)^3](1-\sigma^{-2}x^2)\phi(\sigma^{-1}x)|.\end{aligned}$$

The continuous mapping theorem applies because the function $\phi : (l_\infty(I\!R), \|\cdot\|_\infty) \to I\!R$ defined by $\phi(x) = \sup_{t\in I\!R} |x(t)|$ is continuous.

Since g has symmetric distribution, the expression in (1.1) is smaller in a certain sense when $a = 1$ ($m_n = n$). Recall that given two r.v.'s X and Y, it is said that X is smaller than Y in the usual stochastic order (or strong order), if for each $t \in I\!R$, $\Pr\{X \geq t\} \leq \Pr\{Y \geq t\}$ (see page 1 in Shaked and Shanthikumar (1994)). This is denoted by $X \leq_{\mathrm{st}} Y$. It is easy to see that $F \leq^{st} G$ if and only if for each nondecreasing function ϕ, $E[\phi(X)] \leq E[\phi(Y)]$ (see page 6 in Shaked and Shanthikumar (1994)). We claim that given bounded measurable functions $a(x)$ and $b(x)$, for each $t \geq 0$,

$$\Pr\left\{\sup_{x\in I\!R} |a(x)g| \geq t\right\} \leq \Pr\left\{\sup_{x\in I\!R} |a(x)g + b(x)| \geq t\right\}. \tag{1.2}$$

By the Anderson inequality (Corollary 2 in Anderson (1955)), for each $x \in I\!R$ and each $t \geq 0$, $\Pr\{|g| \geq t\} \leq \Pr\{|g + x| \geq t\}$. This implies that for each $x \in I\!R$ and each $t \geq 0$,

$$\Pr\{|a(x)g| \geq t\} \leq \Pr\{|a(x)g + b(x)| \geq t\} \leq \Pr\left\{\sup_{x\in I\!R} |a(x)g + b(x)| \geq t\right\}.$$

Taking the supremum over x, we get (1.2). It follows from (1.2) that the smallest distribution in (1.1) is obtained when $m_n = n$. If $a = 1$ we get that the bootstrap removes the effect of the skeweness of the distribution. When either $a = 1$ or $E[(X - \mu)^3] = 0$, we get a limit distribution which is stochastically as smallest as possible. So, we conclude that for non-lattice distributions, the bootstrap is more accurate when the bootstrap sample size is precisely the sample size.

For lattice distributions the expansions are more complicated.

THEOREM 1.2. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.i.d.r.v.'s with a lattice distribution with span h and support $\{b + kh : k \in Z\}$. Suppose that $E[X^4] < \infty$.*

(i) *If $\frac{m_n}{n} \to 0$, then*

$$\{m_n^{1/2}(\Pr^*\{m_n^{1/2}(\bar{X}^*_{n,m_n} - \bar{X}_n) \leq x\} - \Pr\{n^{1/2}(\bar{X}_n - \mu) \leq x\}) \\ - \sigma^{-1} h R(h^{-1}(m_n^{1/2}x - (b - \bar{X}_n)))\phi(\sigma_n^{-1}x) : x \in I\!R\}$$

converges weakly to

$$\{6^{-1}\sigma^{-3}E[(X-\mu)^3](1 - \sigma^{-2}x^2)\phi(\sigma^{-1}x) : x \in I\!R\},$$

where $R(y) = [y] - y + 2^{-1}$ and $\sigma_n^2 = n^{-1}\sum_{j=1}^n (X_j - \bar{X}_n)^2$.

(ii) *If $\frac{n}{m_n} \to a$, for some $0 \leq a < \infty$, then*

$$\{n^{1/2}(\Pr^*\{m_n^{1/2}(\bar{X}^*_{n,m_n} - \bar{X}_n) \leq x\} - \Pr\{n^{1/2}(\bar{X}_n - \mu) \leq x\}) \\ - a\sigma_n^{-1} h R(h^{-1}(m_n^{1/2}x - (b - \bar{X}_n)))\phi(\sigma_n^{-1}x) \\ + \sigma^{-1} h R(h^{-1}(n^{1/2}x - (b - \mu)))\phi(\sigma^{-1}x) : x \in I\!R\}$$

converges weakly to

$$\{2^{-1}\sigma^{-3}(\mathrm{Var}((X-\mu)^2))^{1/2} g x \phi(\sigma^{-1}x) \\ + (a^{1/2} - 1)(1 - \sigma^{-2}x^2)\phi(\sigma^{-1}x)6^{-1}\sigma^{-3}E[(X-\mu)^3] : x \in I\!R\},$$

where g is a standard normal distribution.

The function R is discontinuous at the integers. As n increases, $R(h^{-1}(n^{1/2}x - (b-\mu)))$ has more and more discontinuity points. We only can get convergence of the bootstrap stochastic process when the term containing the function R is removed from the bootstrap stochastic processes. Anyhow, as in the case before, the limit of the bootstrap stochastic process is less disperse when $a = 1$. Notice also that when $m_n = n$, the expressions

$$a\sigma_n^{-1}hR(h^{-1}(m_n^{1/2}x - (b - \bar{X}_n)))\phi(\sigma_n^{-1}x)$$

and

$$\sigma^{-1}hR(h^{-1}(n^{1/2}x - (b-\mu)))\phi(\sigma^{-1}x)$$

differ only by the approximations $\bar{X}_n \simeq \mu$ and $\sigma_n \simeq \sigma$.

Next, we consider the bootstrap of quantiles. Given $0 < p < 1$, the p-th quantile is defined as $\xi_0 = \inf\{t \in \mathbb{R} : F(x) \geq p\}$. The sample p-th quantile is defined by

$$\xi_n := F_n^{-1}(p) = \inf\{t \in \mathbb{R} : F_n(t) \geq p\}. \tag{1.3}$$

The asymptotic accuracy of the bootstrap has been considered by Singh (1981), Falk and Reiss (1989) and Falk (1990) in the case $m_n = n$. Falk and Reiss (1989) and Falk (1990) proved that

$$n^{1/4} \sup_{t \in \mathbb{R}} |\Pr{}^*\{n^{1/2}(\xi_n^* - \xi_n) \leq t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \leq t\}|,$$

converges in distribution to a non-degenerate distribution, where ξ_n^* is the bootstrap quantile.

Given two sequences of positive numbers $\{a_n\}$ and $\{b_n\}$, we say that $a_n \ll b_n$, if $a_n b_n^{-1} \to 0$.

By a Poisson process $\{N(t) : t \in \mathbb{R}\}$, we mean that $N(t) = N_1(t)$, for each $t > 0$, and that $N(t) = -N_2(-t)$, for each $t < 0$, where $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are two independent Poisson processes.

THEOREM 1.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d.r.v.'s. Let $0 < p < 1$. Supppose that $F(\xi_0) = p$ and that F is second differentiable at ξ_0, where F is the cdf of X. Let $\xi_n = \inf\{t : F_n(t) \geq p\}$, where F_n is the empirical cdf of $X_1, \ldots, X_n$. Let $\xi_{n,m_n}^* = \inf\{t : F_{n,m_n}^*(t) \geq p\}$, where F_{n,m_n}^* is the bootstrap cdf of $X_{n,1}^*, \ldots, X_{n,m_n}^*$. Let*

$$D_{n,m_n}^*(t) := \Pr{}^*\{m_n^{1/2}(\xi_{n,m_n}^* - \xi_n) \leq t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \leq t\}.$$

(i) *If $m_n \ll n^{2/3}$, then*

$$\begin{aligned}\sup_{t\in\mathbb{R}} \Bigg| & m_n^{1/2} D_{n,m_n}^*(t) - \frac{t^2 F''(\xi_0)}{2\sigma_p}\phi\left(\frac{tF'(\xi_0)}{\sigma_p}\right) - \frac{(2p-1)t^2(F'(\xi_0))^2}{2\sigma_p^3}\phi\left(\frac{tF'(\xi_0)}{\sigma_p}\right) \\ & - 6^{-1}\sigma_p^{-1}(2p-1)\psi\left(\frac{tF'(\xi_0)}{\sigma_p}\right) \\ & - \sigma_p^{-1}R(m_n(p - F_n(\xi_0 + m_n^{-1/2}t)) + F_n(\xi_0 + m_n^{-1/2}t))\phi\left(\frac{tF'(\xi_0)}{\sigma_p}\right)\Bigg| \xrightarrow{\Pr} 0,\end{aligned}$$

where $\psi(x) = (1 - x^2)\phi(x)$ *and* $\sigma_p = (p(1-p))^{1/2}$.

(ii) *If* $n^{-2/3} m_n \to a$, *for some constant* $0 < a < \infty$, *then*

$$\Big\{ m_n^{1/2} D^*_{n,m_n}(t) - \frac{t^2 F''(\xi_0)}{2\sigma_p}\phi\left(\frac{tF'(\xi_0)}{\sigma_p}\right) - \frac{(2p-1)t^2(F'(\xi_0))^2}{2\sigma_p^3}\phi\left(\frac{tF'(\xi_0)}{\sigma_p}\right)$$

$$- 6^{-1}\sigma_p^{-1}(2p-1)\psi\left(\frac{tF'(\xi_0)}{\sigma_p}\right)$$

$$- \sigma_p^{-1} R(m_n(p - F_n(\xi_0 + m_n^{-1/2}t)) + F_n(\xi_0 + m_n^{-1/2}t))\phi\left(\frac{tF'(\xi_0)}{\sigma_p}\right) : t \in \mathbb{R}\Big\}$$

$$\xrightarrow{\mathrm{w}} \{\sigma_p^{-1} a^{3/4} (F'(\xi_0))^{1/2} B(t) \phi(\sigma_p^{-1} t F'(\xi_0)) : t \in \mathbb{R}\},$$

where $\{B(t) : t \in \mathbb{R}\}$ *is a Brownian motion.*

(iii) *If* $n^{2/3} \ll m_n \ll n^2$, *then*

$$\{m_n^{-1/4} n^{1/2} D^*_{n,m_n}(t) : t \in \mathbb{R}\} \xrightarrow{\mathrm{w}} \{\sigma_p^{-1} B(t) \phi(\sigma_p^{-1} t F'(\xi_0)) : t \in \mathbb{R}\}.$$

(iv) *If* $m_n n^{-2} \to a$, *for some* $0 < a < \infty$, *then*

$$\sup_{t \in \mathbb{R}} |D^*_{n,m_n}(t) - \Phi(a^{1/2}\sigma_p^{-1}([-np] - np + N(a^{-1/2} t F'(\xi_0)))) + \Phi(\sigma_p^{-1} t F'(\xi_0))| \xrightarrow{\Pr} 0,$$

where $\{N(t) : t \in \mathbb{R}\}$ *is a Poisson process.*

(v) *If* $n^2 \ll m_n$, *then*

$$\sup_{t \in \mathbb{R}} |D^*_{n,m_n}(t) - \Phi(n^{-1} m^{1/2} \sigma_p^{-1}([-np] - np)) + \Phi(\sigma_p^{-1} t F'(\xi_0))| \xrightarrow{\Pr} 0.$$

In the previous theorem, the faster rate of convergence for the bootstrap of quantiles is obtained when $m_n \simeq n^{2/3}$. In cases (iv) and (v) in the previous theorem, the bootstrap is not consistent.

In Sakov and Bickel (2000), the Edgeworth expansion for the m out of n bootstrapped median is improved by extrapolation. Using this correction to the Edgeworth expansion, the authors are able to get an approximation of the order $n^{-2/5}$. The best possible order given in the previous theorem is $n^{-1/3}$.

2. Proofs

c will denote a universal constant which may vary from line to line. We will simplify $m_n = m$. We will use the following lemma:

LEMMA 2.1. (Esseen lemma (Lemma 16.3.2 in Feller (1966))) *Let* F *be a distribution function and let* G *be a differentiable function such that* $G(-\infty) = 0$ *and* $G(\infty) = 1$. *Let* $\varphi(\xi) = \int_{-\infty}^{\infty} e^{itx} dF(x)$ *and let* $\gamma(\xi) = \int_{-\infty}^{\infty} e^{itx} G'(x) d(x)$. *Suppose that* γ *is continuously differentiable. Then, for each* $0 < T < \infty$,

$$\sup_{x \in \mathbb{R}} |F(x) - G(x)| \le \pi^{-1} \int_{-T}^{T} |\xi|^{-1} |\varphi(\xi) - \gamma(\xi)| d\xi + \pi^{-1} T^{-1} 24 \sup_{x \in \mathbb{R}} |G'(x)|.$$

We will use the well known fact that if X is a random variable, then

$$\left|E\left[e^{itX}-\sum_{j=0}^{k}(j!)^{-1}(itX)^j\right]\right|\le (k+1)!^{-1}|t|^{k+1}E[|X|^{k+1}]. \tag{2.1}$$

We will also use that, under finite first moment, characteristic functions converge uniformly on compact sets, i.e. for each $0<M<\infty$,

$$\sup_{|\xi|\le M}\left|n^{-1}\sum_{j=1}^{n}e^{i\xi X_j}-E[e^{i\xi X}]\right|\to 0 \qquad \text{a.s.}$$

This follows directly from the Blum-DeHart law of the large numbers (see Theorem 7.1.5 in Dudley (1999)).

LEMMA 2.2. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with a nonlattice distribution. Suppose that $E[X^4]<\infty$. Then,*

$$m_n^{1/2}\sup_{x\in\mathbb{R}}|\Pr{}^*\{m^{1/2}\sigma_n^{-1}(\bar X^*_{n,m_n}-\bar X_n)\le x\}-\Phi(x)$$
$$-6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(1-x^2)\phi(x)|\overset{\Pr}{\to}0,$$

where $\sigma_n^2=n^{-1}\sum_{j=1}^n(X_j-\bar X_n)^2$ and $\mu_{3,n}=n^{-1}\sum_{j=1}^n(X_j-\bar X_n)^3$.

PROOF. We apply Esseen's lemma (Lemma 2.1) with

$$G_n(x)=\Phi(x)+6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{n,3}(1-x^2)\phi(x).$$

Then, its Fourier transform is

$$\gamma_n(\xi):=\int_{-\infty}^{\infty}e^{i\xi x}dG_n(x)=e^{-2^{-1}\xi^2}(1+6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{n,3}(i\xi)^3).$$

For each $0<\delta<a$, we have that

$$\begin{aligned}
&m^{1/2}\sup_{x\in\mathbb{R}}\left|\Pr{}^*\left\{\sigma_n^{-1}m^{-1/2}\sum_{j=1}^{m_n}(X^*_{n,j}-\bar X_n)\le x\right\}-G_n(x)\right|\\
&\le \pi^{-1}a^{-1}24\sup_x|G_n'(x)|+\pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}}|\xi|^{-1}|\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)-\gamma_n(\xi)|d\xi\\
&\quad+\pi^{-1}m^{1/2}\int_{\delta m^{1/2}\le|\xi|\le a m^{1/2}}|\xi|^{-1}|\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)-\gamma_n(\xi)|d\xi\\
&=:I+II+III,
\end{aligned}$$

where $\varphi_n(t)=n^{-1}\sum_{j=1}^n e^{it(X_j-\bar X_n)}$. Take $0<\delta<(3/4)\sigma^3(E[|X-EX|^3])^{-1}$ such that

$$\sum_{k=2}^{\infty}k^{-1}\delta^{2k-2}\le 2^{-3}.$$

Now, $I \xrightarrow{\Pr} \pi^{-1}a^{-1}24\sup_x |\phi(x)|$, which can be made arbitrarily small, by taking a large enough. Using that for complex numbers a and b, we have that

$$|e^a - e^b| \le |a-b|e^{\max(|a|,|b|)}$$

and

$$|e^a - 1 - a| \le 2^{-1}|a|^2 e^{|a|},$$

we get that

$$\begin{aligned}
II &\le \pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}} |\xi|^{-1}e^{-2^{-1}\xi^2}|e^{m\log\varphi_n(m^{-1/2}\sigma_n^{-1}\xi)+2^{-1}\xi^2} \\
&\qquad\qquad - 1 - 6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3|d\xi \\
&\le \pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}} |\xi|^{-1}e^{-2^{-1}\xi^2}|e^{m\log\varphi_n(m^{-1/2}\sigma_n^{-1}\xi)+2^{-1}\xi^2} \\
&\qquad\qquad - e^{6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3}|d\xi \\
&\quad + \pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}} |\xi|^{-1}e^{-2^{-1}\xi^2}|e^{6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3} \\
&\qquad\qquad - 1 - 6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3|d\xi \\
&\le \pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}} |\xi|^{-1}e^{-2^{-1}\xi^2}|m\log\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) + 2^{-1}\xi^2 \\
&\qquad\qquad - 6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3| \\
&\quad \times \exp(\max(|m\log\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) + 2^{-1}\xi^2|, 6^{-1}\sigma_n^{-3}m^{-1/2}|\mu_{3,n}||\xi|^3))d\xi \\
&\quad + \pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}} |\xi|^{-1}e^{-2^{-1}\xi^2}2^{-1}|6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3|^2 \\
&\qquad\qquad \times e^{6^{-1}\sigma_n^{-3}m^{-1/2}|\mu_{3,n}||\xi|^3}d\xi \\
&=: II_1 + II_2.
\end{aligned}$$

By (2.1), for n large enough and δ small enough and $|\xi| \le \delta m^{1/2}$,

$$\begin{aligned}
&|m\log\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) + 2^{-1}\xi^2 - 6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(i\xi)^3| \\
&\quad \le m|\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) - 1 + 2^{-1}m^{-1}\xi^2 - 6^{-1}\mu_{3,n}(m^{-1/2}\sigma_n^{-1}i\xi)^3| \\
&\qquad + m\sum_{k=2}^{\infty} k^{-1}|\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) - 1|^k \\
&\quad \le cm^{-1}\xi^4
\end{aligned}$$

and

$$\begin{aligned}
(2.2)\qquad &|m\log\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) + 2^{-1}\xi^2| \\
&\quad \le m|\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) - 1 + 2^{-1}m^{-1}\xi^2| + m\sum_{k=2}^{\infty}|\varphi_n(m^{-1/2}\sigma_n^{-1}\xi) - 1|^k \\
&\quad \le 6^{-1}m|m^{-1/2}\sigma_n^{-1}\xi|^3 n^{-1}\sum_{i=1}^{n}|X_i - \bar{X}_n|^3 + \sum_{k=2}^{\infty} m^{-(k-1)}k^{-1}|\xi|^{2k}
\end{aligned}$$

$$\leq \left(6^{-1}\sigma_n^{-3}\delta n^{-1}\sum_{i=1}^{n}|X_i-\bar{X}_n|^3+\sum_{k=2}^{\infty}k^{-1}\delta^{2k-2}\right)\xi^2$$
$$\leq 2^{-2}\xi^2.$$

Hence, for n large enough and δ small enough,

$$II_1 \leq cm^{-1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}}|\xi|^3 e^{-2^{-2}\xi^2}d\xi \leq cm^{-1/2}.$$

Similarly, for n large enough and δ small enough,

$$II_2 \leq cm^{-1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}}|\xi|^5 e^{-2^{-2}\xi^2}d\xi \leq cm^{-1/2}.$$

We also have that

$$\begin{aligned}
III \;\leq\; & \pi^{-1}m^{1/2}\int_{\delta m^{1/2}\leq|\xi|\leq am^{1/2}}|\xi|^{-1}|\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)|d\xi\\
&+\pi^{-1}m^{1/2}\int_{\delta m^{1/2}\leq|\xi|\leq am^{1/2}}|\xi|^{-1}e^{-2^{-1}\xi^2}(1+6^{-1}\sigma_n^{-3}m^{-1/2}|\mu_{3,n}||\xi|^3)d\xi\\
\leq\; & \pi^{-1}\delta^{-1}am^{1/2}\sup\{|\varphi_n(\xi)|^m : \delta\sigma_n^{-1}\leq|\xi|\leq a\sigma_n^{-1}\}\\
&+2\pi^{-1}\delta^{-1}\int_{\delta m^{1/2}}^{\infty}e^{-2^{-1}\xi^2}(1+6^{-1}\sigma_n^{-3}m^{-1/2}|\mu_{3,n}||\xi|^3)d\xi\\
=:\; & III_1+III_2.
\end{aligned}$$

Since $\sigma_n \to (\mathrm{Var}(X))^{1/2}$ a.s. and $\mu_{3,n}\to E[(X-\mu)^3]$ a.s., $III_2\to 0$ a.s. We also have that for each $0<c<\infty$, $\sup_{|x|\leq c}|\varphi_n(t)-\varphi(t)|\to 0$ a.s., where $\varphi(t)=E[e^{it(X-EX)}]$. Since X has a nonlattice distribution, $\sup_{2^{-1}\delta\sigma\leq|x|\leq 2a\sigma^{-1}}|\varphi(t)|=: c<1$. This implies that $III_1\to 0$ a.s. □

PROOF OF THEOREM 1.1. By Theorem 16.4.1 in Feller (1966)

$$n^{1/2}\sup_{x\in\mathbb{R}}|\Pr\{n^{1/2}\sigma^{-1}(\bar{X}_n-\mu)\leq x\} - \Phi(x)-6^{-1}\sigma^{-3}n^{-1/2}\mu_3(1-x^2)\phi(x)|\to 0. \tag{2.3}$$

By Lemma 2.1 and (2.3), in the case (i), we have that

$$\begin{aligned}
&m_n^{1/2}(\Pr^*\{m^{1/2}(\bar{X}^*_{n,m_n}-\bar{X}_n)\leq x\}-\Pr\{n^{1/2}(\bar{X}_n-\mu)\leq x\})\\
&\simeq m_n^{1/2}(\Phi(\sigma_n^{-1}x)-\Phi(\sigma^{-1}x))+6^{-1}\sigma_n^{-3}\mu_{3,n}(1-\sigma_n^{-2}x^2)\phi(\sigma_n^{-1}x)\\
&\quad-6^{-1}\sigma^{-3}m_n^{1/2}n^{-1/2}\mu_3(1-\sigma^{-2}x^2)\phi(\sigma^{-1}x)\\
&\simeq 6^{-1}\sigma^{-3}\mu_3(1-\sigma^{-2}x^2)\phi(\sigma^{-1}x).
\end{aligned}$$

Proceeding similarly, in the case (ii), we have that

$$\begin{aligned}
&n^{1/2}(\Pr^*\{m^{1/2}(\bar{X}^*_{n,m_n}-\bar{X}_n)\leq x\}-\Pr\{n^{1/2}(\bar{X}_n-\mu)\leq x\})\\
&\simeq n^{1/2}(\Phi(\sigma_n^{-1}x)-\Phi(\sigma^{-1}x))\\
&\quad+n^{1/2}m_n^{-1/2}6^{-1}\sigma_n^{-3}\mu_{3,n}(1-\sigma_n^{-2}x^2)\phi(\sigma_n^{-1}x)
\end{aligned}$$

$$
\begin{aligned}
&- 6^{-1}\sigma^{-3}\mu_3(1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) \\
&\simeq n^{1/2}(\sigma_n^{-1}-\sigma^{-1})x\phi(\sigma^{-1}x) \\
&\quad + (a^{1/2}-1)6^{-1}\sigma^{-3}E[(X-\mu)^3](1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) \\
&\simeq -n^{1/2}\sigma_n^{-1}\sigma^{-1}(\sigma_n^{-1}+\sigma^{-1})^{-1}(\sigma_n^2-\sigma^2)x\phi(\sigma^{-1}x) \\
&\quad + (a^{1/2}-1)6^{-1}\sigma^{-3}E[(X-\mu)^3](1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) \\
&\xrightarrow{\mathrm{d}} 2^{-1}\sigma^{-3}(\mathrm{Var}((X-\mu)^2))^{1/2}gx\phi(\sigma^{-1}x) \\
&\quad + (a^{1/2}-1)6^{-1}\sigma^{-3}E[(X-\mu)^3](1-\sigma^{-2}x^2)\phi(\sigma^{-1}x),
\end{aligned}
$$

where g is a standard normal r.v., being the convergence uniformly in x. □

LEMMA 2.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with a lattice distribution with span h and support $\{b+kh : k \in Z\}$. Suppose that $E[X^4] < \infty$. Then,*

$$
\begin{aligned}
m_n^{1/2} \sup_{x\in\mathbb{R}} |\, &\mathrm{Pr}^*\{m^{1/2}\sigma_n^{-1}(\bar{X}^*_{n,m_n}-\bar{X}_n) \le x\} - \Phi(x) \\
&- 6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(1-x^2)\phi(x) \\
&- \sigma_n^{-1}m^{-1/2}hR(h^{-1}(\sigma_n m^{1/2}x-(b-\bar{X}_n)))\phi(x)| \xrightarrow{\mathrm{Pr}} 0.
\end{aligned}
$$

PROOF. Let

$$G_n(x) = \Phi(x) + 6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{n,3}(1-x^2)\phi(x)$$

and let

$$H_n(x) = \mathrm{Pr}^*\{m^{1/2}\sigma_n^{-1}(\bar{X}^*_{n,m_n}-\bar{X}_n) \le x\}.$$

We apply Esseen's lemma (Lemma 2.1) to $G_n^{\#}$ and $H_n^{\#}$, where $G_n^{\#}$ and $H_n^{\#}$ are the convolutions of G_n and H_n with uniform distribution on $(-2^{-1}\sigma_n^{-1}m^{-1/2}h, 2^{-1}\sigma_n^{-1}m^{-1/2}h)$, in order words,

$$G_n^{\#}(x) = h^{-1}\sigma_n m^{1/2}\int_{-2^{-1}\sigma_n^{-1}m^{-1/2}h}^{2^{-1}\sigma_n^{-1}m^{-1/2}h} G_n(x-y)dy,$$

and a similar formula holds for $H_n^{\#}(x)$. By the Esseen lemma, for each $0<\delta<a$, we have that

$$
\begin{aligned}
&m^{1/2}\sup_{x\in\mathbb{R}}|H_n^{\#}(x)-G_n^{\#}(x)| \\
&\le \pi^{-1}a^{-1}24\sup_x|G_n^{\#\prime}(x)| \\
&\quad + \pi^{-1}m^{1/2}\int_{-\delta m^{1/2}}^{\delta m^{1/2}}|\xi|^{-1}|\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)-\gamma_n(\xi)|\frac{|\sin(2^{-1}\sigma_n^{-1}m^{-1/2}h\xi)|}{|2^{-1}\sigma_n^{-1}m^{-1/2}h\xi|}d\xi \\
&\quad + \pi^{-1}m^{1/2}\int_{\delta m^{1/2}\le|\xi|\le a m^{1/2}}|\xi|^{-1}|\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)-\gamma_n(\xi)| \\
&\qquad\qquad \cdot\frac{|\sin(2^{-1}\sigma_n^{-1}m^{-1/2}h\xi)|}{|2^{-1}\sigma_n^{-1}m^{-1/2}h\xi|}d\xi \\
&=: I + II + III,
\end{aligned}
$$

where $\varphi_n(t) = n^{-1}\sum_{j=1}^{n} e^{it(X_j-\bar{X}_n)}$.

$I \xrightarrow{\Pr} \pi^{-1}a^{-1}24\sup_x|\phi(x)|$, which can be made arbitrarily small, by taking a large enough. By the argument in the proof of Lemma 2.1, $II \xrightarrow{\Pr} 0$.

We also have that

$$\begin{aligned} III &\le \pi^{-1}m^{1/2}\int_{\delta m^{1/2}\le|\xi|\le am^{1/2}} |\xi|^{-1}|\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)|\frac{|\sin(2^{-1}\sigma^{-1}m^{-1/2}h\xi)|}{|2^{-1}\sigma^{-1}m^{-1/2}h\xi|}d\xi \\ &\quad +\pi^{-1}m^{1/2}\int_{\delta m^{1/2}\le|\xi|\le am^{1/2}} |\xi|^{-1}e^{-2^{-1}\xi^2}(1+6^{-1}\sigma_n^{-3}m^{-1/2}|\mu_{3,n}||\xi|^3)d\xi \\ &=: III_1 + III_2. \end{aligned}$$

Since $\psi_n(\xi) = |\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)\sin(2^{-1}\sigma_n^{-1}m^{-1/2}h\xi)|$ is an even periodic function with periodic with period $2\sigma_n m^{1/2}h^{-1}\pi$ such that $\psi_n(\xi) = \psi_n(2\sigma_n m^{1/2}h^{-1}\pi - \xi)$, for each ξ, we have that

$$\begin{aligned} III_1 &\le c\int_{\delta m^{1/2}\le|\xi|\le am^{1/2}} |\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)\sin(2^{-1}\sigma_n^{-1}m^{-1/2}h\xi)|d\xi \\ &\le c\int_{\delta m^{1/2}}^{am^{1/2}} |\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)\sin(2^{-1}\sigma_n^{-1}m^{-1/2}h\xi)|d\xi \\ &\le cm^{-1/2}\int_{\delta m^{1/2}}^{am^{1/2}} |\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)\xi|d\xi. \end{aligned}$$

By (2.2), there exists a $\delta' > 0$, such that for n large enough and $|\xi| \le \delta m^{1/2}$,

$$|\varphi_n(m^{-1/2}\sigma_n^{-1}\xi)|^m \le e^{-2^{-2}\xi^2}.$$

Hence,

$$m^{-1/2}\int_0^{\delta' m^{1/2}} |\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)\xi|d\xi \to 0 \quad \text{a.s.}$$

By a change of variables,

$$\begin{aligned} m^{-1/2}&\int_{\delta' m^{1/2}}^{am^{1/2}} |\varphi_n^m(m^{-1/2}\sigma_n^{-1}\xi)\xi|d\xi \\ &\le cm^{1/2}\int_{\delta'}^{a} |\varphi_n^m(\sigma_n^{-1}\xi)|d\xi. \end{aligned}$$

Now, $\sup_{\delta'\le\xi\le a}|\varphi_n(\xi) - \varphi(\xi)| \to 0$ a.s. and $\sup_{\delta'\le\xi\le a}|\varphi(\xi)| < 1$. Hence,

$$m^{1/2}\int_{\delta'}^{a} |\varphi_n^m(\sigma_n^{-1}\xi)|d\xi \to 0 \quad \text{a.s.}$$

This implies that $III_1 \to 0$ a.s.

By the argument in the proof of Lemma 2.1, $III_2 \xrightarrow{\Pr} 0$. Therefore, we have that

$$m^{1/2}\sup_{x\in\mathbb{R}}|G_n^{\#}(x) - H_n^{\#}(x)| \xrightarrow{\Pr} 0.$$

It is easy to see that

$$m^{1/2}\sup_{x\in\mathbb{R}}|G_n(x)-G_n^{\#}(x)|\to 0.$$

This implies that

$$m_n^{1/2}\sup_{x\in\mathbb{R}}|H_n^{\#}(x)-\Phi(x)-6^{-1}\sigma_n^{-3}m^{-1/2}\mu_{3,n}(1-x^2)\phi(x)|\xrightarrow{\Pr}0. \tag{2.4}$$

Since $H_n(x)$ is a constant in the interval $(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+jh),\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+1)h))$, for each integer j, we have that

$$H_n^{\#}(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1}h)))=H_n(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1}h))).$$

Given x, there exists an integer j such that

$$\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+jh)\le x<\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+1)h).$$

Then,

$$\begin{aligned}
&m^{1/2}(H_n(x)-\Phi(x))\\
&\quad=m^{1/2}(H_n^{\#}(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1}h)))\\
&\qquad\quad-\Phi(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1})h)))\\
&\qquad+m^{1/2}(\Phi(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1})h))-\Phi(x))\\
&\quad\simeq 6^{-1}\sigma_n^{-3}\mu_{3,n}\Psi(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1}h)))\\
&\qquad+m^{1/2}(\sigma_n^{-1}m^{-1/2}(b-\bar{X}_n+(j+2^{-1})h)-x)\phi(x)\\
&\quad\simeq 6^{-1}\sigma_n^{-3}\mu_{3,n}\Psi(x)+\sigma_n^{-1}hR(h^{-1}(\sigma_n m^{1/2}x-(b-\bar{X}_n)))\phi(x),
\end{aligned}$$

where $\Psi(x)=(1-x^2)\phi(x)$. □

PROOF OF THEOREM 1.2. By Theorem 16.4.2 in Feller (1966)

$$\begin{aligned}
n^{1/2}\sup_{x\in\mathbb{R}}|&\Pr\{n^{1/2}\sigma^{-1}(\bar{X}_n-\mu)\le x\}-\Phi(x)\\
&-6^{-1}\sigma^{-3}n^{-1/2}\mu_3(1-x^2)\phi(x)\\
&-\sigma^{-1}n^{-1/2}hR(h^{-1}(\sigma n^{1/2}x-(b-\mu)))\phi(x)|\to 0.
\end{aligned} \tag{2.5}$$

In the case (i), we have that

$$\begin{aligned}
&m_n^{1/2}(\Pr{}^*\{m^{1/2}(\bar{X}_{n,m_n}^*-\bar{X}_n)\le x\}-\Pr\{m^{1/2}(\bar{X}_n-\mu)\le x\})\\
&\quad\simeq m_n^{1/2}(\Phi(\sigma_n^{-1}x)-\Phi(\sigma^{-1}x))\\
&\qquad+6^{-1}\sigma_n^{-3}\mu_{3,n}(1-\sigma_n^{-2}x^2)\phi(\sigma_n^{-1}x)-6^{-1}\sigma^{-3}m_n^{1/2}n^{-1/2}\mu_3(1-\sigma^{-1}x^2)\phi(\sigma^{-1}x)\\
&\qquad+\sigma_n^{-1}hR(h^{-1}(m^{1/2}x-(b-\bar{X}_n)))\phi(\sigma_n^{-1}x)\\
&\qquad-m_n^{1/2}n^{-1/2}\sigma^{-1}hR(h^{-1}(n^{1/2}x-(b-\mu)))\phi(\sigma^{-1}x)\\
&\quad\simeq 6^{-1}\sigma^{-3}\mu_3(1-\sigma^{-2}x^2)\phi(\sigma^{-1}x)+\sigma^{-1}hR(h^{-1}(m^{1/2}x-(b-\bar{X}_n)))\phi(\sigma^{-1}x).
\end{aligned}$$

In the case (ii), we proceed similarly

$$\begin{aligned}
&n^{1/2}(\Pr^*\{m^{1/2}(\bar{X}^*_{n,m_n} - \bar{X}_n) \le x\} - \Pr\{m^{1/2}(\bar{X}_n - \mu) \le x\}) \\
&\quad \simeq n^{1/2}(\Phi(\sigma_n^{-1}x) - \Phi(\sigma^{-1}x)) \\
&\qquad + n^{1/2}m_n^{-1/2}6^{-1}\sigma_n^{-3}\mu_{3,n}(1-\sigma_n^{-2}x^2)\phi(\sigma_n^{-1}x) - 6^{-1}\sigma^{-3}\mu_3(1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) \\
&\qquad + n^{1/2}m_n^{-1/2}\sigma_n^{-1}hR(h^{-1}(m^{1/2}x - (b-\bar{X}_n)))\phi(\sigma_n^{-1}x) \\
&\qquad - \sigma^{-1}hR(h^{-1}(n^{1/2}x - (b-\mu)))\phi(\sigma^{-1}x) \\
&\quad \simeq \sigma_n^{-1}\sigma^{-1}(\sigma_n^{-1} + \sigma^{-1})n^{1/2}(\sigma_n^2 - \sigma^2)x\phi(\sigma^{-1}x) \\
&\qquad + (a^{1/2} - 1)6^{-1}\sigma^{-3}\mu_3(1-\sigma^{-2}x^2)\phi(\sigma^{-1}x) \\
&\qquad + a^{1/2}\sigma^{-1}hR(h^{-1}(m^{1/2}x - (b-\bar{X}_n)))\phi(\sigma_n^{-1}x) \\
&\qquad - \sigma^{-1}hR(h^{-1}(n^{1/2}x - (b-\mu)))\phi(\sigma^{-1}x). \qquad \square
\end{aligned}$$

LEMMA 2.4. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with df F. Suppose that $F(\xi_0) = p$, where $0 < p < 1$, and $F'(\xi_0)$ exists. Then,*

(i) *For each $c > 0$,*

$$\begin{aligned}
\sup_{|t| \le c(\log n)^{1/2}} n^{1/2} \Bigg| & P\{n^{1/2}(\xi_n - \xi_0) \le t\} - \Phi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F(\xi_0))}{\sigma_n(t)}\right) \\
& - \frac{(2F(\xi_0 + n^{-1/2}t) - 1)}{6\sigma_n(t)n^{1/2}}\psi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F(\xi_0))}{\sigma_n(t)}\right) \\
& + \frac{r(n(F(\xi_0) - F(\xi_0 + n^{-1/2}t)) + F(\xi_0 + n^{-1/2}t))}{n^{1/2}\sigma_n(t)} \\
& \cdot \phi(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F(\xi_0))}{\sigma_n(t)}) \Bigg| \to 0,
\end{aligned}$$

where

$$\sigma_n(t) = (F(\xi_0 + n^{-1/2}t)(1 - F(\xi_0 + n^{-1/2}t)))^{1/2}$$

and $\psi(x) = (1 - x^2)\phi(x)$.

(ii) *For each $c > \sigma_p/F'(\xi_0)$, where $\sigma_p^2 = p(1-p)$,*

$$n^{1/2}P\{n^{1/2}|\xi_n - \xi_0| \ge c(\log n)^{1/2}\} \to 0.$$

PROOF. We have that

$$\begin{aligned}
\Pr\{n^{1/2}(\xi_n - \xi_0) \le t\} &= \Pr\{\xi_n \le \xi_0 + n^{-1/2}t\} = \Pr\{F_n(\xi_0 + n^{-1/2}t) \ge p\} \\
&= \Pr\{n^{1/2}(F_n(\xi_0 + n^{-1/2}t) - F(\xi_0 + n^{-1/2}t)) \ge n^{1/2}(p - F(\xi_0 + n^{-1/2}t))\}.
\end{aligned}$$

By the Esseen inequality applied to

$$H_n(x,t) = \Pr\left\{\frac{n^{1/2}(F_n(\xi_0 + n^{-1/2}t) - F(\xi_0 + n^{-1/2}t))}{\sigma_n(t)} \le x\right\}$$

we get that

$$\sup_{|t|\le c(\log n)^{1/2}} \sup_{x\in \mathbb{R}} \left| H_n(x,t) - \Phi(x) - \frac{\mu_{n,3}(t)(1-x^2)\phi(x)}{6\sigma_n^3(t)n^{1/2}} \right. \\ \left. - \frac{R(\sigma_n(t)n^{1/2}F(\xi_0) + F(\xi_0 + n^{-1/2}t))\phi(x)}{n^{1/2}\sigma_n(t)} \right| \to 0,$$

where $\mu_{n,3}(t) = E[(I(X \le \xi_0 + n^{-1/2}t) - F(\xi_0 + n^{-1/2}t\xi_0 + n^{-1/2}t))^3]$. Hence, for $|t| \le c(\log n)^{1/2}$, we have that

$$\begin{aligned}
\Pr\{n^{1/2}(\xi_n - \xi_0) \le t\} &= 1 - H_n\left(\frac{n^{1/2}(F_n(\xi_0) - F(\xi_0 + n^{-1/2}t))}{\sigma_n(t)}\right) \\
&\simeq \Phi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F_n(\xi_0))}{\sigma_n(t)}\right) \\
&\quad - \frac{\mu_n(t)}{6\sigma_n^3(t)n^{1/2}}\psi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F_n(\xi_0))}{\sigma_n(t)}\right) \\
&\quad - \frac{R(n(F(\xi_0) - F(\xi_0 + n^{-1/2}t)) + F(\xi_0 + n^{-1/2}t))}{n^{1/2}\sigma_n(t)} \\
&\quad \cdot \phi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F_n(\xi_0))}{\sigma_n(t)}\right) \\
&= \Phi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F(\xi_0))}{\sigma_n(t)}\right) \\
&\quad + \frac{(2F(\xi_0 + n^{-1/2}t) - 1)}{6\sigma_n(t)n^{1/2}}\psi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F(\xi_0))}{\sigma_n(t)}\right) \\
&\quad - \frac{R(n(F(\xi_0) - F(\xi_0 + n^{-1/2}t)) + F(\xi_0 + n^{-1/2}t))}{n^{1/2}\sigma_n(t)} \\
&\quad \cdot \phi\left(\frac{n^{1/2}(F(\xi_0 + n^{-1/2}t) - F(\xi_0))}{\sigma_n(t)}\right).
\end{aligned}$$

Now, by the Bernstein's inequality,

$$\begin{aligned}
&n^{1/2}\Pr\{n^{1/2}(\xi_n - \xi_0) \le -c(\log n)^{1/2}\} \\
&= n^{1/2}\Pr\{n^{1/2}(F_n(\xi_0) - cn^{-1/2}(\log n)^{1/2}) - F(\xi_0 - cn^{-1/2}(\log n)^{1/2})) \\
&\qquad \ge n^{1/2}(p - F(\xi_0 - cn^{-1/2}(\log n)^{1/2}))\} \\
&\le n^{1/2} \\
&\quad \cdot \exp\left(\frac{-n(p - F(\xi_0 - cn^{-1/2}(\log n)^{1/2}))^2}{2F(\xi_0 - cn^{-1/2}(\log n)^{1/2})(1 - F(\xi_0 - cn^{-1/2}(\log n)^{1/2})) + (2/3)n^{-1/2}|p - F(\xi_0 - cn^{-1/2}(\log n)^{1/2})|}\right) \\
&\simeq n^{1/2}\exp(-2^{-1}\sigma_p^{-2}c^2(F'(\xi_0)^2\log n)) \to 0. \qquad \square
\end{aligned}$$

Next lemma follows similarly to Lemma 2.4 and its proof is omitted.

Lemma 2.5. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with df F. Suppose that $F(\xi_0) = p$, where $0 < p < 1$. Then,*

(i) *For each* $c > 0$,

$$\sup_{|t|\le c(\log m)^{1/2}} m^{1/2} \Bigg| \Pr{}^*\{m^{1/2}(\xi_{n,m} - \xi_n) \le t\} - \Phi\left(\frac{m^{1/2}(F_n(\xi_n + m^{-1/2}t) - p)}{\sigma^*_{n,m}(t)}\right)$$
$$- \frac{(2F_n(\xi_n + m^{-1/2}t) - 1)}{6\sigma^*_{n,m}(t)m^{1/2}}\psi\left(\frac{m^{1/2}(F_n(\xi_n + m^{-1/2}t) - p)}{\sigma^*_{n,m}(t)}\right)$$
$$+ \frac{R(m(p - F_n(\xi_n + m^{-1/2}t)) + F_n(\xi_n + m^{-1/2}t))}{m^{1/2}\sigma^*_{n,m}(t)}$$
$$\cdot \phi(\frac{m^{1/2}(F_n(\xi_n + m^{-1/2}t) - p)}{\sigma^*_{n,m}(t)})\Bigg| \to 0,$$

where

$$\sigma^*_{n,m}(t) = (F_n(\xi_n + m^{-1/2}t)(1 - F_n(\xi_n + m^{-1/2}t)))^{1/2}.$$

(ii) *For each* $c > \sigma_p/F'(\xi_0)$,

$$m^{1/2}\Pr{}^*\{m^{1/2}|\xi^*_{n,m} - \xi_n| \ge c(\log m)^{1/2}\} \xrightarrow{\Pr} 0.$$

To obtain certain limit theorems for empirical processes, we will use the theory of VC subgraph of functions in Dudley (1999) and Pollard (1990). More specifically, we need the extension of these theorems to manageable triangular arrays of functions in Pollard (1990) and Arcones (1999). We need the last reference, since in some cases the limit is not a Gaussian process. Given a set S and a collection of subsets $\mathcal{C}$, for $A \subset S$, let $\Delta^{\mathcal{C}}(A) = \text{card}\{A \cap C : C \in \mathcal{C}\}$, let $m^{\mathcal{C}}(n) = \max\{\Delta^{\mathcal{C}}(A) : \text{card}(A) = n\}$ and let $s(\mathcal{C}) = \sup\{n : m^{\mathcal{C}}(n) = 2^n\}$. $\mathcal{C}$ is said to be a VC class of sets if $s(\mathcal{C}) < \infty$. Given a function $f : S \to \mathbb{R}$, the subgraph of f is the set $\{(x, y) \in S \times \mathbb{R} : 0 \le y \le f(x)$ or $f(x) \le y \le 0\}$. A class of functions $\mathcal{F}$ is a VC subgraph class if the collection of subgraphs of $\mathcal{F}$ is a VC class. The interest of these classes of functions lies in their good properties with respect to covering numbers. Given a pseudometric space (T, d), the ϵ-covering number $N(\epsilon, T, d)$ is defined by

$$N(\epsilon, T, d) = \min\{m : \text{there exists a covering of } T \text{ by } m \text{ balls of radius} \le \epsilon\}.$$

Given a positive measure μ on $(S, \mathcal{S})$ we define $N_2(\epsilon, \mathcal{F}, \mu) = N(\epsilon, \mathcal{F}, \|\cdot\|_{L_2(\mu)})$. If $\mathcal{F}$ is a VC subgraph class (Pollard (1990), Proposition II. 25), there are finite constants A and v such that, for each probability measure μ with $\mu F^2 < \infty$,

$$N_2(\epsilon, \mathcal{F}, \mu) \le A((\mu F^2)^{1/2}/\epsilon)^v,$$

where $F(x) = \sup_{f \in \mathcal{F}} |f(x)|$ and A and v can be chosen depending only on $s(\mathcal{F})$.

We also will need the following lemma.

LEMMA 2.6. (Reiss (1986); Lemma 1.4 in Falk and Reiss (1989)) *Let* $\{X_n\}_{n=1}^\infty$ *be a sequence of i.i.d.r.v.'s. Let* $0 < p < 1$. *Suppose that* $F(\xi_0) = p$ *and that* F *is differentiable at* ξ_0. *Let* $\xi_n = \inf\{t : F_n(t) \ge p\}$. *Then, the conditional distribution of* *the process* $\{F_n(\xi_n + m^{-1/2}t) : t \in \mathbb{R}\}$ *given that* $n^{1/2}(\xi_n - \xi_0) = u$ *is*

$$W_{n,u}(t) = \begin{cases} n^{-1}(-[-np]-1)G_{-[-np]-1}\left(\dfrac{F(\xi_0 + n^{-1/2}u + m^{-1/2}t)}{F(\xi_0 + n^{-1/2}u)}\right) & \text{if } t < 0 \\ -n^{-1}[-np] + n^{-1}(n + [-np])H_{n+[-np]} & \\ \quad \cdot \left(\dfrac{F(\xi_0 + n^{-1/2}u + m^{-1/2}t) - F(\xi_0 + n^{-1/2}u)}{1 - F(\xi_0 + n^{-1/2}u)}\right) & \text{if } t \geq 0 \end{cases}$$

where $G_{-[-np]-1}$ *and* $H_{n-[-np]}$ *denote the empirical d.f.'s of two independent samples of i.i.d.r.v.'s with uniform distribution on the interval* (0,1).

LEMMA 2.7. *Under the conditions in the previous lemma:*
(i) *If* $m_n \ll n^{2/3}$, *then*

$$\sup_{t\in\mathbb{R}} m_n|F_n(\xi_n + m_n^{-1/2}t) - F_n(\xi_n) - F(\xi_n + m_n^{-1/2}t) + F(\xi_n)|\phi(\sigma_p^{-1}tF'(\xi_0)) \xrightarrow{\Pr} 0.$$

(ii) *If* $n^{-2/3}m_n \to a$, *for some constant* $0 < a < \infty$, *then*

$$\{m_n(F_n(\xi_n + m_n^{-1/2}t) - F_n(\xi_n) - F(\xi_n + m_n^{-1/2}t) + F(\xi_n))\phi(\sigma_p^{-1}tF'(\xi_0)) : t \in \mathbb{R}\}$$
$$\xrightarrow{\text{w}} \{a^{3/4}(F'(\xi_0))^{1/2}B(t)\phi(\sigma_p^{-1}tF'(\xi_0)) : t \in \mathbb{R}\}.$$

(iii) *If* $n^{2/3} \ll m \ll n^2$, *then*

$$\begin{aligned}\{m_n^{1/4}n^{1/2}(F_n(\xi_n &+ m_n^{-1/2}t) - F_n(\xi_n) \\ &- F(\xi_n + m_n^{-1/2}t) + F(\xi_n))\phi(\sigma_p^{-1}tF'(\xi_0)) : t \in \mathbb{R}\} \\ &\xrightarrow{\text{w}} \{a^{3/4}(F'(\xi_0))^{1/2}B(t)\phi(\sigma_p^{-1}tF'(\xi_0)) : t \in \mathbb{R}\}.\end{aligned}$$

(iv) *If* $n^{-2}m_n \to a$, *for some constant* $0 < a < \infty$, *then for each* $0 < M < \infty$,

$$\{m_n^{1/2}(F_n(\xi_n + m_n^{-1/2}t) - F_n(\xi_n)) : |t| \leq M\},$$
$$\xrightarrow{\text{w}} \{a^{1/2}N(a^{-1/2}tF'(\xi_0)) : |t| \leq M\},$$

where $\{N(t) : t \in \mathbb{R}\}$ *is a Poisson process.*
(v) *If* $n^2 \ll m_n$, *then for each* $0 < M < \infty$,

$$\sup_{|t|\leq M} m_n^{1/2}|F_n(\xi_n + m_n^{-1/2}t) - F_n(\xi_n)| \xrightarrow{\Pr} 0.$$

PROOF. By Lemma 2.6, it suffices to consider the convergence of the transformed processes over uniform distributions. We only consider the part where $t \geq 0$. The other part can be dealt similarly. By an abuse of notation, in this proof $\{X_j\}$ will denote a sequence of i.i.d.r.v.'s uniformly distributed over the interval $(0, 1)$. Its empirical d.f. is denoted by H_n.

We only consider the case (i) with full detail. The rest of the cases are similar. It suffices to prove that, for each $0 < M < \infty$ and $s \in \mathbb{R}$,

$$\begin{aligned}\sup_{t\geq 0}\Bigg| m \Bigg(H_{n+[-np]} &\left(\frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)}\right) \\ &- \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)}\Bigg) e^{-bt^2}\Bigg| \xrightarrow{\Pr} 0,\end{aligned}$$

where $b = 2^{-1}\sigma_p^2(F'(\xi_0))^2$. We apply Theorem 2.1 in Arcones (1999).

We consider the class of functions $\mathcal{F}_n := \{f_n(x,t) : t \geq 0\}$, where

$$f_n(x,t) = \left\{ I\left(X \leq \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)} \right) e^{-bt} : t \geq 0 \right\}.$$

The subgraph of $f_n(\cdot, t)$ is

$$A_n(t) = \left\{ (x,y) : 0 \leq \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)}, 0 \leq y \leq e^{-bt} \right\}.$$

Let

$$B_n(t) = \left\{ (x,y) : 0 \leq \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)} \right\}$$

and let

$$C_n(t) = \{(x,y) : 0 \leq y \leq e^{-bt}\}.$$

Since the classes $\{B_n(t) : t \geq 0\}$ and $\{C_n(t) : t \geq 0\}$ are linearly ordered by inclusion, by Theorem 4.5.7 in Dudley (1992), $s(\{C_n(t) : t \geq 0\}) \leq 2$. This implies that the triangular array of functions $\{f_{n,j}(x,t) : 1 \leq j \leq n + [-np], t \geq 0\}$ is manageable in the sense of Definition 1.1 in Arcones (1999), where $f_{n,j}(x,t) = f_n(x,t)$.

Since

$$mn^{-1} \sup_{t \geq 0} I\left(X \leq \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)} \right) e^{-bt} \to 0,$$

conditions (iii), (iv) and (vii) in Theorem 2.1 in Arcones (1999) hold.

To check condition (v) in this theorem, we need that

$$n^{-1}m^2 E\left[\sup_{t \geq 0} I\left(X \leq \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)} \right) e^{-2bt} \right] \to 0.$$

We have that

$$\begin{aligned}
&n^{-1}m^2 E\left[\sup_{t \geq 0} I\left(X \leq \frac{F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s)}{1 - F(\xi_0 + n^{-1/2}s)} \right) e^{-2bt} \right] \\
&\quad = n^{-1}m^2 E[\exp(-bm(F^{-1}(F(\xi_0 + n^{-1/2}s) \\
&\qquad\qquad + X(1 - F(\xi_0 + n^{-1/2}s)) - \xi_0 - n^{-1/2}s))^2)] \\
&\quad \leq n^{-1}m^2 E[\exp(-bm\tau^{-1}(X(1 - F(\xi_0 + n^{-1/2}s)))^2)] \\
&\quad = n^{-1}m^2 E[\exp(-cmX^2)] \\
&\quad \leq cn^{-1}m^{3/2} \int_0^\infty \exp(-cx^2)dx,
\end{aligned}$$

where $\tau = \sup_{0 \leq x \leq 1} x^{-1}(F(\xi_0 + n^{-1/2}s + m^{-1/2}t) - F(\xi_0 + n^{-1/2}s))$. This implies condition (v) in Theorem 2.1 in Arcones (1999).

Condition (vi) in this theorem is obvious.

Cases (ii) and (iii) follow similarly. In the case (iv), we need to prove that

$$\left\{\sum_{j=1}^{n+[-np]} I\left(X_j \leq \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t)-F(\xi_0+n^{-1/2}u)}{1-F(\xi_0+n^{-1/2}s)}\right) : t \geq 0\right\}$$
$$\xrightarrow{\mathrm{w}} \{N(a^{-1/2}tF'(\xi_0)) : t \geq 0\}.$$

To get the convergence of the finite dimensional distributions, we need that for $0 < t_1 < t_2 < \cdots < t_h$ and $\lambda, \ldots, \lambda \in \mathbb{R}$,

$$E\left[\exp\left(\sum_{k=1}^{h}\lambda_k \sum_{j=1}^{n+[-np]} I\left(X_j \leq \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t)-F(\xi_0+n^{-1/2}s)}{1-F(\xi_0+n^{-1/2}s)}\right)\right)\right]$$
$$\xrightarrow{\mathrm{w}} E\left[\exp\left(\sum_{k=1}^{h}\lambda_k N(a^{-1/2}t_k F'(\xi_0))\right)\right].$$

We have that

$$E\left[\exp\left(\sum_{k=1}^{h}\lambda_k \sum_{j=1}^{n+[-np]} I\left(X_j \leq \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t_k)-F(\xi_0+n^{-1/2}u)}{1-F(\xi_0+n^{-1/2}s)}\right)\right)\right]$$
$$= \left(E\left[\exp\left(\sum_{l=1}^{h}\sum_{k=l}^{h}\lambda_k I\left(\frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t_{l-1})-F(\xi_0+n^{-1/2}u)}{1-F(\xi_0+n^{-1/2}s)} \leq X\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\leq \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t_l)-F(\xi_0+n^{-1/2}s)}{1-F(\xi_0+n^{-1/2}s)}\right)\right)\right]\right)^{n+[-np]}$$
$$= \left(\sum_{l=1}^{h} e^{\sum_{k=l}^{h}\lambda_k}\frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t_l)-F(\xi_0+n^{-1/2}s)+m^{-1/2}t_{l-1}}{1-F(\xi_0+n^{-1/2}s)}\right.$$
$$\left.+1-\frac{F(\xi_0+n^{-1/2}u+m^{-1/2}t_h)-F(\xi_0+n^{-1/2}u)}{1-F(\xi_0+n^{-1/2}s)}\right)^{n+[-np]}$$
$$\to \exp\left(\sum_{l=1}^{h}(e^{\sum_{k=l}^{h}\lambda_k}-1)a^{-1/2}(t_l-t_{l-1})F'(\xi_0)\right)$$
$$= E\left[\exp\left(\sum_{k=1}^{h}\lambda_k N(a^{-1/2}t_k F'(\xi_0))\right)\right].$$

So, condition (i) in Theorem 2.1 in Arcones (1999) follows. The rest of the conditions in this theorem follow similarly to those conditions in the case (i).

Observe that in the case (v), for each $s, t \in \mathbb{R}$,

$$\sum_{j=1}^{n+[-np]} I\left(X_j \leq \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t)-F(\xi_0+n^{-1/2}s)}{1-F(\xi_0+n^{-1/2}s)}\right) \xrightarrow{\Pr} 0.$$

We have that

$$\Pr\left\{\sum_{j=1}^{n+[-np]} I\left(X_j \le \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t)-F(\xi_0+n^{-1/2}s)}{1-F(\xi_0+n^{-1/2}s)}\right) > 0\right\}$$
$$\le n\Pr\left\{X \le \frac{F(\xi_0+n^{-1/2}s+m^{-1/2}t)-F(\xi_0+n^{-1/2}s)}{1-F(\xi_0+n^{-1/2}s)}\right\} \le cnm^{-1/2} \to 0. \quad \square$$

PROOF OF THEOREM 1.3. We only need to consider $|t| \le c\min((\log n)^{1/2}, (\log m)^{1/2})$. In the case (i), by Lemmas 2.4 and 2.5, we have that

$$\begin{aligned}
&m^{1/2}(\Pr{}^*\{m^{1/2}(\xi^*_{n,m}-\xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n-\xi_0) \le t\})\\
&\simeq m^{1/2}\left(\Phi\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\right)\right.\\
&\qquad\left.-\Phi\left(\frac{n^{1/2}(F(\xi_0+n^{-1/2}t)-p)}{(F(\xi_0+n^{-1/2}t)(1-F(\xi_0+n^{-1/2}t)))^{1/2}}\right)\right)\\
&\quad+\frac{(2F_n(\xi_n+m^{-1/2}t)-1)}{6(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\\
&\quad\cdot\psi\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\right)\\
&\quad-\frac{R(m(p-F_n(\xi_n+m^{-1/2}t))+F_n(\xi_n+m^{-1/2}t))}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\\
&\quad\cdot\phi\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\right)\\
&=: I+II+III.
\end{aligned}$$

We have that

$$\begin{aligned}
I &\simeq m^{1/2}\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\right.\\
&\qquad\left.-\frac{n^{1/2}(F(\xi_0+n^{-1/2}t)-F(\xi_0))}{(F(\xi_0+n^{-1/2}t)(1-F(\xi_0+n^{-1/2}t)))^{1/2}}\right)\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\simeq m^{1/2}\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}-\sigma_p^{-1}tF'(\xi_0)\right)\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\simeq \frac{m(F_n(\xi_n+m^{-1/2}t)-F_n(\xi_n)-F(\xi_n+m^{-1/2}t)+F(\xi_n))}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad+\frac{m^{1/2}(m^{1/2}(F(\xi_n+m^{-1/2}t)-F(\xi_n))-tF'(\xi_0))}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad+m^{1/2}\left(\frac{1}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\right.\\
&\qquad\left.-\frac{1}{(p(1-p))^{1/2}}\right)tF'(\xi_0)\phi(\sigma_p^{-1}tF'(\xi_0))\\
&=: I_1+I_2+I_3.
\end{aligned}$$

By Lemma 2.7, $I_1 \xrightarrow{\Pr} 0$. By the second differentiability of the df,

$$I_2 \xrightarrow{\Pr} 2^{-1}\sigma_p^{-1}t^2F''(\xi_0)\phi(\sigma_p^{-1}tF'(\xi_0)).$$

As to the last term

$$\begin{aligned}
I_3 &\simeq m^{1/2}\left(\frac{(p(1-p))^{1/2}-(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}{\sigma_p^2}\right)tF'(\xi_0)\\
&\quad\cdot\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\simeq m^{1/2}\left(\frac{p-p^2-F_n(\xi_n+m^{-1/2}t)+F_n^2(\xi_n+m^{-1/2}t)}{2\sigma_p^3}\right)tF'(\xi_0)\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\simeq 2^{-1}\sigma_p^{-3}(2p-1)(tF'(\xi_0))^2\phi(\sigma_p^{-1}tF'(\xi_0)).
\end{aligned}$$

We also have that

$$II \simeq -6^{-1}\sigma_p^{-1}(2p-1)(1-\sigma_p^{-2}t^2(F'(\xi_0))^2)\phi(\sigma_p^{-1}tF'(\xi_0))$$

and

$$III \simeq -\sigma_p^{-1}R(m(p-F_n(\xi_n+m^{-1/2}t))+F_n(\xi_n+m^{-1/2}t))\phi(\sigma_p^{-1}tF'(\xi_0)).$$

Therefore,

$$\begin{aligned}
&m^{1/2}(\Pr{}^*\{m^{1/2}(\xi_{n,m}^*-\xi_n)\le t\}-\Pr\{n^{1/2}(\xi_n-\xi_0)\le t\})\\
&\quad\simeq 2^{-1}\sigma_p^{-1}t^2F''(\xi_0)\phi(\sigma_p^{-1}t^2F'(\xi_0))\\
&\qquad+2^{-1}\sigma_p^{-3}(2p-1)(tF'(\xi_0))^2\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\qquad-6^{-1}\sigma_p^{-1}(2p-1)(1-\sigma_p^{-2}t^2(tF'(\xi_0)^2)\phi(\sigma_p^{-1}tF'(\xi_0)))\\
&\qquad-\sigma_p^{-1}R(m(F_n(\xi_n)-F_n(\xi_n+m^{-1/2}t))+F_n(\xi_n+m^{-1/2}t))\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad=2^{-1}\sigma_p^{-1}t^2F''(\xi_0)\phi(\sigma_p^{-1}t^2F'(\xi_0))\\
&\qquad+(1/3)\sigma_p^{-3}(2p-1)(tF'(\xi_0))^2\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\qquad+6^{-1}\sigma_p^{-1}(2p-1)\phi(\sigma_p^{-1}tF'(\xi_0)s)\\
&\qquad-\sigma_p^{-1}R(m(F_n(\xi_n)-F_n(\xi_n+m^{-1/2}t))+F_n(\xi_n+m^{-1/2}t))\phi(\sigma_p^{-1}tF'(\xi_0)).
\end{aligned}$$

The case (ii) follows similarly to case (i). But, in this case the term I_1:

$$\left\{\frac{m(F_n(\xi_n+m^{-1/2}t)-F_n(\xi_n)-F(\xi_n+m^{-1/2}t)+F(\xi_n))}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\phi(\sigma_p^{-1}tF'(\xi_0)s):t\in\mathbb{R}\right\}$$

converges weakly to

$$\{\sigma_p^{-1}a^{3/4}B(t)(F'(\xi_0))^{1/2}\phi(\sigma_p^{-1}a^{3/2}tF'(\xi_0)):t\in\mathbb{R}\}.$$

In the case (iii), we have that

$$\begin{aligned}
&m^{-1/4}n^{1/2}(\Pr{}^*\{m^{1/2}(\xi_{n,m}^*-\xi_n)\le t\}-\Pr\{n^{1/2}(\xi_n-\xi_0)\le t\})\\
&\quad\simeq m^{-1/4}n^{1/2}\left(\Phi\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)}{\sigma_{n,m}^*(t)}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\Phi\left(\frac{n^{1/2}(F(\xi_0+n^{-1/2}t)-p)}{\sigma_n(t)}\right)\bigg)\\
&\simeq m^{-1/4}n^{1/2}\left(\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-F_n(\xi_n))}{\sigma^*_{n,m}(t)}-\frac{tF'(\xi_0)}{\sigma_n(t)}\right)\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\simeq m^{-1/4}n^{1/2}\frac{m^{1/2}(F_n(\xi_n+m^{-1/2}t)-F_n(\xi_n)-F(\xi_n+m^{-1/2}t)+F(\xi_n))}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\\
&\quad\cdot\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad+m^{-1/4}n^{1/2}\frac{m^{1/2}(F(\xi_n+m^{-1/2}t)+F(\xi_n))-F'(\xi_0)t}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad+m^{-1/4}n^{1/2}\\
&\quad\cdot\left(\frac{1}{(F_n(\xi_n+m^{-1/2}t)(1-F_n(\xi_n+m^{-1/2}t)))^{1/2}}\right.\\
&\qquad\qquad\left.-\frac{1}{(p(1-p))^{1/2}}\right)tF'(\xi_0)\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad+m^{-1/4}n^{1/2}\left(\frac{1}{(p(1-p))^{1/2}}\right.\\
&\qquad\qquad\left.-\frac{1}{(F(\xi_0+n^{-1/2}t)(1-F(\xi_0+n^{-1/2}t)))^{1/2}}\right)tF'(\xi_0)\\
&\quad\cdot\phi(\sigma_p^{-1}tF'(\xi_0))\\
&\quad+m^{-1/4}n^{1/2}\frac{tF'(\xi_0)-n^{1/2}(F(\xi_0+n^{-1/2}t)-F(\xi_0))}{(F(\xi_0+n^{-1/2}t)(1-F(\xi_0+n^{-1/2}t)))^{1/2}}\phi(\sigma_p^{-1}tF'(\xi_0))\\
&=: IV+V+VI+VII+VIII.
\end{aligned}
$$

It is easy to see that $V, VI, VII, VIII \xrightarrow{\Pr} 0$. The term IV converges weakly to

$$\{\sigma_p^{-1}B(t)(F'(\xi_0))^{1/2}\phi(\sigma_p^{-1}tF'(\xi_0)) : t\in \mathbb{R}\}.$$

Cases (iv) and (v) follow by noticing that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}|D^*_{n,m_n}(t)-\Phi((\sigma^*_{n,m}(t))^{-1}m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p))\\
&\qquad\qquad+\Phi((\sigma_n(t))^{-1}n^{1/2}(F(\xi_0+n^{-1/2}t)-p))|\xrightarrow{\Pr}0,\\
&F_n(\xi_n)=-n^{-1}[-np]
\end{aligned}
$$

and

$$\sup_{t\in\mathbb{R}}|\Phi((\sigma_n(t))^{-1}n^{1/2}(F(\xi_0+n^{-1/2}t)-p))-\Phi(\sigma_p^{-1}tF'(\xi_0))|\xrightarrow{\Pr}0.$$

In the case (iv), we have that

$$
\begin{aligned}
&(\sigma^*_{n,m}(t))^{-1}m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)\\
&\quad\simeq\sigma_p^{-1}m^{1/2}n^{-1}([-np]-np)+\sigma_p^{-1}m^{1/2}(F_n(\xi_n+m^{-1/2}t)-F_n(\xi_n+m^{-1/2}t))\\
&\quad\simeq\sigma_p^{-1}a^{1/2}([-np]-np)+\sigma_p^{-1}a^{1/2}N(a^{-1/2}tF'(\xi_0)).
\end{aligned}
$$

Analogously, in the case (iv),

$$\begin{aligned}
&(\sigma^*_{n,m}(t))^{-1}m^{1/2}(F_n(\xi_n+m^{-1/2}t)-p)\\
&\quad\simeq \sigma_p^{-1}m^{1/2}n^{-1}([-np]-np)+\sigma_p^{-1}m^{1/2}(F_n(\xi_n+m^{-1/2}t)-F_n(\xi_n+m^{-1/2}t))\\
&\quad\simeq \sigma_p^{-1}m^{1/2}n^{-1}([-np]-np). \qquad \square
\end{aligned}$$

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