Ann. Inst. Statist. Math.
Vol. 55, No. 3, 499–506 (2003)

CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION BY STOCHASTIC ORDERING PROPERTIES OF THE GEOMETRIC COMPOUND

CHIN-YUAN HU[1] AND GWO DONG LIN[2]

[1] *Department of Business Education, National Changhua University of Education, Changhua 50058, Taiwan, R.O.C.*
[2] *Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan, R.O.C.,* e-mail: gdlin@stat.sinica.edu.tw

(Received August 3, 2001; revised May 21, 2002)

Abstract. Under the reliability NBU/NWU conditions, the exponential distribution is characterized by stochastic ordering properties which link the geometric compound with minimum order statistics or spacings of order statistics. This somewhat answers a question posed by Kakosyan, Klebanov and Melamed (1984, *Characterization of Distributions by the Method of Intensively Monotone Operators*, Springer, New York). We also show the related results based on the residual life in a renewal process and on record values. Finally, some fundamental properties of the NBUC/NWUC classes of life distributions are investigated.

Key words and phrases: Characterization, exponential distribution, geometric compounding model, stochastic order, increasing convex order, order statistics, record values, residual life.

1. Introduction

Consider a sequence of independent and identically distributed (i.i.d.) nonnegative and nondegenerate random variables $X, X_1, X_2, \dots, X_n, \dots$ with common distribution $F(x) = P(X \le x)$, $x \ge 0$. Assume that ν, independent of $\{X_n\}_{n=1}^{\infty}$, is a geometric random variable with parameter $p \in (0,1)$, namely, $P(\nu = n) = p(1-p)^{n-1}$ for $n = 1, 2, \dots$. Then the random sum $S_\nu \equiv \sum_{n=1}^{\nu} X_n$ is called a geometric compound of the sequence $\{X_n\}_{n=1}^{\infty}$. The geometric compounding model is useful in many fields, such as risk theory, queueing theory, reliability and distribution theory (see Hu and Lin (2001) and the references therein). In this paper, we shall consider the characterization of exponential distribution through the properties of geometric compound. A remarkable characterization result can be stated as follows (Arnold (1973) and Azlarov *et al.* (1972)):

(AADS) Under the geometric compounding model, pS_ν has the same distribution as X (denoted $pS_\nu \overset{d}{=} X$) if and only if F is exponential.

On the other hand, let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the corresponding order statistics of $\{X_k\}_{k=1}^{n}$ defined above. Then under the condition

(C) $\lim_{x \to 0^+} F(x)/x = \lambda$ for some $\lambda \in (0, \infty)$,

the identity $nX_{1,n} \overset{d}{=} X$ (for some $n \ge 2$) characterizes F to be exponential (Gupta (1973); note that without the condition (C), the conclusion fails). Based on this fact and the above AADS result, Kakosyan *et al.* ((1984), p. 41) posed the question:

(KKM) Under the condition (C), is it true that the identity $pS_\nu \stackrel{d}{=} nX_{1,n}$ (for some $n \geq 2$) is a characteristic property for the exponential distribution?

The KKM question still remains unanswered. But, surprisingly, assuming the reliability NWU condition on F instead of the above crucial condition (C), the identity in the question does characterize F to be an exponential distribution. Moreover, in the next section we are able to weaken the identity condition to a stochastic inequality (Theorem 2.1). The exponential distribution is also characterized by using stochastic ordering relationships between the geometric compound and spacings of order statistics (Theorem 2.2). We show some related results based on the residual life in a renewal process and on record values (Theorems 2.3 and 2.4). Finally, some useful properties (which play crucial roles in Section 2) are further investigated in Section 3 (Theorems 3.1 and 3.2).

2. Characterizations by stochastic inequalities

To state the main results, we need more notations. Let Y and Z be two nonnegative random variables with respective distributions G and H. For convenience, denote $\overline{G} = 1 - G$ and $\overline{H} = 1 - H$. Then we say that Y is smaller than Z in the stochastic order (denoted $Y \leq_{st} Z$) if $\overline{G}(t) \leq \overline{H}(t)$ for all $t \geq 0$, that Y is smaller than Z in the increasing convex order (denoted $Y \leq_{icx} Z$) if $\int_x^\infty \overline{G}(t)dt \leq \int_x^\infty \overline{H}(t)dt$ for all $x \geq 0$, and that Y is smaller than Z in the Laplace transform order (denoted $Y \leq_L Z$) if $E(\exp(-sY)) \leq E(\exp(-sZ))$ for all $s \geq 0$. The three kinds of ordering satisfy the implications: (i) if $Y \leq_{st} Z$ then $Y \leq_{icx} Z$ and $Z \leq_L Y$, and (ii) if $Y \leq_{icx} Z$ with $E(Y) = E(Z) < \infty$, then $Y \leq_L Z$ (see, e.g., Stoyan (1983), pp. 8–9, and Shaked and Shanthikumar (1994), pp. 83–93). Also, a distribution F on $[0, \infty)$ is said to be new better than used (NBU) if $\overline{F}(x + y) \leq \overline{F}(x)\overline{F}(y)$ for all $x, y \geq 0$, while F is new worse than used (NWU) if $\overline{F}(x+y) \geq \overline{F}(x)\overline{F}(y)$ for all $x, y \geq 0$. Using stochastic inequalities, we have the following characterization results.

THEOREM 2.1. *Under the geometric compounding model, assume that* $N \geq 1$, *independent of* $\{X_n\}_{n=1}^\infty$, *is an integer-valued random variable and let* $X_{1,N} = \min\{X_1, \ldots, X_N\}$.

(a) *If* (i) F *is* NWU *and* (ii) $pS_\nu \leq_{st} NX_{1,N}$, *then* F *is exponential.*

(b) *If* (i) F *is* NBU, (ii) $E(X) < \infty$ *and* (iii) $NX_{1,N} \leq_{st} pS_\nu$, *then* F *is exponential.*

COROLLARY 2.1. *Under the geometric compounding model,*

(a) *if* (i) F *is* NWU *and* (ii) $pS_\nu \leq_{st} nX_{1,n}$ *for some* $n \geq 2$, *then* F *is exponential*;

(b) *if* (i) F *is* NBU, (ii) $E(X) < \infty$ *and* (iii) $nX_{1,n} \leq_{st} pS_\nu$ *for some* $n \geq 2$, *then* F *is exponential.*

To prove Theorem 2.1, we need the following lemmas. The first crucial lemma is a refinement of Lin and Hu's (2001) Theorem 2 and will be used in the sequel. It extends the above AADS result; for further extension see Theorem 3.1 below. Recall that a distribution F on $[0, \infty)$ is said to be in the $\mathcal{L}$-class of life distributions if its mean μ is finite and if its Laplace transform L satisfies the relation: $L(s) \leq 1/(1 + \mu s)$ for all $s \geq 0$. For applications and properties of the $\mathcal{L}$-class of life distributions, see, e.g., Klefsjö (1983) and Lin (1998).

LEMMA 2.1. *Under the geometric compounding model,*

(a) *if* $X \leq_L pS_\nu$, *then* $F \in \mathcal{L}$ *and* $E(X^2) < \infty$;

(b) (i) *if* $pS_\nu \leq_{icx} X$ *and if the coefficient of variation* (CV) *of* X *is equal to one, then* F *is exponential*; (ii) *if* $X \leq_{icx} pS_\nu$ *and* $\mathrm{CV}(X) = 1$, *then* F *is exponential*;

(c) *if* $pS_\nu \leq_{st} X$, *then* F *is exponential.*

PROOF. (a) As in the proof of Lin and Hu's (2001) Theorem 2, set $k(s) = 1/L(s) - 1$, $s \geq 0$, where $L(s) = E(\exp(-sX))$. Then it follows from the assumption $X \leq_L pS_\nu$ that for fixed $s > 0$ and $n \geq 1$,

$$\frac{k(s)}{s} \geq \frac{k(ps)}{ps} \geq \cdots \geq \frac{k(p^n s)}{p^n s} = k'(\theta_n p^n s), \qquad \text{where} \quad \theta_n \in (0,1).$$

The last term tends to $k'(0^+) \equiv \lim_{s\to 0^+} k'(s) = -L'(0^+) = E(X)$ as $n \to \infty$ (Royden (1988), p. 265). Therefore $E(X) \leq k(s)/s < \infty$, or, equivalently, $L(s) \leq 1/(1 + sE(X))$ for $s \geq 0$. This means that $F \in \mathcal{L}$ and hence $E(X^2) < \infty$ (see, e.g., Lin (1998), Theorem 5).

(b) Suppose that $pS_\nu \leq_{icx} X$ and $\mathrm{CV}(X) = 1$. Then we have $E\{(pS_\nu)^2\} = E(X^2) < \infty$ (Hu and Lin (2001), Lemma 1) and $pS_\nu \stackrel{d}{=} X$ by Theorem 2.1 of Huang and Lin (1999). Using the AADS result we conclude that F is exponential. The second part can be proved similarly.

(c) Suppose $pS_\nu \leq_{st} X$. Then $pS_\nu \leq_{icx} X$ and $X \leq_L pS_\nu$. The latter implies that $E(X^2) < \infty$ by part (a), and the former together with the fact $E(pS_\nu) = E(X) < \infty$ implies that $pS_\nu \leq_L X$. Hence pS_ν and X are equal in Laplace transform. This means that $pS_\nu \stackrel{d}{=} X$ and hence F is exponential. (The proof is somewhat different from the previous one.)

LEMMA 2.2. *Let* X, $\{X_n\}_{n=1}^\infty$ *and* N *be the same as in Theorem* 2.1.

(a) *If* F *is* NWU, *then* $NX_{1,N} \leq_{st} X$.

(b) *If* F *is* NBU, *then* $X \leq_{st} NX_{1,N}$.

PROOF. Note that for $x \geq 0$, we have

$$P(NX_{1,N} > x) = \sum_{n=1}^{\infty} P(N = n)P(nX_{1,n} > x) = \sum_{n=1}^{\infty} P(N = n)(\overline{F}(x/n))^n. \tag{2.1}$$

Suppose F is NWU. Then $(\overline{F}(x/n))^n \leq \overline{F}(x)$. Applying this result to identity (2.1) yields that $P(NX_{1,N} > x) \leq \overline{F}(x) = P(X > x)$ for $x \geq 0$; namely, $NX_{1,N} \leq_{st} X$. This proves part (a). Part (b) can be proved by a similar argument.

PROOF OF THEOREM 2.1. (a) Suppose that F is NWU and $pS_\nu \leq_{st} NX_{1,N}$. Then $NX_{1,N} \leq_{st} X$ by Lemma 2.2(a), and hence $pS_\nu \leq_{st} X$. This implies that F is exponential by Lemma 2.1(c).

(b) Suppose that F is NBU, $E(X) < \infty$ and $NX_{1,N} \leq_{st} pS_\nu$. Then by Lemma 2.2(b), we have $X \leq_{st} NX_{1,N} \leq_{st} pS_\nu$ and $E(X) = E(pS_\nu) < \infty$. This implies that $X \stackrel{d}{=} pS_\nu$ (see, e.g., Shaked and Shanthikumar (1994), p. 8) and hence F is exponential by the AADS result. The proof of the theorem is complete.

For order statistics $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$, define the normalized spacings $D_{k,n} = (n-k+1)(X_{k,n} - X_{k-1,n})$ for $k = 1, 2, \ldots, n$, where $X_{0,n} \equiv 0$. It is known that the identity $D_{n,n} \stackrel{d}{=} X$ (for some $n \ge 2$) is a characteristic property of the exponential distribution among the continuous distributions. Puri and Rubin (1970) proved this characterization result for the case $n = 2$. As for general results, see, e.g., Rao and Shanbhag ((1994), Theorem 8.2.5). We shall now investigate the characterization of exponential distribution by stochastic ordering relationships between the geometric compound and the random spacing $D_{N,N}$. Namely, using $D_{N,N}$ instead of $NX_{1,N}$, we have the counterpart of Theorem 2.1.

THEOREM 2.2. *Under the geometric compounding model, assume that $N \ge 1$, independent of $\{X_n\}_{n=1}^{\infty}$, is an integer-valued random variable and that F is a continuous distribution. Further, let $D_{N,N} = X_{N,N} - X_{N-1,N}$, where $X_{N,N} = \max\{X_1, \ldots, X_N\}$.*
(a) *If* (i) *F is* NBU *and* (ii) *$pS_\nu \le_{st} D_{N,N}$, then F is exponential.*
(b) *If* (i) *F is* NWU, (ii) *$E(X) < \infty$ and* (iii) *$D_{N,N} \le_{st} pS_\nu$, then F is exponential.*

To prove Theorem 2.2, we need the following lemma.

LEMMA 2.3. *Let $X, \{X_n\}_{n=1}^{\infty}$, N and $D_{N,N}$ be the same as in Theorem* 2.2.
(a) *If F is* NBU, *then $D_{N,N} \le_{st} X$.*
(b) *If F is* NWU, *then $X \le_{st} D_{N,N}$.*

PROOF. For $x \ge 0$, we have

$$P(D_{N,N} > x) = P(N=1)P(X > x) + \sum_{n=2}^{\infty} P(N=n)P(X_{n,n} - X_{n-1,n} > x).$$

Since F is continuous, the order statistics $\{X_{k,n}\}_{k=1}^{n}$ form a Markov chain (see, e.g., Galambos and Kotz (1978), p. 38) and hence we can write, for $n \ge 2$ and $x \ge 0$,

$$P(X_{n,n} - X_{n-1,n} > x) = \int_0^\infty n(n-1)(F(t))^{n-2}\overline{F}(t+x)dF(t). \tag{2.2}$$

Suppose that F is NBU. Then it follows from identity (2.2) that for $n \ge 2$ and $x \ge 0$,

$$P(X_{n,n} - X_{n-1,n} > x) \le \int_0^\infty n(n-1)(F(t))^{n-2}\overline{F}(t)\overline{F}(x)dF(t) = \overline{F}(x).$$

Therefore $P(D_{N,N} > x) \le \sum_{n=1}^{\infty} P(N=n)\overline{F}(x) = \overline{F}(x) = P(X > x)$ for $x \ge 0$. Namely, $D_{N,N} \le_{st} X$. This proves part (a). Part (b) can be proved by a similar argument.

PROOF OF THEOREM 2.2. (a) Suppose that F is NBU and $pS_\nu \le_{st} D_{N,N}$. Then by Lemma 2.3(a), we have $pS_\nu \le_{st} D_{N,N} \le_{st} X$, and hence F is exponential due to Lemma 2.1(c).

(b) Suppose that F is NWU, $E(X) < \infty$ and $D_{N,N} \le_{st} pS_\nu$. Then by Lemma 2.3(b), we have $X \le_{st} D_{N,N} \le_{st} pS_\nu$ and $E(X) = E(pS_\nu) < \infty$. Therefore $X \stackrel{d}{=} pS_\nu$. This implies that F is exponential by the AADS result. The proof of the theorem is complete.

Next, we shall show two more characteristic properties of the exponential distribution through the residual life in a renewal process defined below. Let $X, X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. nonnegative and nondegenerate random variables with common distribution F. Define the renewal process $\{S_n\}_{n=1}^{\infty}$ by $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$. For $t > 0$, let $N(t)$ be the number of points S_n in the interval $[0, t]$ and let $\gamma(t)$ be the residual life at time t, i.e., $\gamma(t) \equiv S_{N(t)+1} - t$. If $E(X) < \infty$, the identity $\gamma(t) \stackrel{d}{=} X$ (for some $t > 0$) characterizes the exponential distribution among the *continuous* distributions (see Isham *et al.* (1975) or Galambos and Kotz (1978), p. 94). It is natural to ask the question: Under what conditions, is it true that the identity $pS_\nu \stackrel{d}{=} \gamma(t)$ (for some $t > 0$) characterizes the exponential distribution? We shall try to answer this question. In the next result we don't require the continuity condition on the underlying distribution F; instead, we assume the reliability NBU/NWU property of F.

THEOREM 2.3. *Under the geometric compounding model,*

(a) *if F is* NBU *and $pS_\nu \leq_{st} \gamma(t)$ for some $t > 0$, then F is exponential*;

(b) *if F is* NWU, *$E(X) < \infty$ and $\gamma(t) \leq_{st} pS_\nu$ for some $t > 0$, then F is exponential.*

PROOF. Recall that the residual life $\gamma(t)$ has the properties:

(i) if F is NBU, then $\gamma(t) \leq_{st} X$ for each $t > 0$;

(ii) if F is NWU, then $X \leq_{st} \gamma(t)$ for each $t > 0$

(see, e.g., Barlow and Proschan (1981), p. 169). These together with Lemma 2.1 imply the required results. The proof is complete.

Finally, we give two characteristic properties of the exponential distribution through the spacings of record values defined below. Let $X, X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. nonnegative random variables with common *continuous* distribution F. Define the record times $\{T(n)\}_{n=0}^{\infty}$ by $T(0) = 1$ and $T(n) = \min\{m : m > T(n-1), X_m > X_{T(n-1)}\}$ for $n \geq 1$. Then $\{X_{T(n)}\}_{n=0}^{\infty}$ is called the record values of the sequence $\{X_n\}_{n=1}^{\infty}$. It is known that if $X_{T(n)} - X_{T(n-1)} \stackrel{d}{=} X$ for some $n \geq 1$, then F is exponential (see Lau and Rao (1982) and Witte (1988)). As before, using the geometric compound we have the following result.

THEOREM 2.4. *Under the geometric compounding model with F being continuous,*

(a) *if F is* NBU *and $pS_\nu \leq_{st} X_{T(n)} - X_{T(n-1)}$ for some $n \geq 1$, then F is exponential*;

(b) *if F is* NWU, *$E(X) < \infty$ and $X_{T(n)} - X_{T(n-1)} \leq_{st} pS_\nu$ for some $n \geq 1$, then F is exponential.*

PROOF. Recall the Markov property of record values: $P(X_{T(n)} > x \mid X_{T(n-1)} = y) = \overline{F}(x)/\overline{F}(y)$ for $x \geq y$ (see Shorrock (1972) or Azlarov and Volodin (1986), p. 27). Then we have

$$P(X_{T(n)} - X_{T(n-1)} > x) = \int_Q P(X_{T(n)} > x + z \mid X_{T(n-1)} = z) dH_{n-1}(z)$$

$$= \int_Q \frac{\overline{F}(x+z)}{\overline{F}(z)} dH_{n-1}(z) \quad \text{for all} \quad x > 0, \tag{2.3}$$

in which H_{n-1} denotes the distribution of $X_{T(n-1)}$ and Q is its support. Therefore,

(i) if F is NBU, then $X_{T(n)} - X_{T(n-1)} \le_{st} X$ for all $n \ge 1$;

(ii) if F is NWU, then $X \le_{st} X_{T(n)} - X_{T(n-1)}$ for all $n \ge 1$.

Lemma 2.1(c) then completes the proof.

3. Extensions

Lemma 2.1 plays a crucial role in the previous results. We shall now consider its possible extension. In the geometric compounding model, replace the random variable ν by the Pascal (negative binomial) random variable ν_κ having mass function

$$P(\nu_\kappa = n) = \binom{n-1}{\kappa-1} p^\kappa (1-p)^{n-\kappa}, \qquad n = \kappa, \kappa+1, \ldots,$$

where $p \in (0,1)$ and κ is a positive integer. The resulting Pascal compounding model has realistic applications in sickness and accident insurance (Grandell (1997), Preface). We now extend Lemma 2.1 to the following.

THEOREM 3.1. *Under the Pascal compounding model,*

(a) *if* $\sum_{n=1}^{\kappa} X_n \le_L pS_{\nu_\kappa}$, *then* $F \in \mathcal{L}$ *and* $E(X^2) < \infty$;

(b) $pS_{\nu_\kappa} \le_{st} \sum_{n=1}^{\kappa} X_n$ *if and only if* F *is exponential.*

PROOF. Observe that $E\{\exp(-spS_{\nu_\kappa})\} = (E\{\exp(-spS_\nu)\})^\kappa$ and $E\{\exp(-s\sum_{n=1}^{\kappa} X_n)\} = (E\{\exp(-sX)\})^\kappa$ for $s \ge 0$. Then part (a) follows immediately from Lemma 2.1(a).

Next, we consider part (b). The proof of the sufficiency part is trivial and omitted. It remains to prove the necessity part. Suppose that $pS_{\nu_\kappa} \le_{st} \sum_{n=1}^{\kappa} X_n$. Then $\sum_{n=1}^{\kappa} X_n \le_L pS_{\nu_\kappa}$ and hence $E(X^2) < \infty$ by part (a). Further, $E(pS_{\nu_\kappa}) = E(\sum_{n=1}^{\kappa} X_n) = \kappa E(X) < \infty$. This together with the stochastic inequality assumption implies that $pS_{\nu_\kappa} \stackrel{d}{=} \sum_{n=1}^{\kappa} X_n$ and hence $pS_\nu \stackrel{d}{=} X$ due to the above observation. By the AADS result, F is exponential. The proof is complete.

Finally, we investigate the NBUC/NWUC classes of life distributions defined below. A distribution F on $[0,\infty)$ is said to be new better than used in convex ordering (NBUC) if

$$\int_x^\infty \overline{F}(y+z)dz \le \overline{F}(y) \int_x^\infty \overline{F}(z)dz \quad \text{for all} \quad x, y \ge 0 \tag{3.1}$$

(see Cao and Wang (1991)). Similarly, F is new worse than used in convex ordering (NWUC) if the inequality in (3.1) is reversed. Clearly, NBU $\Rightarrow$ NBUC and NWU $\Rightarrow$ NWUC. We have the following convex-ordering inequalities for the NBUC/NWUC classes, which are counterparts of the above-mentioned stochastic-ordering inequalities for the NBU/NWU classes. For more properties and applications of these classes of life distributions, see Hendi *et al.* (1993), Li *et al.* (2000) and Willmot and Lin ((2001), p. 95).

THEOREM 3.2. *Let* $X, X_1, X_2, \ldots, X_n, \ldots$ *be a sequence of i.i.d. nonnegative random variables with common distribution* F, *and let* $D_{N,N}$, $\gamma(t)$ *and* $X_{T(n)}$ *be the same as in Theorems* 2.2–2.4.

(a) *If F is continuous and* NBUC, *then* $D_{N,N} \leq_{icx} X$.
(b) *If F is continuous and* NWUC, *then* $X \leq_{icx} D_{N,N}$.
(c) *If F is* NBUC, *then* $\gamma(t) \leq_{icx} X$ *for each* $t > 0$.
(d) *If F is* NWUC, *then* $X \leq_{icx} \gamma(t)$ *for each* $t > 0$.
(e) *If F is continuous and* NBUC, *then* $X_{T(n)} - X_{T(n-1)} \leq_{icx} X$ *for each* $n \geq 1$.
(f) *If F is continuous and* NWUC, *then* $X \leq_{icx} X_{T(n)} - X_{T(n-1)}$ *for each* $n \geq 1$.

PROOF. To prove part (a), recall from identity (2.2) that for $n \geq 2$ and $x \geq 0$,

$$\int_x^\infty P(X_{n,n} - X_{n-1,n} > z)dz = \int_0^\infty n(n-1)(F(t))^{n-2} \int_x^\infty \overline{F}(t+z)dzdF(t).$$

If, in addition, F is NBUC, then for $x \geq 0$,

$$\int_x^\infty P(X_{n,n} - X_{n-1,n} > z)dz \leq \int_x^\infty \overline{F}(z)dz \int_0^\infty n(n-1)(F(t))^{n-2}\overline{F}(t)dF(t) \tag{3.2}$$

$$= \int_x^\infty \overline{F}(z)dz. \tag{3.3}$$

Namely, $D_{n,n} \leq_{icx} X$ for each $n \geq 2$. Next, we have that for $x \geq 0$,

$$\begin{aligned}
&\int_x^\infty P(D_{N,N} > z)dz \\
&= P(N=1)\int_x^\infty \overline{F}(z)dz + \sum_{n=2}^\infty P(N=n)\int_x^\infty P(D_{n,n} > z)dz \\
&\leq P(N=1)\int_x^\infty \overline{F}(z)dz + \sum_{n=2}^\infty P(N=n)\int_x^\infty \overline{F}(z)dz \qquad (3.4)\\
&= \int_x^\infty \overline{F}(z)dz,
\end{aligned}$$

in which the inequality is due to (3.2) and (3.3). Therefore, $D_{N,N} \leq_{icx} X$. This proves part (a). If F is NWUC, a similar argument applies with inequalities in (3.2) and (3.4) reversed, so part (b) is proved. As for parts (c) and (d), we can apply Barlow and Proschan's ((1981), p. 169) argument, while for parts (e) and (f) we apply the argument in (2.3). The proof is complete.

Acknowledgements

The authors would like to thank the referees for helpful comments which improve the presentation of the paper.

REFERENCES

Arnold, B. C. (1973). Some characterizations of the exponential distribution by geometric compounding, *SIAM J. Appl. Math.*, **24**, 242–244.

Azlarov, T. A. and Volodin, N. A. (1986). *Characterization Problems Associated with the Exponential Distribution* (translated from the Russian by Margaret Stein, translation edited by Ingram Olkin), Springer, New York.

Azlarov, T. A., Dzamirzaev, A. A. and Sultanova, M. M. (1972). Characterizing properties of the exponential distribution, and their stability, *Random Processes and Statistical Inference* (Russian), No. II, **94**, 10–19, Izdat. "Fan" Uzbek, SSR, Tashkent. [MR 48 (1974) 3150]

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing. Probability Models*, 2nd ed., To Begin With, Silver Spring, Maryland.

Cao, J. H. and Wang, Y. D. (1991). The NBUC and NWUC classes of life distributions, *J. Appl. Probab.*, **28**, 473–479 (Correction: *ibid.* (1992). **29**, p. 753).

Galambos, J. and Kotz, S. (1978). *Characterizations of Probability Distributions. A Unified Approach with an Emphasis on Exponential and Related Models*, Lecture Notes in Math., **675**, Springer, New York.

Grandell, J. (1997). *Mixed Poisson Processes*, Monogr. Statist. Appl. Probab., **77**, Chapman & Hall, New York.

Gupta, R. C. (1973). A characteristic property of the exponential distribution, *Sankhyā Ser. B*, **35**, 365–366.

Hendi, M. I., Mashhour, A. F. and Montasser, M. A. (1993). Closure of the NBUC class under formulation of parallel systems, *J. Appl. Probab.*, **30**, 975–978.

Hu, C.-Y. and Lin, G. D. (2001). On the geometric compounding model with applications, *Probab. Math. Statist.*, **21**, 135–147.

Huang, J. S. and Lin, G. D. (1999). Equality in distribution in a convex ordering family, *Ann. Inst. Statist. Math.*, **51**, 345–349.

Isham, V., Shanbhag, D. N. and Westcott, M. (1975). A characterization of the Poisson process using forward recurrence times, *Math. Proc. Cambridge Philos. Soc.*, **78**, 513–516.

Kakosyan, A. V., Klebanov, L. B. and Melamed, J. A. (1984). *Characterization of Distributions by the Method of Intensively Monotone Operators*, Lecture Notes in Math., **1088**, Springer, New York.

Klefsjö, B. (1983). A useful ageing property based on the Laplace transform, *J. Appl. Probab.*, **20**, 615–626.

Lau, K.-S. and Rao, C. R. (1982). Integrated Cauchy functional equation and characterizations of the exponential law, *Sankhyā Ser. A*, **44**, 72–90 (Corrigendum: *ibid.* (1982). **44**, p. 452).

Li, X., Li, Z. and Jing, B.-Y. (2000). Some results about the NBUC class of life distributions, *Statist. Probab. Lett.*, **46**, 229–237.

Lin, G. D. (1998). Characterizations of the $\mathcal{L}$-class of life distributions, *Statist. Probab. Lett.*, **40**, 259–266.

Lin, G. D. and Hu, C.-Y. (2001). Characterizations of distributions via the stochastic ordering property of random linear forms, *Statist. Probab. Lett.*, **51**, 93–99.

Puri, P. S. and Rubin, H. (1970). A characterization based on the absolute difference of two i.i.d. random variables, *Ann. Math. Statist.*, **41**, 2113–2122.

Rao, C. R. and Shanbhag, D. N. (1994). *Choquet-Deny Type Functional Equations with Applications to Stochastic Models*, Wiley, New York.

Royden, H. L. (1988). *Real Analysis*, 3rd ed., Prentice-Hall, New Jersey.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and Their Applications*, Academic Press, New York.

Shorrock, R. W. (1972). A limit theorem for inter-record times, *J. Appl. Probab.*, **9**, 219–223 (Correction: *ibid.* (1972). **9**, p. 877).

Stoyan, D. (1983). *Comparison Methods for Queues and Other Stochastic Models* (translation from the German edited by Daryl J. Daley), Wiley, New York.

Willmot, G. E. and Lin, X. S. (2001). *Lundberg Approximations for Compound Distributions with Insurance Applications*, Lecture Notes in Statist., **156**, Springer, New York.

Witte, H.-J. (1988). Some characterizations of distributions based on the integrated Cauchy functional equation, *Sankhyā Ser. A*, **50**, 59–63.