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BOOTSTRAP FOR THE CONDITIONAL DISTRIBUTION FUNCTION WITH TRUNCATED AND CENSORED DATA

M. C. IGLESIAS PÉREZ[1] AND W. GONZÁLEZ MANTEIGA[2]

[1] *Departamento de Estadística e Investigación Operativa, Escuela Universitaria de Ingeniería Técnica Forestal, Universidad de Vigo, Campus de Pontevedra, Pontevedra 36005, Spain*
[2] *Departamento de Estadística e Investigación Operativa, Facultad de Matemáticas, Universidad de Santiago de Compostela, Campus Sur, Santiago de Compostela 15706, Spain*

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Abstract. We propose a resampling method for left truncated and right censored data with covariables to obtain a bootstrap version of the conditional distribution function estimator. We derive an almost sure representation for this bootstrapped estimator and, as a consequence, the consistency of the bootstrap is obtained. This bootstrap approximation represents an alternative to the normal asymptotic distribution and avoids the estimation of the complicated mean and variance parameters of the latter.

Key words and phrases: Censored data, truncated data, kernel estimator, generalized product-limit estimator, bootstrapped estimator, asymptotic representation, consistency.

1. Introduction

In this paper we study lifetime data with covariables which are subject to both left truncation and right censorship. Let (X, Y, T, S) be a random vector, where Y is the lifetime, T is the random left truncation time, S denotes the random right censoring time and X is a covariable related with Y. It is assumed that Y, T, S are conditionally independent at $X = x$ and $\alpha(x) = P(T \leq Z \mid X = x) > 0$, where $Z = \min\{Y, S\}$. In this model, one observes (X, Z, T, δ) if $Z \geq T$, where $\delta = 1_{\{Y \leq S\}}$. When $Z < T$ nothing is observed. (We will refer to this model as LTRC model.)

Let $(X_i, Z_i, T_i, \delta_i)$, for $i = 1, 2, \dots, n$, be an i.i.d. random sample from (X, Z, T, δ) which one observes (then $T_i \leq Z_i$, for all i). If $F(y \mid x) = P(Y \leq y \mid X = x)$ denotes the conditional distribution function of Y when $X = x$, a nonparametric estimator of $F(\cdot \mid x)$, called generalized product-limit estimator (GPLE), $\hat{F}_h(\cdot \mid x)$, is defined in Iglesias Pérez and González Manteiga (1999), as follows:

$$\hat{F}_h(y \mid x) = 1 - \prod_{i=1}^{n} \left[1 - \frac{1_{\{Z_i \leq y\}} \delta_i B_{hi}(x)}{\sum_{j=1}^{n} 1_{\{T_j \leq Z_i \leq Z_j\}} B_{hj}(x)} \right] \tag{1.1}$$

where $\{B_{hi}(x)\}_{i=1}^{n}$ is a sequence of nonparametric weights (specifically, Nadaraya and Watson weights) and $h = h_n$ is the bandwidth parameter. Moreover, some important properties about this estimator are provided: an almost sure asymptotic representation of $\hat{F}_h(\cdot \mid x)$, the asymptotic normality of $(nh)^{1/2}(\hat{F}_h(y \mid x) - F(y \mid x))$ and the weak convergence of the corresponding process.

Note that the GPLE reduces to a GPLE with left truncation (see LaValley and Akritas (1994)) when there is no right censoring ($\delta = 1, Z = Y$) and to a GPLE with right censorship when there is no left truncation ($T = 0$). The GPLE with censored data has been further studied by Beran (1981), Dabrowska (1989), González-Manteiga and Cadarso-Suárez (1994), Akritas (1994) or Van Keilegom and Veraverbeke (1997), among others. These latter authors, in addition with other results, define a bootstrap procedure for approximating the distribution of $(nh)^{1/2}(\hat{F}_h(y \mid x) - F(y \mid x))$ and show the validity of their proposed bootstrap procedure.

The aim of this paper is to study a bootstrap method for left truncated and right censored data (LTRC data) with covariables to obtain a bootstrapped estimator of the conditional distribution function.

Bootstrap methods for LTRC data without covariables have been studied by Wang (1991), Bilker and Wang (1997) and Gross and Lai (1996). Wang generalizes Efron's "obvious" bootstrap method for RC (Efron (1981)) to LTRC data under certain assumptions about the identifiability of F (the distribution function of Y) and assuming that the variables Y and (T, S) are independent, $S \geq T$ and $D = S - T$ is independent of T. Her "obvious" method starts by estimating the unknown distribution functions of Y, T and D, denoted by F, L and Q, respectively, through the corresponding non parametric maximum likelihood estimators (NPMLE): $\hat{F}$, $\hat{L}$ and $\hat{Q}$. Immediately after, it generates independent random variables: Y_i^* from $\hat{F}$, T_i^* from $\hat{L}$ and D_i^* from $\hat{Q}$ and defines $Z_i^* = \min\{Y_i^*, S_i^*\}$ where $S_i^* = D_i^* + T_i^*$ and $\delta_i^* = 1_{\{Z_i^* = Y_i^*\}}$. The observation $(T_i^*, Z_i^*, \delta_i^*)$ is retained if and only if $T_i^* \leq Z_i^*$, so the bootstrap sample consists of n random vectors $(T_1^*, Z_1^*, \delta_1^*), \ldots, (T_n^*, Z_n^*, \delta_n^*)$ with $T_i^* \leq Z_i^*$. The validity of this "obvious" bootstrap method is proved by Bilker and Wang (1997). These authors also point out that the previous "obvious" method is not equivalent to the "simple" bootstrap for LTRC data, again using Efron's (1981) term, which places equal weight, $1/n$, at each of the observed triplets $\{(T_i, Z_i, \delta_i)\}_{i=1}^n$. Gross and Lai (1996) define the "simple" bootstrap for LTRC data in a model with covariables, so that, the bootstrap sample consists of n random vectors $(X_1^*, T_1^*, Z_1^*, \delta_1^*), \ldots, (X_n^*, T_n^*, Z_n^*, \delta_n^*)$ obtained by sampling with replacement and placing equal mass at each of the observed vectors $\{(X_i, T_i, Z_i, \delta_i)\}_{i=1}^n$. They develop an asymptotic theory of the simple bootstrap method for this extended model, showing that the simple bootstrap approximations to the sampling distributions of various statistics from these data are accurate to the order of $O_P(n^{-1})$. It is worth pointing out that between the mentioned statistics the only that have covariables is an estimator of the parameter β, where β is the minimizer of $E\{\rho(Y - \beta^T X)1_{\{a \leq Y \leq b\}}\}$ in some region D, where $[a, b]$ is an appropriate interval and ρ is a convex and differentiable real function.

In Section 3 we propose a "conditional obvious" bootstrap method for LTRC data with covariables in which the conditional independence of Y, T, S given $X = x$ is assumed. This new assumption is a bit more stringent than that adopted by Wang in one-sample case but is more convenient for the development of theory and methods (see also Remark 1 in Section 3).

2. Notation and assumptions

To define and study our bootstrap procedure, which is done in Sections 3 and 4, we need to introduce some notation and assumptions. The following curves are defined:

(i) $M(x) = P(X \leq x)$, represents the distribution function of X.

(ii) $G(y \mid x) = P(S \leq y \mid X = x)$, is the conditional distribution function of S at $X = x$.

(iii) $L(y \mid x) = P(T \leq y \mid X = x)$, is the conditional distribution function of T at $X = x$.

(iv) $H(y \mid x) = P(Z \leq y \mid X = x)$, is the conditional distribution function of Z at $X = x$.

(v) $H_1(y \mid x) = P(Z \leq y, \delta = 1 \mid X = x)$, is the conditional subdistribution function (when $Z = Y$) of Z at $X = x$.

(vi) The conditional cumulative hazard rate function of Y at $X = x$, which is

$$\Lambda(y \mid x) = \int_{-\infty}^{y} \frac{dF(t \mid x)}{1 - F(t^- \mid x)}. \tag{2.1}$$

This function can be written as

$$\Lambda(y \mid x) = \int_{-\infty}^{y} \frac{dH_1^{\#}(t \mid x)}{C(t \mid x)} \tag{2.2}$$

where $C(y \mid x) = P(T \leq y \leq Z \mid X = x, T \leq Z)$, (see equation (5) in Iglesias Pérez and González Manteiga (1999)). The relations given in (2.1) and (2.2) are crucial in the definition of the GPLE (1.1) as was shown in the above mentioned paper.

(vii) Remember $F(y \mid x) = P(Y \leq y \mid X = x)$, the conditional distribution function of Y when $X = x$, and

(viii) $\alpha(x) = P(T \leq Z \mid X = x)$, the conditional probability of absence of truncation at $X = x$.

Moreover, for any distribution function $W(t) = P(\eta \leq t)$, we denote the left and right support endpoints by $a_W = \inf\{t \,/\, W(t) > 0\}$ and $b_W = \inf\{t \,/\, W(t) = 1\}$, respectively. Specifically, we will use the notation: $a_{L(\cdot|x)}$, $a_{H(\cdot|x)}$, $b_{L(\cdot|x)}$ and $b_{H(\cdot|x)}$ for the support endpoints of functions $L(y \mid x)$ and $H(y \mid x)$, considering L and H as functions of the variable y for a fixed x value.

Finally, for a distribution function W, we define $W^{\#}(t) = P(\eta \leq t \mid T \leq Z)$. Then we will consider: $M^{\#}(x) = P(X \leq x \mid T \leq Z)$, $L^{\#}(y \mid x) = P(T \leq y \mid X = x, T \leq Z)$, $H^{\#}(y \mid x) = P(Z \leq y \mid X = x, T \leq Z)$ and $H_1^{\#}(y \mid x) = P(Z \leq y, \delta = 1 \mid X = x, T \leq Z)$. We also define the function $H_0^{\#}(y \mid x) = H^{\#}(y \mid x) - H_1^{\#}(y \mid x)$.

To formulate our results, we will use some of the hypotheses listed below:

• The model assumptions:

(H1) X, Y, T and S are absolutely continuous random variables.

(H2) a) The variable X takes values in an interval $I = [x_1, x_2]$ contained in the support of $m^{\#}$ (density of $M^{\#}$ (see Remark 1 in Iglesias Pérez and González Manteiga (1999))), such that

$$0 < \gamma = \inf[m^{\#}(x) : x \in I_\varepsilon] < \sup[m^{\#}(x) : x \in I_\varepsilon] = \Gamma < \infty$$

for some $I_\varepsilon = [x_1 - \varepsilon, x_2 + \varepsilon]$ with $\varepsilon > 0$ and $0 < \varepsilon\Gamma < 1$.

And for all $x \in I_\varepsilon$ the r.v. Y, T, S are conditionally independent at $X = x$.

b) Moreover, as regards the Y, T and S variables, we consider:

i) $a_{L(\cdot|x)} \leq a_{H(\cdot|x)}$, $b_{L(\cdot|x)} \leq b_{H(\cdot|x)}$, for all $x \in I_\varepsilon$. (Compare with Woodroofe's results (Woodroofe (1985)) about identifiability of F for truncated data without covariables.)

ii) The variable Y moves in an interval $[a,b]$ such that $\inf[\alpha^{-1}(x)(1-H(b \mid x))L(a \mid x) : x \in I_\varepsilon] \geq \theta > 0$ (note that, if $a_{L(\cdot|x)} < y < b_{H(\cdot|x)}$ then $C(y \mid x) = \alpha^{-1}(x)(1-H(y \mid x)) \times L(y \mid x) > 0$, therefore condition ii) say that $C(y \mid x) \geq \theta > 0$ in $[a,b] \times I_\varepsilon$).

(H3) $a < a_{H(\cdot|x)}$ and $b_{L(\cdot|x)} < b_{H(\cdot|x)}$, for all $x \in I_\varepsilon$.

(H4) The corresponding (improper) densities of the distribution (subdistribution) functions $L(y)$, $H(y)$ and $H_1(y)$ are bounded away from 0 in $[a,b]$.

- The smoothness hypotheses:

(H5) The first derivatives with respect to x of functions $m(x)$ and $\alpha(x)$ exist and are continuous in $x \in I_\varepsilon$ and the first derivatives with respect to x of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous and bounded in $(y,x) \in [0,\infty) \times I_\varepsilon$.

(H6) The second derivatives with respect to x of functions $m(x)$ and $\alpha(x)$ exist and are continuous in $x \in I_\varepsilon$ and the second derivatives with respect to x of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous and bounded in $(y,x) \in [0,\infty) \times I_\varepsilon$.

(H7) The first derivatives with respect to y of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous in $(y,x) \in [a,b] \times I_\varepsilon$.

(H8) The second derivatives with respect to y of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous in $(y,x) \in [a,b] \times I_\varepsilon$.

(H9) The derivatives first with respect to x and second with respect to y of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous in $(y,x) \in [a,b] \times I_\varepsilon$.

(H10) The third derivatives with respect to x of functions $m(x)$ and $\alpha(x)$ exist and are continuous in $x \in I_\varepsilon$, and the third derivatives with respect to x of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous and bounded in $(y,x) \in [0,\infty) \times I_\varepsilon$.

(H11) The fourth derivatives with respect to x of functions $m(x)$ and $\alpha(x)$ exist and are continuous in $x \in I_\varepsilon$ and the fourth derivatives with respect to x of functions $L(y \mid x)$, $H(y \mid x)$ and $H_1(y \mid x)$ exist and are continuous and bounded in $(y,x) \in [0,\infty) \times I_\varepsilon$.

- The kernel function assumptions:

(H12) The kernel function, K, is a symmetrical density vanishing outside $(-1,1)$ and the total variation of K is less than some $\lambda < +\infty$.

(H13) The kernel function, K, is twice continuously differentiable and with bounded first derivative.

- The bandwidth parameter hypothesis:

(H14) The bandwidth parameter $h = (h_n)$ verifies: $h \to 0$, $\ln n/(nh) \to 0$ and $nh^5/\ln n = O(1)$.

Finally, we work with nonnegative variables as is usual in survival analysis.

3. The bootstrap procedure

In this section we develop a bootstrap procedure for approximating the distribution of the statistic $(nh)^{1/2}[\hat{F}_h(y \mid x) - F(y \mid x)]$, which will be called *conditional obvious bootstrap method for LTRC data*. In order to do so, we need to dispose of appropriate estimators for the conditional distributions of T and S. Hence, we discuss this question previously.

We can consider the model with variables: X, Z and T, in which we observe (X,Z,T) if $Z \geq T$ and if $Z < T$ nothing is observed. This model is a truncation model with covariables and, under the hypothesis H2, we have that:

1) Y, T, S are conditionally (mutually) independent at $X = x$ (for all $x \in I_\varepsilon$) which implies that T and $Z = \min\{Y,S\}$ are conditionally independent at $X = x$.

2) $a_{L(\cdot|x)} \leq a_{H(\cdot|x)}$, $b_{L(\cdot|x)} \leq b_{H(\cdot|x)}$, for all $x \in I_\varepsilon$.

These two conditions are the identifiability conditions for the conditional distributions of T and Z at $X = x$, and the mentioned distributions are uniquely determined by the following equalities (see Woodroofe (1985) for the unconditional case):

$$\int_0^y \frac{dH(t \mid x)}{1 - H(t \mid x)} = \int_0^y \frac{dH^{\#}(t \mid x)}{C(t \mid x)}$$

and

$$\int_y^{+\infty} \frac{dL(t \mid x)}{L(t \mid x)} = \int_y^{+\infty} \frac{dL^{\#}(t \mid x)}{C(t \mid x)}.$$

Therefore, it is possible to define and study both GPLE of $H(y \mid x)$, $\hat{H}_h(y \mid x)$, and of $L(y \mid x)$, $\hat{L}_h(y \mid x)$ (in the same way as $\hat{F}_h(y \mid x)$ was derived). These estimators are

$$\hat{H}_h(y \mid x) = 1 - \prod_{i=1}^{n} \left[1 - \frac{1_{\{Z_i \le y\}} B_{hi}(x)}{\sum_{j=1}^{n} 1_{\{T_j \le Z_i \le Z_j\}} B_{hj}(x)} \right] \tag{3.1}$$

and

$$\hat{L}_h(y \mid x) = \prod_{i=1}^{n} \left[1 - \frac{1_{\{T_i > y\}} B_{hi}(x)}{\sum_{j=1}^{n} 1_{\{T_j \le T_i \le Z_j\}} B_{hj}(x)} \right], \tag{3.2}$$

respectively. As before, $\{B_{hi}(x)\}_{i=1}^{n}$ is a sequence of nonparametric weights and $h = h_n$ is the bandwidth parameter. Specifically, we use Nadaraya and Watson weights, which are given by $B_{hi}(x) = K(\frac{x - X_i}{h}) / \sum_{j=1}^{n} K(\frac{x - X_j}{h})$, $i = 1, 2, \ldots, n$, where K denotes a kernel function.

Note that the GPL estimators given by (3.1) and (3.2) equals to PL estimators of the distributions functions of Z and T, respectively, studied by Woodroofe (1985) when one is in absence of a covariables situation ($B_{hi}(x) = 1/n$, for all i). Consistency of the both PLE is proved in Woodroofe's paper (1985), among other properties.

On the other hand, the hypothesis that Y, T and S are conditionally (mutually) independent given $X = x$, allows us to define the GPLE of $G(y \mid x)$, $\hat{G}_h(y \mid x)$, by taking into account that the variable S plays a similar role to Y but with an indicator variable $(1 - \delta_i)$ instead of δ_i in the LTRC model (defined in p. 1). So, $\hat{G}_h(y \mid x)$ is obtained by substituting in formula (1.1) δ_i by $(1 - \delta_i)$, that is:

$$\hat{G}_h(y \mid x) = 1 - \prod_{i=1}^{n} \left[1 - \frac{1_{\{Z_i \le y\}} (1 - \delta_i) B_{hi}(x)}{\sum_{j=1}^{n} 1_{\{T_j \le Z_i \le Z_j\}} B_{hj}(x)} \right]. \tag{3.3}$$

The consistency of $\hat{G}_h(y \mid x)$ is a consequence of the consistency of $\hat{F}_h(y \mid x)$ and the relationship $H_0(y \mid x) = H(y \mid x) - H_1(y \mid x)$. Moreover, note that: $(1 - \hat{G}_h(y \mid x)) \times (1 - \hat{F}_h(y \mid x)) = (1 - \hat{H}_h(y \mid x))$, which is the empirical version of the equality: $(1 - G(y \mid x))(1 - F(y \mid x)) = (1 - H(y \mid x))$.

A very interesting property of all GPLE's is the fact that the jumps of these functions are easy to be calculated (see p. 218 in Iglesias Pérez and González Manteiga (1999)). Particularly, the jump of $\hat{F}_h(y \mid x)$ at $y = Z_i$ is given by

$$B_{hi}(x) \delta_i \frac{1 - \hat{F}_h(Z_i^- \mid x)}{\hat{C}_h(Z_i \mid x)} \tag{3.4}$$

where

$$\hat{C}_h(y \mid x) = \sum_{j=1}^{n} 1_{\{T_j \le y \le Z_j\}} B_{hj}(x), \tag{3.5}$$

the jump of $\hat{L}_h(y \mid x)$ at $y = T_i$ is given by

$$B_{hi}(x) \frac{\hat{L}_h(T_i \mid x)}{\hat{C}_h(T_i \mid x)}, \tag{3.6}$$

and the jump of $\hat{H}_h(y \mid x)$ at $y = Z_i$ is equal to

$$B_{hi}(x) \frac{1 - \hat{H}_h(Z_i^- \mid x)}{\hat{C}_h(Z_i \mid x)}. \tag{3.7}$$

Remark 1. As can be read in Part 3.2 of Gross and Lai (1996), when the Y_i are subject to right censoring in addition to left truncation, difficulties in estimating the joint censoring-truncation distribution increase substantially, although the distribution function of Y can be consistently estimated by the PLE under certain assumptions. Some additional hypothesis will be necessary for consistent estimation of the mentioned joint distribution. In that sense, we would like to emphasize that the assumption of conditionally independence between Y, T and S given X is crucial for consistent estimation of the conditional joint censoring-truncation distribution in the present work. This assumption generalizes the hypothesis of independence between Y, T and S when no covariables are present (assumed by Zhou (1996), among other authors), and, of course, it is not the only proposal to do that. Another one, for example, would be to generalize the model proposed by Wang (1991) and Bilker and Wang (1997) in absence of covariables to a conditional context with covariables.

In order to illustrate how our bootstrap method is based on consistent estimators, we present a simple simulation study to show graphically the consistence of the PLE's of the distribution functions L, F and G without covariables. The three mentioned distributions are simulated independently from exponential distributions with means 0.1, 1 and 4, for L, F and G, respectively. (This produces a 11% of truncation and a 20% of censorship.) The triplets (T, Y, S) were drawn independently until n of them satisfied the condition $T \le Z = \min\{Y, S\}$. In this way, censored and truncated samples $(T_1, Z_1, \delta_1), \ldots, (T_n, Z_n, \delta_n)$ of size n were obtained for $n = 5$, $n = 50$ and $n = 500$, and PLE's of L, F and G were calculated using these samples. Moreover, 1000 replications of the three PLE's were obtained for each n. Finally, we present the averages of the replicated PLE's (APLE's) in Figs. 1, 2 and 3. Note the good performance of APLE's of L and F (that's why $n = 500$ is not displayed because graphical differences are imperceptible) and the strong influence of censorship on S (80%) in the estimation of G.

We are now able to enumerate the conditional obvious bootstrap method for LTRC data: Let $\{(X_i, T_i, Z_i, \delta_i)\}_{i=1}^{n}$ be the observed sample,

1. For X_i, independently obtain: Y_i^* from $\hat{F}_g(y \mid X_i)$, S_i^* from $\hat{G}_g(y \mid X_i)$ and T_i^* from $\hat{L}_g(y \mid X_i)$ where $\hat{F}_g(y \mid X_i)$, $\hat{G}_g(y \mid X_i)$ and $\hat{L}_g(y \mid X_i)$ are the estimators defined in (1.1), (3.3) and (3.2) respectively, but with a bandwidth sequence $g = (g_n)$ (see Remark 2 below).

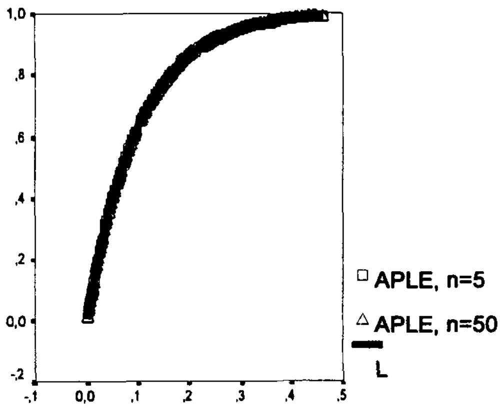

Fig. 1. Average PLE's of L.

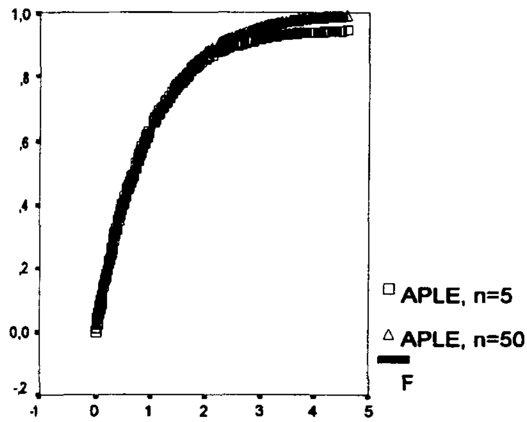

Fig. 2. Average PLE's of F.

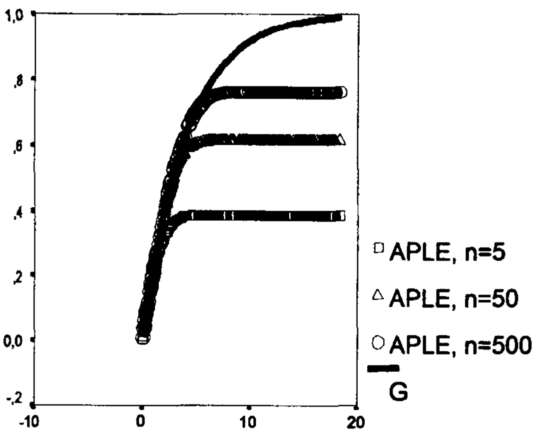

Fig. 3. Average PLE's of G.

Define $Z_i^* = \min\{Y_i^*, S_i^*\}$ and $\delta_i^* = 1_{\{Y_i^* \le S_i^*\}}$. If $T_i^* \le Z_i^*$ then keep the observation $(X_i, T_i^*, Z_i^*, \delta_i^*)$, otherwise discard due to truncation and repeat the process.

Continue until the bootstrap sample $(X_1, T_1^*, Z_1^*, \delta_1^*), \ldots, (X_n, T_n^*, Z_n^*, \delta_n^*)$ is completed.

2. Based on the bootstrap sample, obtain the bootstrap analogue of the GPLE of $F(y \mid x)$ in (1.1), which is given by:

$$\hat{F}_h^*(y \mid x) = 1 - \prod_{i=1}^{n} \left[1 - \frac{1_{\{Z_i^* \le y\}} \delta_i^* B_{hi}(x)}{\sum_{j=1}^{n} 1_{\{T_j^* \le Z_i^* \le Z_j^*\}} B_{hj}(x)} \right]. \tag{3.8}$$

3. Approximating the distribution of $(nh)^{1/2}[\hat{F}_h(y \mid x) - F(y \mid x)]$ by the bootstrap distribution of $(nh)^{1/2}[\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x)]$.

Remark 2. The bandwidth sequence $g = (g_n)$, defined in the first step of the resampling scheme, is typically asymptotically larger than $h = (h_n)$. This oversmoothing in an initial pilot bandwidth aims to the bootstrap bias and the bootstrap variance are asymptotically good estimators for the bias and variance terms. Due to this, we consider the following hypothesis:

(H15) $g = (g_n)$ verifies that $g \to 0$ and $\ln n/(ng) \to 0$. Moreover $g/h \to \infty$, $\frac{ng^5}{\ln n} \to \infty$ and $\frac{ng^5}{\ln n}\frac{h}{g} = O(1)$.

Remark 3. Note that the above defined conditional obvious bootstrap lets us obtain the bootstrap version of different estimators for conditional functions of interest, in addition to $F(y \mid x)$. Some important estimators which will be used in the next section are the kernel nonparametric estimators of the functions $H_1^{\#}(y \mid x)$ and $C(y \mid x)$ with bandwidth g, given by

$$\hat{H}_{1g}^{\#}(y \mid x) = \sum_{i=1}^{n} 1_{\{Z_i \le y, \delta_i = 1\}} B_{gi}(x) \tag{3.9}$$

and

$$\hat{C}_g(y \mid x) = \sum_{i=1}^{n} 1_{\{T_i \le y \le Z_i\}} B_{gi}(x) \tag{3.10}$$

respectively, and the corresponding estimators, with bandwidth h, made with the bootstrap sample, defined as

$$\hat{H}_{1h}^{\#*}(y \mid x) = \sum_{i=1}^{n} 1_{\{Z_i^* \le y, \delta_i^* = 1\}} B_{hi}(x) \tag{3.11}$$

and

$$\hat{C}_h^*(y \mid x) = \sum_{i=1}^{n} 1_{\{T_i^* \le y \le Z_i^*\}} B_{hi}(x). \tag{3.12}$$

We will also use the following estimator of $\Lambda(y \mid x)$ (see (2.2)):

$$\hat{\Lambda}_g(y \mid x) = \int_0^y \frac{d\hat{H}_{1g}^{\#}(t \mid x)}{\hat{C}_g(t \mid x)} = \sum_{i=1}^{n} \frac{1_{\{Z_i \le y\}} \delta_i B_{gi}(x)}{\sum_{j=1}^{n} 1_{\{T_j \le Z_i \le Z_j\}} B_{gj}(x)} \tag{3.13}$$

and its bootstrap version with bandwidth h, given by

$$\hat{\Lambda}_h^*(y \mid x) = \int_0^y \frac{d\hat{H}_{1h}^{\#*}(t \mid x)}{\hat{C}_h^*(t \mid x)} = \sum_{i=1}^{n} \frac{1_{\{Z_i^* \le y\}} \delta_i^* B_{hi}(x)}{\sum_{j=1}^{n} 1_{\{T_j^* \le Z_i^* \le Z_j^*\}} B_{hj}(x)}. \tag{3.14}$$

4. Main results and proofs

Firstly, we present the main results about this conditional obvious bootstrap method for LTRC data: an almost sure representation, and, as a consequence, the uniform weak consistency of the bootstrap.

THEOREM 4.1. (Bootstrap almost sure representation) *Suppose that conditions* (H1)–(H9) *and* (H12)–(H15) *hold. Then, for* $x \in I$ *and* $y \in [a, b]$, *it follows that*:

a) $\hat{\Lambda}_h^*(y \mid x) - \hat{\Lambda}_g(y \mid x)$

$$= \sum_{i=1}^{n} B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - \sum_{i=1}^{n} B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x) + R\Lambda_n^*(y \mid x)$$

where

$$\xi(Z,T,\delta,y,x) = \frac{1_{\{Z\le y,\delta=1\}}}{C(Z \mid x)} - \int_0^y \frac{1_{\{T\le u\le Z\}}}{C^2(u \mid x)} dH_1^{\#}(u \mid x)$$

and

$$\sup_{[a,b]\times I} |R\Lambda_n^*(y \mid x)| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \qquad a.s.\ (P).$$

b) $\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x)$

$$= (1 - F(y \mid x))\left[\sum_{i=1}^n B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - \sum_{i=1}^n B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x)\right]$$
$$+ R_n^*(y \mid x)$$

where

$$\sup_{[a,b]\times I} |R_n^*(y \mid x)| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \qquad a.s.\ (P).$$

THEOREM 4.2. (Uniform weak consistency) *Under the assumptions of Theorem* 4.1, (H10) *and* (H11), *for $x \in I$ and $y \in [a,b]$, we have that*

$$\sup_{t\in\mathbb{R}} |P^*[(nh)^{1/2}(\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x)) \le t] - P[(nh)^{1/2}(\hat{F}_h(y \mid x) - F(y \mid x)) \le t]|$$

converges to zero in probability.

Note that the symbol $*$ after expectations, variances and probabilities means that the corresponding statistical operators are conditioned on the observed sample $\{(X_i, T_i, Z_i, \delta_i)\}_{i=1}^n$.

It is interesting to point out that the bootstrap representation of $\hat{F}_h^*(y \mid x)$ given by Theorem 4.1 comes down to the bootstrap representation given by Van Keilegom and Veraverbeke (1997) when there is no left truncation ($T = 0$), although these latter authors work at fixed design context with Gasser Müller type weights. Moreover, when one is in an absence of covariables situation ($B_{hi}(x) = 1/n$, for all i) and there is no left truncation ($T = 0$) Theorem 4.1 b) comes down to the bootstrap representation obtained by Lo and Singh (1986) for the bootstrap version of the Kaplan-Meier estimator.

In what follows, we present the proofs of the above mentioned theorems.

The first step is to establish the lemmas which are necessary to prove Theorem 4.1. The proof scheme of this theorem is similar to the same one of Theorem 2 in Iglesias Pérez and González Manteiga (1999). So, several results about uniform strong consistency (as Theorem 1 in Iglesias Pérez and González Manteiga (1999)) and about the almost sure behavior of the modulus of continuity (as Lemma 6 in the latter mentioned paper) have to be derived, but according to the present situation. Thus, in Lemmas 4.2 and 4.3 below, we deal with the uniform consistency of some estimators of the conditional functions $H_1^{\#}(y \mid x)$ and $C(y \mid x)$: nonparametric estimators with bandwidth g given by

(3.9) and (3.10) respectively (Lemma 4.2), and the corresponding bootstrap estimators defined by (3.11) and (3.12) (Lemma 4.3). An analogous scheme, but as regards the results about the modulus of continuity of the mentioned estimators, leads to Lemmas 4.4 and 4.5. Moreover, we establish the Lemma 4.1 previously because it is necessary in Lemmas 4.3 and 4.5 to analyze the expectations of the bootstrap estimators $\hat{H}^{\#*}_{1g}(y \mid x)$ and $\hat{C}^*_g(y \mid x)$, (see again (3.11) and (3.12)).

LEMMA 4.1. a) *For the GPLE of $L(y \mid x)$ with bandwidth g, defined according to (3.2) and denoted as $\hat{L}_g(y \mid x)$, the jump at $y = T_j$, $d\hat{L}_g(T_j \mid x)$, is equal to*

$$\frac{B_{gj}(x)/(1-\hat{H}_g(T_j^- \mid x))}{\sum_{i=1}^n B_{gi}(x)/(1-\hat{H}_g(T_i^- \mid x))}. \tag{4.1}$$

If $B_{gj}(x) = 0$ the latter quantity means 0.

b) *For the GPLE of $H(y \mid x)$ with bandwidth g, defined according to (3.1) and denoted as $\hat{H}_g(y \mid x)$, the jump at $y = Z_j$, $d\hat{H}_g(Z_j \mid x)$, is equal to*

$$\frac{B_{gj}(x)/\hat{L}_g(Z_j \mid x)}{\sum_{i=1}^n B_{gi}(x)/\hat{L}_g(Z_i \mid x)}.$$

If $B_{gj}(x) = 0$ the latter quantity means 0.

c) *Moreover, we have*

$$\sum_{i=1}^n \frac{B_{gi}(x)}{1-\hat{H}_g(T_i^- \mid x)} = \sum_{i=1}^n \frac{B_{gi}(x)}{\hat{L}_g(Z_i \mid x)}. \tag{4.2}$$

PROOF. a) To prove that $d\hat{L}_g(T_j \mid x)$ is given by (4.1), it is suffices to show, first, that

$$\frac{(1-\hat{H}_g(T_j^- \mid x))d\hat{L}_g(T_j \mid x)}{B_{gj}(x)} \tag{4.3}$$

(denoted by $s(T_j)$) is a constant C', for $j = 1, \ldots, n$ and, second, that this constant is equal to

$$\left(\sum_{i=1}^n \frac{B_{gi}(x)}{1-\hat{H}_g(T_i^- \mid x)}\right)^{-1}. \tag{4.4}$$

To prove (4.3) we will see that the expression $s(T_j)$ for an arbitrary j ($j = 1, \ldots, n-1$) and for the corresponding $j+1$ are equal. Let's consider the sample $\{(X_{[i]}, T_{(i)}, Z_{[i]}, \delta_{[i]})\}_{i=1}^n$ where $T_{(1)} \le T_{(2)} \le \cdots \le T_{(n)}$ are the ordered T-values, $X_{[i]}$, $Z_{[i]}$ and $\delta_{[i]}$ denotes the concomitants associated with $T_{(i)}$, and $B_{g[i]}(x)$ is the weight corresponding to $X_{[i]}$. For an arbitrary j between 1 and $n-1$ it is easy to see, because (3.6), that $s(T_{(j)}) = (1-\hat{H}_g(T_{(j)}^- \mid x))\hat{L}_g(T_{(j)} \mid x)/\hat{C}_g(T_{(j)} \mid x)$. It is also easy to show, by using the definitions in (3.1), (3.2) and (3.5) but with bandwidth g, that

$$s(T_{(j+1)}) = s(T_{(j)}) \left[\prod_{T_{(j)} \le Z_i < T_{(j+1)}} \left(1 - \frac{B_{gi}(x)}{\hat{C}_g(Z_i \mid x)}\right)\right] \left[\frac{\hat{C}_g(T_{(j)} \mid x)}{\hat{C}_g(T_{(j+1)} \mid x) - B_{g[j+1]}(x)}\right],$$

thus, the proof of (4.3) finishes if we show that

$$\left[\prod_{T_{(j)}\le Z_i<T_{(j+1)}}\left(1-\frac{B_{gi}(x)}{\hat{C}_g(Z_i\mid x)}\right)\right]\left[\frac{\hat{C}_g(T_{(j)}\mid x)}{\hat{C}_g(T_{(j+1)}\mid x)-B_{g[j+1]}(x)}\right]=1. \tag{4.5}$$

In order to obtain this result, we analyze the set $\{Z_i \,/\, T_{(j)} \le Z_i < T_{(j+1)}\}$. Suppose that this set contain k ordered elements denoted by $(T_{(j)} \le) Z_{[m_1]} \le Z_{[m_2]} \le \cdots \le Z_{[m_k]} (< T_{(j+1)})$. Then, because the definition of $\hat{C}_g(y \mid x)$, we have:

$$\begin{aligned}
\hat{C}_g(T_{(j)}\mid x) &= \sum_{i=1}^{n} 1_{\{T_{(i)}\le T_{(j)}\le Z_{[i]}\}} B_{g[i]}(x)\\
\hat{C}_g(Z_{[m_1]}\mid x) &= \hat{C}_g(T_{(j)}\mid x)\\
\hat{C}_g(Z_{[m_2]}\mid x) &= \hat{C}_g(Z_{[m_1]}\mid x) - B_{g[m_1]}(x)\\
&\;\;\vdots\\
\hat{C}_g(Z_{[m_k]}\mid x) &= \hat{C}_g(Z_{[m_{(k-1)}]}\mid x) - B_{g[m_{(k-1)}]}(x)\\
\hat{C}_g(T_{(j+1)}\mid x) &= \hat{C}_g(Z_{[m_k]}\mid x) - B_{g[m_k]}(x) + B_{g[j+1]},
\end{aligned} \tag{4.6}$$

and these expressions lead to (4.5).

The result of $C' = (4.4)$ is obtained taking into account that (4.3) leads to $\sum_{j=1}^n d\hat{L}_g(T_j \mid x) = C' \sum_{j=1}^n (B_{gj}(x)/1 - \hat{H}_g(T_j^- \mid x))$, an expression which is equivalent to $C' = (4.4)$.

b) Analogous to the proof of a). First, it is shown that the expression

$$\frac{\hat{L}_g(Z_j\mid x)d\hat{H}_g(Z_j\mid x)}{B_{gj}(x)} \text{ is a constant } C'', \text{ for } j=1,2,\ldots,n, \tag{4.7}$$

and, second, it is obtained that $C'' = \sum_{i=1}^n (B_{gi}(x)/\hat{L}_g(Z_i \mid x))$.

c) The equation (4.2) means that $C' = C''$, and this result can be demonstrated (due to (4.3) and (4.7)) by proving that for certain indexes i, j in $\{1, 2, \ldots, n\}$ it verifies that:

$$\frac{(1-\hat{H}_g(T_j^-\mid x))d\hat{L}_g(T_j\mid x)}{B_{gj}(x)} = \frac{\hat{L}_g(Z_i\mid x)d\hat{H}_g(Z_i\mid x)}{B_{gi}(x)}.$$

This equality is straightforwardly proved for the pair $(T_{(n)}, Z_{[n]})$.

Remark 4. Lemma 4.1 gives the jumps of the GPLE's of $L(y \mid x)$ and $H(y \mid x)$ in an alternative form to (3.6) and (3.7), respectively. In fact this lemma consists in the empirical version of these theoretical relations:

$$dL(y\mid x) = \frac{dL^{\#}(y\mid x)}{1-H(y^-\mid x)}\alpha(x), \qquad dH(y\mid x) = \frac{dH^{\#}(y\mid x)}{L(y\mid x)}\alpha(x)$$

and

$$\alpha(x)^{-1} = \int_{\mathbb{R}} \frac{dH^{\#}(y\mid x)}{L(y\mid x)} = \int_{\mathbb{R}} \frac{dL^{\#}(y\mid x)}{1-H(y^-\mid x)},$$

for parts a), b) and c), respectively.

LEMMA 4.2. *Under the hypothesis* (H1), (H2a), (H5), (H6), (H12), (H14) *and* (H15), *it follows that:*

$$\sup_{[a,b]\times I} |\hat{H}^{\#}_{1g}(y \mid x) - H^{\#}_1(y \mid x)| = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad a.s.\ (P) \tag{4.8}$$

$$\sup_{[a,b]\times I} |\hat{C}_g(y \mid x) - C(y \mid x)| = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad a.s.\ (P). \tag{4.9}$$

PROOF. It is easy to see that the assumptions in this lemma lead to the assumptions in Lemma 5 and Theorem 1 of Iglesias Pérez and González Manteiga (1999) for the estimators $\hat{H}^{\#}_{1g}(y \mid x)$ and $\hat{C}_g(y \mid x)$. So, applying these results we obtain the order $O((\frac{\ln n}{ng})^{1/2}) + O(g^2)$ for (4.8) and (4.9). The latter order is a $O((\frac{\ln n}{nh})^{1/2})$ because the rates of convergence about h and g (see (H15)).

LEMMA 4.3. *Under the hypothesis* (H1)–(H3), (H5)–(H7) *and* (H12)–(H15), *it follows that*:

a) $$\sup_{[a,b]\times I} |\hat{H}^{\#*}_{1h}(y \mid x) - \hat{H}^{\#}_{1g}(y \mid x)| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad a.s.\ (P)$$

b) $$\sup_{[a,b]\times I} |\hat{C}^*_h(y \mid x) - \hat{C}_g(y \mid x)| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad a.s.\ (P).$$

PROOF. a) We study the expression $|\hat{H}^{\#*}_{1h}(y \mid x) - \hat{H}^{\#}_{1g}(y \mid x)|$ which can be upper-bounded by:

$$|\hat{H}^{\#*}_{1h}(y \mid x) - E^*\hat{H}^{\#*}_{1h}(y \mid x)| \tag{4.10}$$

$$+ |E^*\hat{H}^{\#*}_{1h}(y \mid x) - H^{\#}_1(y \mid x)| \tag{4.11}$$

$$+ |\hat{H}^{\#}_{1g}(y \mid x) - H^{\#}_1(y \mid x)|. \tag{4.12}$$

To analyze (4.10) and (4.11) we will use that

$$E^*\hat{H}^{\#*}_{1h}(y \mid x) = \sum_{i=1}^{n} B_{hi}(x)\hat{H}^{\#}_{1g}(y \mid X_i), \tag{4.13}$$

result that is proved now. As $E^*\hat{H}^{\#*}_{1h}(y \mid x) = \sum_{i=1}^n B_{hi}(x)E^*[1_{\{Z_i^* \le y, \delta_i^* = 1\}}]$ we calculate:

$$E^*[1_{\{Z_i^* \le y, \delta_i^*=1\}}] = \sum_{k=1}^{n} 1_{\{Z_k \le y\}} P^*(Z_i^* = Z_k, \delta_i^* = 1)$$

$$= \frac{\sum_k 1_{\{Z_k \le y\}} (\sum_j 1_{\{T_j \le Z_k\}} d\hat{L}_g(T_j \mid X_i))(1 - \hat{G}_g(Z_k \mid X_i)) d\hat{F}_g(Z_k \mid X_i)}{\sum_{k=1}^n \sum_{j=1}^n 1_{\{T_j \le Z_k\}} d\hat{L}_g(T_j \mid X_i) d\hat{H}_g(Z_k \mid X_i)}. \tag{4.14}$$

By taking into account the jumps of $\hat{F}_g(\cdot \mid X_i)$ and $\hat{H}_g(\cdot \mid X_i)$ at $y = Z_k$ (see (3.4) and (3.7) but with bandwidth g) and the fact of $1 - \hat{H}_g(\cdot \mid X_i) = (1 - \hat{F}_g(\cdot \mid X_i))(1 - \hat{G}_g(\cdot \mid X_i))$, we can write that

$$(4.14) = \frac{\sum_{k=1}^n 1_{\{Z_k \le y\}} \hat{L}_g(Z_k \mid X_i)\delta_k d\hat{H}_g(Z_k \mid X_i)}{\sum_{k=1}^n \hat{L}_g(Z_k \mid X_i) d\hat{H}_g(Z_k \mid X_i)},$$

where, via Lemma 4.1, the above expression equals to $\hat{H}^*_{1g}(y \mid X_i)$. This proves (4.13).

Now we analyze the term (4.10). To study the $\sup_{y\in[a,b]}(4.10)$ we use a idea of Lo and Singh (1986). If we partition the interval $[a,b]$ into $c_n \sim (\frac{\ln n}{nh})^{-1/2}$ subintervals $[y_j, y_{j+1}]$ with $j = 1, \ldots, c_n$, where $y_1 = a$ and $y_{c_n+1} = b$, we have that $\sup_{y\in[a,b]}(4.10)$ is bounded above by

$$(4.15) \qquad \max_j |\hat{H}^{\#*}_{1h}(y_j \mid x) - E^*\hat{H}^{\#*}_{1h}(y_j \mid x)|$$

$$(4.16) \qquad + \max_j |E^*\hat{H}^{\#*}_{1h}(y_{j+1} \mid x) - E^*\hat{H}^{\#*}_{1h}(y_j \mid x)|$$

where we have taken into account that $\hat{H}^{\#*}_{1h}(y \mid x)$ and $E^*\hat{H}^{\#*}_{1h}(y \mid x)$ are nondecreasing functions. The term (4.15) is a $O_{P^*}((\frac{\ln n}{nh})^{1/2})$ a.s. (P) by using that $P(\bigcup_i A_i) \le \sum_i P(A_i)$ and by applying the Bernstein inequality (in a bootstrap context) to $[1_{\{Z^*_i \le y_j, \delta^*_i = 1\}} - \hat{H}^{\#}_{1g}(y_j \mid X_i)]B_{hi}(x)$. A further step is to prove that

$$(4.17) \qquad \sup_{x\in I} D^*_h(x) = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P)$$

where $D^*_h(x) = (4.15)$. To do this, we use an idea of Cheng and Cheng (1987) which consists in consider a set $E_n \subset I$ such that for all $x \in I$, exists $\hat{x}_n \in E_n$ which verifies that $|x - \hat{x}_n| \le c_I/n^2$ (c_I is a constant) and there are at most $n^2 + 1$ such elements in E_n. So, it is easy to see that $\sup_{x\in I} D^*_h(x)$ is smaller than

$$(4.18) \qquad \max_{\hat{x}_n \in E_n} D^*_h(\hat{x}_n) + \sup_{x\in I}\max_j |\hat{H}^{\#*}_{1h}(y_j \mid x) - E^*\hat{H}^{\#*}_{1h}(y_j \mid x) - \hat{H}^{\#*}_{1h}(y_j \mid \hat{x}_n) + E^*\hat{H}^{\#*}_{1h}(y_j \mid \hat{x}_n)|.$$

The first term above is $O_{P^*}((\frac{\ln n}{nh})^{1/2})$ a.s. (P) bearing in mind the analysis of (4.15) and the inequality $P(\bigcup_i A_i) \le \sum_i P(A_i)$. As to the second term, it can be written as

$$\sup_{x\in I}\max_j \left|\sum_{i=1}^n [B_{hi}(x) - B_{hi}(\hat{x}_n)](1_{\{Z^*_i \le y_j, \delta^*_i = 1\}} - \hat{H}^{\#}_{1g}(y \mid X_i))\right|$$
$$\le \sup_{x\in I}\sum_{i=1}^n |B_{hi}(x) - B_{hi}(\hat{x}_n)| = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P),$$

where the latter order is straightforwardly obtained by using the definition of the Nadaraya-Watson weights and the following known results and assumptions: $\sup_{x\in I} |\hat{m}^{\#}_h(x) - m^{\#}(x)| = O((\frac{\ln n}{nh})^{1/2})$ a.s. (P), $m^{\#}(x) \ge \gamma > 0$ for $x \in I$, the first derivative of the kernel function K is bounded and $|x - \hat{x}_n| \le c_I/n^2$. Thus, (4.17) has been shown.

As regards (4.16), we have that $\sup_{x\in I}(4.16)$ is bounded above by

$$2\sup_{x\in I}\sup_{y\in[a,b]}\left|\sum_{i=1}^{n}B_{hi}(x)\hat{H}_{1g}^{\#}(y\mid X_i)-H_1^{\#}(y\mid x)\right| \tag{4.19}$$

$$+\sup_{x\in I}\max_{j}|H_1^{\#}(y_{j+1}\mid x)-H_1^{\#}(y_j\mid x)| \tag{4.20}$$

where, as we show immediately, $(4.19)=O((\frac{\ln n}{nh})^{1/2})$ a.s. (P) and $(4.20)=O((\frac{\ln n}{nh})^{1/2})$. So, $\sup_{x\in I}(4.16)=O((\frac{\ln n}{nh})^{1/2})$ a.s. (P) and this result together with (4.17) lead to

$$\sup_{[a,b]\times I}(4.10)=O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right)\qquad \text{a.s. } (P). \tag{4.21}$$

The proof of $(4.20)=O((\frac{\ln n}{nh})^{1/2})$ follows from a Taylor expansion, the hypothesis (H7) and the conditions about the partition on $[a,b]$. To study (4.19) we use the following upper bound of it:

$$2\sup_{x\in I}\sup_{y\in[a,b]}\left|\sum_{i=1}^{n}B_{hi}(x)[\hat{H}_{1g}^{\#}(y\mid X_i)-H_1^{\#}(y\mid X_i)]\right| \tag{4.22}$$

$$+2\sup_{x\in I}\sup_{y\in[a,b]}\left|\sum_{i=1}^{n}B_{hi}(x)H_1^{\#}(y\mid X_i)-H_1^{\#}(y\mid x)\right|, \tag{4.23}$$

where

$$\begin{aligned}(4.22)&\le 2\sup_{x\in I}\sup_{\tilde{x}\in(x-h,x+h)}\sup_{y\in[a,b]}\left|\sum_{i=1}^{n}B_{hi}(x)\hat{H}_g^{\#}(y\mid\tilde{x})-H^{\#}(y\mid\tilde{x})\right|\\&=O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right)\qquad\text{a.s. } (P)\end{aligned}$$

(the latter order is consequence of $h\to 0$ and Lemma 4.2), and $(4.23)=O((\frac{\ln n}{nh})^{1/2})$ a.s. (P), since Lemma 5 in Iglesias Pérez and González Manteiga (1999) can be applied, lightly adapted, to the estimator $\sum_{i=1}^{n}B_{hi}(x)H_1^{\#}(y\mid X_i)$.

The expression (4.11), because (4.13) and the order of (4.19), is such that

$$\sup_{[a,b]\times I}(4.11)=O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right)\qquad\text{a.s. } (P). \tag{4.24}$$

Finally, $\sup_{[a,b]\times I}(4.12)=O((\frac{\ln n}{nh})^{1/2})$ a.s. (P), by using the Lemma 4.2. Combining this result with (4.21) and (4.24) we conclude the proof of a).

b) We can write:

$$\begin{aligned}\sup_{[a,b]\times I}|\hat{C}_h^*(y\mid x)-\hat{C}_g(y\mid x)|\le&\sup_{[a,b]\times I}|\hat{L}_h^{\#*}(y\mid x)-\hat{L}_g^{\#}(y\mid x)|\\&+\sup_{[a,b]\times I}|\hat{H}_h^{\#*}(y\mid x)-\hat{H}_g^{\#}(y\mid x)|,\end{aligned}$$

where $\hat{L}_h^{\#*}(y \mid x) = \sum_{i=1}^n 1_{\{T_i^* \le y\}} B_{hi}(x)$ and $\hat{L}_g^{\#}(y \mid x) = \sum_{i=1}^n 1_{\{T_i \le y\}} B_{gi}(x)$; $\hat{H}_h^{\#*}(y \mid x) = \sum_{i=1}^n 1_{\{Z_i^* \le y\}} B_{hi}(x)$ and $\hat{H}_g^{\#}(y \mid x) = \sum_{i=1}^n 1_{\{Z_i \le y\}} B_{gi}(x)$.

The analysis of the latter two above $\sup_{[a,b]\times I}$ terms is a copy, step by step, of the proof of part a) in this lemma. Observe only that $E^* \hat{L}_h^{\#*}(y \mid x) = \sum_{i=1}^n B_{hi}(x) \hat{L}_g^{\#}(y \mid X_i)$ and $E^* \hat{H}_h^{\#*}(y \mid x) = \sum_{i=1}^n B_{hi}(x) \hat{H}_g^{\#}(y \mid X_i)$.

LEMMA 4.4. *Under the hypothesis* (H1), (H2a), (H4)–(H9), (H12) *and* (H14)–(H15), *it follows that*:

$$\sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{H}_{1g}^{\#}(t \mid x) - \hat{H}_{1g}^{\#}(s \mid x) - H_1^{\#}(t \mid x) + H_1^{\#}(s \mid x)|$$
$$= O\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \quad a.s.\ (P)$$
$$\sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{C}_g(t \mid x) - \hat{C}_g(s \mid x) - C(t \mid x) + C(s \mid x)|$$
$$= O\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \quad a.s.\ (P)$$

where $b_n = O((\frac{\ln n}{nh})^{1/2})$.

PROOF. Under the present assumptions it is possible to apply Lemma 6 in Iglesias Pérez and González Manteiga (1999) for $W(t \mid x) = H_1^{\#}(y \mid x)$ and obtain the order $O((\frac{\ln n}{ng} b_n)^{1/2}) + O(b_n^2) + O(b_n g)$ a.s. (P) for the first expression above. The latter order is a $O((\frac{\ln n}{nh})^{3/4})$ because: $(\frac{\ln n}{ng} b_n)^{1/2} : (\frac{\ln n}{nh})^{3/4} = (\frac{h}{g})^{1/2} \to 0$, and $b_n g : (\frac{\ln n}{nh})^{3/4} = (\frac{ng^5}{\ln n} \frac{h}{g})^{1/4} = O(1)$ (see (H15)).

The second $\sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}}$ above can be upper-bounded by

$$\sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{L}_g^{\#}(t \mid x) - \hat{L}_g^{\#}(s \mid x) - L^{\#}(t \mid x) + L^{\#}(s \mid x)|$$
$$+ \sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{H}_g^{\#}(t \mid x) - \hat{H}_g^{\#}(s \mid x) - H^{\#}(t \mid x) + H^{\#}(s \mid x)|$$

which have order $O((\frac{\ln n}{nh})^{3/4})$ a.s. (P), arguing in the same way that before, but for $L^{\#}(y \mid x)$ and $H^{\#}(y \mid x)$, respectively.

LEMMA 4.5. *Under the hypothesis* (H1)–(H9) *and* (H12)–(H15), *it follows that*:

$$\sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{H}_{1h}^{\#*}(t \mid x) - \hat{H}_{1h}^{\#*}(s \mid x) - \hat{H}_{1g}^{\#}(t \mid x) + \hat{H}_{1g}^{\#}(s \mid x)|$$

and

$$\sup_{x\in I} \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{C}_h^*(t \mid x) - \hat{C}_h^*(s \mid x) - \hat{C}_g(t \mid x) + \hat{C}_g(s \mid x)|,$$

where $b_n \sim (\frac{\ln n}{nh})^{1/2}$, *are* $O_{P^*}((\frac{\ln n}{nh})^{3/4})$ *a.s.* (P).

PROOF. We study $|\hat{H}_{1h}^{\#*}(t \mid x) - \hat{H}_{1h}^{\#*}(s \mid x) - \hat{H}_{1g}^{\#}(t \mid x) + \hat{H}_{1g}^{\#}(s \mid x)|$, which is smaller than

$$|\hat{H}_{1h}^{\#*}(t \mid x) - \hat{H}_{1h}^{\#*}(s \mid x) - E^*\hat{H}_{1h}^{\#*}(t \mid x) + E^*\hat{H}_{1h}^{\#*}(s \mid x)| \tag{4.25}$$
$$+ |E^*\hat{H}_{1g}^{\#*}(t \mid x) - E^*\hat{H}_{1g}^{\#*}(s \mid x) - \hat{H}_{1g}^{\#}(t \mid x) + \hat{H}_{1g}^{\#}(s \mid x)|. \tag{4.26}$$

Arguing as in the proof of Lemma A.5 in Van Keilegom and Veraverbeke (1997) we divide the interval $[a, b]$ in subintervals I_i with center t_i and radio b_n, such that $t_0 = a$, $t_m = b$ and, if $|s - t| \leq b_n$ there is an interval I_i, $i = 1, 2, \ldots, m - 1$, with $s, t \in I_i$. Each interval I_i is also divided in subintervals whose limits are called t_{ij}, and such that $t_{ij} = t_i + j\frac{b_n}{a_n}$, with $j = -a_n, \ldots, +a_n$ and $a_n \sim (\frac{\ln n}{nh})^{-1/4}$. Using these partitions and the monotony of the estimators $\hat{H}_{1h}^{\#*}(\cdot \mid x)$ and $E^*\hat{H}_{1h}^{\#*}(\cdot \mid x)$, we have that the $\sup_{\{s,t\in[a,b]/|s-t|\leq b_n\}}(4.25)$ is bounded above by:

$$\max_i \max_{j,k} |\hat{H}_{1h}^{\#*}(t_{ik} \mid x) - \hat{H}_{1h}^{\#*}(t_{ij} \mid x) - E^*\hat{H}_{1h}^{\#*}(t_{ik} \mid x) + E^*\hat{H}_{1h}^{\#*}(t_{ij} \mid x)| \tag{4.27}$$
$$+ 2 \max_i \max_j |E^*\hat{H}_{1h}^{\#*}(t_{ij+1} \mid x) - E^*\hat{H}_{1h}^{\#*}(t_{ij} \mid x)|. \tag{4.28}$$

As regards (4.27), we have that $(4.27) = \max_i \max_{j,k} |\sum_{r=1}^n W_{r(ijk)}^*(x)|$, where $W_{r(ijk)}^*(x) = B_{hr}(x)[1_{\{Z_r^* \leq t_{ik}, \delta_r^*=1\}} - 1_{\{Z_r^* \leq t_{ij}, \delta_r^*=1\}} - \hat{H}_{1g}^{\#}(t_{ik} \mid X_r) + \hat{H}_{1g}^{\#}(t_{ij} \mid X_r)]$. In a bootstrap context, the variables $W_{r(ijk)}^*(x)$ are i.i.d. and $E^*W_{r(ijk)}^*(x) = 0$. Moreover, it is not very difficult to see that: $|W_{r(ijk)}^*(x)| \leq 2\max_r B_{hr}(x) = O(\frac{1}{nh})$ a.s. (P) and $\sum_r \operatorname{Var}^* W_{r(ijk)}^*(x) = O((\frac{\ln n}{nh})^{1/2}\frac{1}{nh})$ a.s. (P). So, we can apply the Bernstein inequality (in a bootstrap context) to the variables $W_{r(ijk)}^*(x)$ (for any $x \in I$). This application (and the inequality $P(\bigcup_i A_i) \leq \sum_i P(A_i)$) leads to

$$(4.27) = \max_i \max_{j,k} \left|\sum_{r=1}^n W_{r(ijk)}^*(x)\right| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \quad \text{a.s. } (P). \tag{4.29}$$

A second step in this proof is to show that

$$\sup_{x\in I}(4.27) = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \quad \text{a.s. } (P). \tag{4.30}$$

To prove (4.30) we arguing as in the study of (4.17). Thus, we can write:

$$\begin{aligned}\sup_{x\in I} S1_h^*(x) \leq{}& \max_{\hat{x}_n\in E_n} S1_h^*(\hat{x}_n)\\ &+ \sup_{x\in I}\max_i \max_{j,k} |\hat{H}_{1h}^{\#*}(t_{ik} \mid x) - \hat{H}_{1h}^{\#*}(t_{ij} \mid x)\\ &- E^*\hat{H}_{1h}^{\#*}(t_{ik} \mid x) + E^*\hat{H}_{1h}^{\#*}(t_{ij} \mid x)\\ &- \hat{H}_{1h}^{\#*}(t_{ik} \mid \hat{x}_n) + \hat{H}_{1h}^{\#*}(t_{ij} \mid \hat{x}_n)\\ &+ E^*\hat{H}_{1h}^{\#*}(t_{ik} \mid \hat{x}_n) - E^*\hat{H}_{1h}^{\#*}(t_{ij} \mid \hat{x}_n)|\end{aligned} \tag{4.31}$$

where $S1_h^*(x)$ denotes (4.27) and the set E_n verifies the conditions of E_n as defined for (4.18).

The $\max_{\hat{x}_n\in E_n} S1_h^*(\hat{x}_n) = O_{P^*}((\frac{\ln n}{nh})^{3/4})$ a.s. (P), bearing in mind (4.29) and the inequality $P(\bigcup_i A_i) \leq \sum_i P(A_i)$.

On the other hand, we can write (4.31) as $\sup_{x\in I} \max_i \max_{j,k} |\sum_{r=1}^n [B_{hr}(x) - B_{hr}(\hat{x}_n)]A^*_{r(ijk)}|$, where $A^*_{r(ijk)} = 1_{\{Z_r^*\leq t_{ik},\delta_r^*=1\}} - 1_{\{Z_r^*\leq t_{ij},\delta_r^*=1\}} - \hat{H}_{1g}^{\#}(t_{ik} \mid X_r) + \hat{H}_{1g}^{\#}(t_{ij} \mid X_r)$. Then, an upper bound for (4.31) is

$$\sup_{x\in I}\max_i\max_{j,k}\frac{1}{\hat{m}_h^{\#}(x)\hat{m}_h^{\#}(\hat{x}_n)}\left|(\hat{m}_h^{\#}(\hat{x}_n)-\hat{m}_h^{\#}(x))\sum_{r=1}^n\frac{1}{nh}K\left(\frac{\hat{x}_n-X_r}{h}\right)A^*_{r(ijk)}\right|$$
$$+\sup_{x\in I}\max_i\max_{j,k}\frac{2}{\hat{m}_h^{\#}(x)}\left|\sum_{r=1}^n\left(\frac{1}{nh}K\left(\frac{x-X_r}{h}\right)-\frac{1}{nh}K\left(\frac{\hat{x}_n-X_r}{h}\right)\right)\right|,$$

where it is easy to show that the second term is a $O(\frac{1}{n^2h^2})$ a.s. (P). The first term above is smaller than

$$\sup_{x\in I}\frac{|\hat{m}_h^{\#}(\hat{x}_n)-\hat{m}_h^{\#}(x)|}{\hat{m}_h^{\#}(x)\hat{m}_h^{\#}(\hat{x}_n)}\max_{\hat{x}_n\in E_n}\left(\max_i\max_{j,k}\left|\sum_{r=1}^n B_{hr}(\hat{x}_n)A^*_{r(ijk)}\right|\right),$$

and it is not very difficult to obtain that the first factor above is $O((\frac{\ln n}{nh})^{1/2})$ a.s. (P) and the second one is $O_{P^*}((\frac{\ln n}{nh})^{3/4})$ a.s. (P) (note that $B_{hr}(\hat{x}_n)A^*_{r(ijk)} = W^*_{r(ijk)}(\hat{x}_n)$). So we have finished the proof of (4.30).

Now, we study (4.28), which equals $2\max_i\max_j |\sum_{r=1}^n B_{hr}(x)(\hat{H}_{1g}^{\#}(t_{ij+1} \mid X_r) - \hat{H}_{1g}^{\#}(t_{ij} \mid X_r))|$. Then, $\sup_{x\in I}$(4.28) can be upper-bounded by

$$2\sup_{x\in I}\max_i\max_j\left|\sum_{r=1}^n B_{hr}(x)(\hat{H}_{1g}^{\#}(t_{ij+1}\mid X_r)-\hat{H}_{1g}^{\#}(t_{ij}\mid X_r)\right.$$
$$\left.-H_1^{\#}(t_{ij+1}\mid X_r)+H_1^{\#}(t_{ij}\mid X_r))\right|$$
$$+2\sup_{x\in I}\max_i\max_j\left|\sum_{r=1}^n B_{hr}(x)(H_1^{\#}(t_{ij+1}\mid X_r)-H_1^{\#}(t_{ij}\mid X_r))\right|.$$

The first term of the latter expression is bounded by

$$2\sup_{x\in I}\sup_{\tilde{x}\in(x-h,x+h)}\sup_{\{s,t\in[a,b]/|s-t|\leq b_n\}}|\hat{H}_{1g}^{\#}(s\mid\tilde{x})-\hat{H}_{1g}^{\#}(t\mid\tilde{x})-H_1^{\#}(s\mid\tilde{x})+H_1^{\#}(t\mid\tilde{x})|,$$

which is $O((\frac{\ln n}{nh})^{3/4})$ a.s. (P) via Lemma 4.4. For the second one we have the upper bound given by $2\sup_{x\in I}\sup_{\tilde{x}\in(x-h,x+h)}\max_i\max_j |(H_1^{\#}(t_{ij+1} \mid \tilde{x}) - H_1^{\#}(t_{ij} \mid \tilde{x}))|$ which is smaller than $C(H_1^{\#'})\max_i\max_j |t_{i,j+1}-t_{ij}| = O((\frac{\ln n}{nh})^{3/4})$. Therefore, we have obtained that $\sup_{x\in I}(4.28) = O((\frac{\ln n}{nh})^{3/4})$ a.s. (P). Moreover, this result and (4.30) lead to

$$\sup_{x\in I}\sup_{\{s,t\in[a,b]/|s-t|\leq b_n\}}(4.25)=O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right)\quad\text{a.s. }(P). \tag{4.32}$$

Secondly, we study the term given by (4.26). This term is bounded above by

$$\left|\sum_{i=1}^{n} B_{hi}(x)(\hat{H}_{1g}^{\#}(t \mid X_i) - \hat{H}_{1g}^{\#}(s \mid X_i) - H_1^{\#}(t \mid X_i) + H_1^{\#}(s \mid X_i))\right| \tag{4.33}$$

$$+\left|\sum_{i=1}^{n} B_{hi}(x)(H_1^{\#}(t \mid X_i) - H_1^{\#}(s \mid X_i)) - H_1^{\#}(t \mid x) + H_1^{\#}(s \mid x)\right| \tag{4.34}$$

$$+\,|H_1^{\#}(t \mid x) - H_1^{\#}(s \mid x) - \hat{H}_{1g}^{\#}(t \mid x) + \hat{H}_{1g}^{\#}(s \mid x)|. \tag{4.35}$$

As regards $\sup_{x\in I}\sup_{\{s,t\in[a,b]/|s-t|\le b_n\}}(4.33)$, one has the following upper bound:

$$\sup_{x\in I}\ \sup_{\tilde{x}\in(x-h,x+h)}\ \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} |\hat{H}_{1g}^{\#}(s \mid \tilde{x}) - \hat{H}_{1g}^{\#}(t \mid \tilde{x}) - H_1^{\#}(s \mid \tilde{x}) + H_1^{\#}(t \mid \tilde{x})|,$$

which is a $O((\frac{\ln n}{nh})^{3/4})$ a.s. (P) from the application of Lemma 4.4 on an certain interval $I_{\varepsilon_1} / I \subset I_{\varepsilon_1} \subset I_{\varepsilon}$. We also have that $\sup_{x\in I}\sup_{\{s,t\in[a,b]/|s-t|\le b_n\}}(4.35) = O((\frac{\ln n}{nh})^{3/4})$ a.s. (P), via Lemma 4.4. Finally, we obtain the same order for $\sup_{x\in I}\cdot \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}}(4.34)$, since Lemma 6 in Iglesias Pérez and González Manteiga (1999) can be applied, lightly adapted, to the estimator $\hat{W}_n(y \mid x) = \sum_{i=1}^{n} B_{hi}(x)H_1^{\#}(y \mid X_i)$ of the function $W(y \mid x) = H_1^{\#}(y \mid x)$. The three latter results lead to

$$\sup_{x\in I}\ \sup_{\{s,t\in[a,b]/|s-t|\le b_n\}} (4.26) = O\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \quad \text{a.s. } (P).$$

This expression, combined with (4.32), implies the first part of this lemma.

The proof of the second part of this lemma is analogous to the proof of the first one, but taking into account the relationship between the functions C, $H^{\#}$ and $L^{\#}$, and also the relations between their non parametric estimators and their bootstrap versions, respectively.

Proof of Theorem 4.1. a) We analyze the difference $\hat{\Lambda}_h^*(y \mid x) - \hat{\Lambda}_g(y \mid x)$ (see (3.13) and (3.14)), which can be written as

$$\int_0^y \frac{d\hat{H}_{1h}^{\#*}(t \mid x) - d\hat{H}_{1g}^{\#}(t \mid x)}{C(t \mid x)} - \int_0^y \left(\frac{\hat{C}_h^*(t \mid x) - \hat{C}_g(t \mid x)}{C^2(t \mid x)}\right) dH_1^{\#}(t \mid x) + R1 + R2 + R3$$

where

$$R1 = \int_0^y \left(\frac{1}{\hat{C}_h^*(t \mid x)} - \frac{1}{\hat{C}_g(t \mid x)}\right)(d\hat{H}_{1h}^{\#*}(t \mid x) - d\hat{H}_{1g}^{\#}(t \mid x)),$$

$$R2 = \int_0^y \left(\frac{1}{\hat{C}_g(t \mid x)} - \frac{1}{C(t \mid x)}\right)(d\hat{H}_{1h}^{\#*}(t \mid x) - d\hat{H}_{1g}^{\#}(t \mid x)),$$

$$R3 = \int_0^y \left(\frac{\hat{C}_h^*(t \mid x) - \hat{C}_g(t \mid x)}{C^2(t \mid x)}\right)(dH_1^{\#}(t \mid x) - d\hat{H}_{1g}^{\#}(t \mid x))$$
$$+ \int_0^y (\hat{C}_h^*(t \mid x) - \hat{C}_g(t \mid x))\left(\frac{1}{C^2(t \mid x)} - \frac{1}{\hat{C}_h^*(t \mid x)\hat{C}_g(t \mid x)}\right) d\hat{H}_{1g}^{\#}(t \mid x).$$

The two first terms in the above given expression are the dominant part of the difference $\hat{\Lambda}_h^*(y \mid x) - \hat{\Lambda}_g(y \mid x)$ and, with the notation of this theorem, can be written as:

$$\sum_{i=1}^n B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - \sum_{i=1}^n B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x).$$

The other three terms are the error part, $R\Lambda_n^*(y \mid x) = R1 + R2 + R3$, which has to be bounded.

The expressions $R1$, $R2$ and the first summand in $R3$, denoted by $R31$, have a similar form to the term $R2_{na}(y \mid x)$ defined in the proof of Theorem 2 in Iglesias Pérez and González Manteiga (1999) and, because Lemmas 4.2, 4.3, 4.4 and 4.5, the respective least upper bounds in $[a, b] \times I$ can be studied as the $\sup_{[a,b]\times I} |R2_{na}(y \mid x)|$. So, we obtain a $O_{P^*}(\frac{\ln n}{nh})^{3/4}$ a.s. (P) for all of them. For the second summand in $R3$, denoted by $R32$, is not difficult to show that $\sup_{[a,b]\times I} |R32| = O_{P^*}(\frac{\ln n}{nh})$ a.s. (P), using Lemmas 4.2 and 4.3. Thus, one has that

$$\sup_{[a,b]\times I} |R\Lambda_n^*(y \mid x)| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) \quad \text{a.s. } (P). \tag{4.36}$$

b) To analyze $\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x)$ we use the identity:

$$\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x) = (1 - \hat{F}_g(y \mid x))(1 - e^{\ln(1-\hat{F}_h^*(y|x)) - \ln(1-\hat{F}_g(y|x))})$$

and a Taylor expansion of $1 - e^{\ln(1-\hat{F}_h^*(y|x)) - \ln(1-\hat{F}_g(y|x))}$ which leads to $(-A - \frac{1}{2}e^{\tau_n}A^2)$, where $A = \ln(1 - \hat{F}_h^*(y \mid x)) - \ln(1 - \hat{F}_g(y \mid x))$ and τ_n is between 0 and A.

Considering the discomposing for $-A$ given by $-A = [-\ln(1 - \hat{F}_h^*(y \mid x)) - \hat{\Lambda}_h^*(y \mid x)] + [\hat{\Lambda}_h^*(y \mid x) - \hat{\Lambda}_g(y \mid x)] + [\hat{\Lambda}_g(y \mid x) + \ln(1 - \hat{F}_g(y \mid x))]$ and using the representation obtained in part a) of this theorem, we find that $\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x)$ is equal to

$$(1 - F(y \mid x))\left[\sum_{i=1}^n B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - \sum_{i=1}^n B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x)\right] + R_n^*(y \mid x)$$

where

$$\begin{aligned} R_n^*(y \mid x) = {} & (1 - F(y \mid x))[-\ln(1 - \hat{F}_h^*(y \mid x)) - \hat{\Lambda}_h^*(y \mid x)] \\ & + (1 - F(y \mid x))[R\Lambda_n^*(y \mid x)] \\ & + (1 - F(y \mid x))[\hat{\Lambda}_g(y \mid x) + \ln(1 - \hat{F}_g(y \mid x))] \\ & + (1 - F(y \mid x))\left[-\frac{1}{2}e^{\tau_n}A^2\right] \\ & + [1 - \hat{F}_g(y \mid x) - (1 - F(y \mid x))]\left(-A - \frac{1}{2}e^{\tau_n}A^2\right). \end{aligned} \tag{4.37}$$

Now, we will study each term of $R_n^*(y \mid x)$. For the first one, we have that

$$\sup_{[a,b]\times I} |-\ln(1 - \hat{F}_h^*(y \mid x)) - \hat{\Lambda}_h^*(y \mid x)| = O_{P^*}\left(\frac{1}{nh}\right) \quad \text{a.s. } (P). \tag{4.38}$$

This result is obtained in a similar way as

$$\sup_{[a,b]\times I} |\hat{\Lambda}_g(y \mid x) + \ln(1 - \hat{F}_g(y \mid x))| = O\left(\frac{1}{ng}\right) \quad \text{a.s. } (P), \tag{4.39}$$

which is shown in Theorem 2.2 of Iglesias Pérez and González Manteiga (1999) (expression 3.22). The order for the second term is given by (4.36), and the order for the third one is shown in (4.39).

The following step in this proof is to show that $\sup_{[a,b]\times I} |A| = O_{P^*}((\frac{\ln n}{nh})^{1/2})$ a.s. (P). From part a) in this theorem, (4.36), (4.38) and (4.39) it follows that

$$\begin{aligned} \sup_{[a,b]\times I} |A| = \sup_{[a,b]\times I} \Bigg| & \sum_{i=1}^{n} B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) \\ & - \sum_{i=1}^{n} B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x) \Bigg| \\ & + O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{3/4}\right) + O\left(\frac{1}{ng}\right) + O_{P^*}\left(\frac{1}{nh}\right) \quad \text{a.s. } (P). \end{aligned} \tag{4.40}$$

Moreover, Lemma 4.3 and the assumptions (H2) and (H3) show that

$$\begin{aligned} & \sup_{[a,b]\times I} \left| \sum_{i=1}^{n} B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - \sum_{i=1}^{n} B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x) \right| \\ & = \sup_{[a,b]\times I} \Bigg| \int_0^y \frac{d\hat{H}_{1h}^{\#*}(t \mid x) - d\hat{H}_{1g}^{\#}(t \mid x)}{C(t \mid x)} \\ & \qquad - \int_0^y \left(\frac{\hat{C}_h^*(t \mid x) - \hat{C}_g(t \mid x)}{C^2(t \mid x)} \right) dH_1^{\#}(t \mid x) \Bigg| \\ & = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P). \end{aligned} \tag{4.41}$$

Thus, (4.40) and (4.41) lead to

$$\sup_{[a,b]\times I} |A| = O_{P^*}\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P). \tag{4.42}$$

To find an upper bound for the fourth and fifth terms of $R_n^*(y \mid x)$ (see (4.37)) we use (4.42), and we also use the inequality given by $e^{\tau_n} \le \frac{1-\hat{F}_h^*(y|x)}{1-\hat{F}_g(y|x)} \le \frac{1}{1-\hat{F}_g(y|x)}$, which is a.s. bounded for all $(y, x) \in [a, b] \times I$, because $F(y \mid x) < 1$ and

$$\sup_{[a,b]\times I} |\hat{F}_g(y \mid x) - F(y \mid x)| = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P). \tag{4.43}$$

(This latter result is easily shown following the proof of Theorem 2.2 part c) in Iglesias Pérez and González Manteiga (1999) and using the assumptions about the bandwidths

h and g.) Hence, we have that the $\sup_{[a,b]\times I}$ of the fourth and of the fifth terms are $O_{P^*}(\frac{\ln n}{nh})$ a.s. (P).

Finally, we have proved that $\sup_{[a,b]\times I} R_n^*(y \mid x) = O_{P^*}((\frac{\ln n}{nh})^{3/4})$ a.s. (P).

Immediately below, we introduce five lemmas which will be used in the proof of Theorem 4.2. To be more specific, this theorem is immediate through Lemmas 4.6 and 4.10, because Lemma 4.6 gives the approximation between the distribution of $(nh)^{1/2}(\hat{F}_h(y \mid x) - F(y \mid x))$ and the normal distribution with parameters $b(y \mid x)$ and $s(y \mid x)$ (defined in the mentioned lemma), and Lemma 4.10 establishes the approximation in probability between the bootstrap distribution of $(nh)^{1/2}(\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x))$ and the same normal distribution. As regards Lemmas 4.7, 4.8 and 4.9, they are necessary to prove Lemma 4.10.

LEMMA 4.6. *Assume* (H1)–(H9), (H12), (H14) *and* $h = Cn^{-1/5}$. *Then, for* $x \in I$ *and* $y \in [a, b]$, *one has*

$$\sup_{t\in\mathbb{R}} \left| P[(nh)^{1/2}(\hat{F}_h(y \mid x) - F(y \mid x)) \le t] - \Phi_N\left(\frac{t - b(y \mid x)}{s(y \mid x)}\right) \right| \longrightarrow 0$$

where

$$b(y \mid x) = C^{5/2}(1 - F(y \mid x))\left(\int z^2 K(z)dz\right)(\Phi''(x)m^{\#}(x) + 2\Phi'(x)m^{\#\prime}(x))/2m^{\#}(x)$$

with

$$\begin{aligned}(4.44)\qquad \Phi(u) &= E[\xi(Z, T, \delta, y, x) \mid T \le Z, X = u] \\ &= \int_0^y \frac{dH_1^{\#}(s \mid u)}{C(s \mid x)} - \int_0^y \frac{C(s \mid u)}{C^2(s \mid x)} dH_1^{\#}(s \mid x), \\ s^2(y \mid x) &= (1 - F(y \mid x))^2\left(\int K^2(z)dz\right)\left(\int_0^y \frac{dH_1^{\#}(s \mid x)}{C^2(s \mid x)}\right)/m^{\#}(x)\end{aligned}$$

and Φ_N *denotes the distribution function of a standard normal random variable.*

PROOF. It is a consequence of Corollary 3b) in Iglesias Pérez and González Manteiga (1999).

LEMMA 4.7. *Assume* (H1)–(H9), (H12)–(H15) *and* $h = Cn^{-1/5}$. *Then, for* $x \in I$ *and* $y \in [a, b]$, *one has*

$$\begin{aligned}\sup_{t\in\mathbb{R}} \Bigg| P^*\Bigg[\Bigg((nh)^{1/2}(1 - F(y \mid x)) \sum_{i=1}^n (B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) \\ - B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x))\Bigg) \le t\Bigg] \\ - \Phi_N\left(\frac{t - b_n^*(y \mid x)}{(s_n^{*2}(y \mid x))^{1/2}}\right)\Bigg| \xrightarrow{a.s.} 0\end{aligned}$$

where

$$b_n^*(y \mid x) = (nh)^{1/2}(1 - F(y \mid x)) \cdot \sum_{i=1}^{n} [B_{hi}(x) E^*(\xi(Z_i^*, T_i^*, \delta_i^*, y, x)) - B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x)]$$

and

$$s_n^{*2}(y \mid x) = nh(1 - F(y \mid x))^2 \sum_{i=1}^{n} B_{hi}^2(x) \operatorname{Var}^*(\xi(Z_i^*, T_i^*, \delta_i^*, y, x)).$$

PROOF. By applying the Berry-Esséen inequality to the variables:

$$V_i^* = (nh)^{1/2}(1 - F(y \mid x)) B_{hi}(x)[\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - E^*(\xi(Z_i^*, T_i^*, \delta_i^*, y, x))],$$

we obtain, for the $\sup_{t \in \mathbb{R}}$ defined in the statement of this lemma the following upper bound:

$$A \frac{\sum_{i=1}^{n} B_{hi}^3(x) E^*(|\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - E^*(\xi(Z_i^*, T_i^*, \delta_i^*, y, x))|^3)}{(\sum_{i=1}^{n} B_{hi}^2(x) \operatorname{Var}^* \xi(Z_i^*, T_i^*, \delta_i^*, y, x))^{3/2}} \tag{4.45}$$

where A denotes an universal constant.

The numerator in (4.45) is a $O(\sum_{i=1}^{n} B_{hi}^3(x)) = O((nh)^{-2})$ a.s. (P) because the bootstrap expectation enclosed therein is smaller than $4[E^*(|\xi(Z_i^*, T_i^*, \delta_i^*, y, x)|^3) + (E^*|\xi(Z_i^*, T_i^*, \delta_i^*, y, x)|)^3]$ and $|\xi(Z_i^*, T_i^*, \delta_i^*, y, x)|$ are uniformly bounded (by $2/\theta^2$), for $(y, x) \in [a, b] \times I$ and for all i. The denominator in (4.45) is a $O((nh)^{-3/2})$ a.s. (P) as a consequence of Lemmas 4.8a) and 4.9 (below). Thus, (4.45) is a $O(\frac{(nh)^{-2}}{(nh)^{-3/2}}) = O((nh)^{-1/2})$ a.s.

LEMMA 4.8. *Assume* (H1)–(H9), (H12)–(H15) *and* $h = Cn^{-1/5}$. *Then, for* $x \in I$ *and* $y \in [a, b]$, *it follows*:

$$\text{a)} \quad s_n^{*2}(y \mid x) - s_n^2(y \mid x) = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad a.s.\ (P)$$

where

$$s_n^2(y \mid x) = (nh)(1 - F(y \mid x))^2 \sum_{i=1}^{n} B_{hi}^2(x) \operatorname{Var}^{Y|X}(\xi(Z_i, T_i, \delta_i, y, x)).$$

b) *Moreover, if* (H10) *and* (H11) *hold, then*

$$b_n^*(y \mid x) - b_n(y \mid x) = O_P((nh^5 g^4)^{1/2})$$

where

$$b_n(y \mid x) = (nh)^{1/2}(1 - F(y \mid x)) \sum_{i=1}^{n} B_{hi}(x) E^{Y|X}(\xi(Z_i, T_i, \delta_i, y, x)).$$

The symbol $Y \mid X$ *after expectations, variances and probabilities means that the corresponding statistical operators are conditioned on the observations* $(X_1, X_2, \ldots, X_n)$.

PROOF. a) Note that $s_n^{*2}(y \mid x) - s_n^2(y \mid x)$ equals to

$$nh(1 - F(y \mid x))^2 \sum_{i=1}^{n} B_{hi}^2(x)[E^* \xi_i^{*2}(y,x) - E^{Y|X}\xi_i^2(y,x)] \tag{4.46}$$

$$+ nh(1 - F(y \mid x))^2 \sum_{i=1}^{n} B_{hi}^2(x)[(E^{Y|X}\xi_i(y,x))^2 - (E^*\xi_i^*(y,x))^2] \tag{4.47}$$

where we have used the notation: $\xi_i^*(y,x)$ for $\xi(Z_i^*, T_i^*, \delta_i^*, y, x)$ and $\xi_i(y,x)$ for $\xi(Z_i, T_i, \delta_i, y, x)$.

For (4.46) we have, after straightforward calculations, that $E^*\xi_i^{*2}(y,x) = A_i^* + B_i^* - 2C_i^*$, where

$$\begin{aligned}
A_i^* &= E^*\left(\frac{1_{\{Z_i^* \le y, \delta_i^* = 1\}}}{C^2(Z_i^* \mid x)}\right) = \int_0^y \frac{d\hat{H}_{1g}^{\#}(u \mid X_i)}{C^2(u \mid x)} \\
&= \frac{\hat{H}_{1g}^{\#}(y \mid X_i)}{C^2(y \mid x)} + 2\int_0^y \frac{\hat{H}_{1g}^{\#}(u \mid X_i)}{C^3(u \mid x)} dC(u \mid x), \\
B_i^* &= E^*\left(\left(\int_0^y \frac{1_{\{T_i^* \le u \le Z_i^*\}}}{C^2(u \mid x)} dH_1^{\#}(u \mid x)\right)^2\right) \\
&= 2\hat{\alpha}_g^{-1}(X_i)\int_0^y \frac{\hat{L}_g(u \mid X_i)}{C^2(u \mid x)}\left(\int_u^y \frac{(1 - \hat{H}_g(v \mid X_i))}{C^2(v \mid x)} dH_1^{\#}(v \mid x)\right) dH_1^{\#}(u \mid x) \\
&= 2\int_0^y \frac{\hat{C}_g(u \mid X_i)}{(1 - \hat{H}_g(u \mid X_i))C^2(u \mid x)}\left(\int_u^y \frac{(1 - \hat{H}_g(v \mid X_i))}{C^2(v \mid x)} dH_1^{\#}(v \mid x)\right) dH_1^{\#}(u \mid x), \\
C_i^* &= E^*\left[\left(\frac{1_{\{Z_i^* \le y, \delta_i^* = 1\}}}{C(Z_i^* \mid x)}\right)\left(\int_0^y \frac{1_{\{T_i^* \le u \le Z_i^*\}}}{C^2(u \mid x)} dH_1^{\#}(u \mid x)\right)\right] \\
&= \hat{\alpha}_g^{-1}(X_i)\int_0^y \left(\hat{L}_g(u \mid X_i)\int_u^y \frac{d\hat{H}_{1g}(v \mid X_i)}{C(v \mid x)}\right)\frac{dH_1^{\#}(u \mid x)}{C^2(u \mid x)} \\
&= \int_0^y \left(\frac{\hat{C}_g(u \mid X_i)}{1 - \hat{H}_g(u \mid X_i)}\int_u^y \frac{(1 - \hat{H}_g(u \mid X_i))d\hat{H}_{1g}^{\#}(z \mid X_i)}{C(z \mid x)\hat{C}_g(z \mid X_i)}\right)\frac{dH_1^{\#}(u \mid x)}{C^2(u \mid x)}
\end{aligned}$$

and where $\hat{\alpha}_g^{-1}(X_i) = \int \hat{L}_g(z \mid X_i)d\hat{H}_g(z \mid X_i)$. Making a similar analysis, we obtain that $E^{Y|X}\xi_i^2(y,x) = A_i + B_i - 2C_i$, where

$$\begin{aligned}
A_i &= E^{Y|X}\left(\frac{1_{\{Z_i \le y, \delta_i = 1\}}}{C^2(Z_i \mid x)}\right) = \int_0^y \frac{dH_1^{\#}(u \mid X_i)}{C^2(u \mid x)} \\
&= \frac{H_1^{\#}(y \mid X_i)}{C^2(y \mid x)} + 2\int_0^y \frac{H_1^{\#}(u \mid X_i)}{C^3(u \mid x)} dC(u \mid x), \\
B_i &= E^{Y|X}\left(\left(\int_0^y \frac{1_{\{T_i \le u \le Z_i\}}}{C^2(u \mid x)} dH_1^{\#}(u \mid x)\right)^2\right) \\
&= 2\int_0^y \frac{C(u \mid X_i)}{(1 - H(u \mid X_i))C^2(u \mid x)}\left(\int_u^y \frac{(1 - H(v \mid X_i))}{C^2(v \mid x)} dH_1^{\#}(v \mid x)\right) dH_1^{\#}(u \mid x)
\end{aligned}$$

and

$$C_i = E^{Y|X}\left[\left(\frac{1_{\{Z_i \le y, \delta_i = 1\}}}{C(Z_i \mid x)}\right)\left(\int_0^y \frac{1_{\{T_i \le u \le Z_i\}}}{C^2(u \mid x)} dH_1^{\#}(u \mid x)\right)\right]$$
$$= \int_0^y \left(\frac{C(u \mid X_i)}{1 - H(u \mid X_i)} \int_u^y \frac{(1 - H(v \mid X_i)) dH_1^{\#}(v \mid X_i)}{C(v \mid X_i) C(v \mid x)}\right) \frac{dH_1^{\#}(u \mid x)}{C^2(u \mid x)}.$$

Consequently, we can write

$$(4.48) \qquad (4.46) = O(1) \sum_{i=1}^{n} B_{hi}(x)[(A_i^* - A_i) + (B_i^* - B_i) + 2(C_i - C_i^*)] \quad \text{a.s. } (P).$$

When $h \to 0$ and taking into account Lemma 4.2 and the fact of $C(y \mid x) > 0$ for $(y, x) \in [a, b] \times I$, we can argue as in (4.22) to show

$$(4.49) \qquad \sup_{[a,b] \times I} \left| \sum_{i=1}^{n} B_{hi}(x)(A_i^* - A_i) \right| = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P).$$

A slightly more difficult than (4.49), because we need to add and substrate the convenient terms, but in a similar way, we obtain the same order for $\sup_{[a,b] \times I} |\sum_{i=1}^{n} B_{hi}(x)(B_i^* - B_i)|$ and for $\sup_{[a,b] \times I} |\sum_{i=1}^{n} B_{hi}(x)(C_i^* - C_i)|$. To study these two latter supremums we also have to use that, when $h \to 0 : 1 - H(y \mid x) > 0$ for $(y, x) \in [a, b] \times I_\varepsilon$ and

$$\sup_{x \in I} \sup_{\tilde{x} \in (x-h, x+h)} \sup_{y \in [a,b]} |\hat{H}_g(y \mid \tilde{x}) - H(y \mid \tilde{x})| = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P).$$

(This result is similar to (4.43) in the particular case of absence of censorship.) All these rates of a.s. convergence and (4.48) lead to

$$(4.50) \qquad (4.46) = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P).$$

As regards (4.47), we have:

$$(4.47) = O(1) \sum_{i=1}^{n} B_{hi}(x)[(E^{Y|X}\xi_i(y, x) - E^*\xi_i^*(y, x))(E^{Y|X}\xi_i(y, x) + E^*\xi_i^*(y, x))] \quad \text{a.s. } (P)$$

where $E^{Y|X}\xi_i(y, x) - E^*\xi_i^*(y, x)$ is equal to

$$\int_0^y \frac{dH_1^{\#}(u \mid X_i) - d\hat{H}_{1g}^{\#}(u \mid X_i)}{C(u \mid x)} - \int_0^y \frac{C(u \mid X_i) - \hat{C}_g(u \mid X_i)}{C^2(u \mid x)} dH_1^{\#}(u \mid x)$$

and $E^{Y|X}\xi_i(y, x) + E^*\xi_i^*(y, x)$ are uniformly bounded, for $(y, x) \in [a, b] \times I$ and for all i. Therefore

$$(4.47) \le O\left(\sup_{[a,b] \times I} \left| \sum_{i=1}^{n} B_{hi}(x)(E^{Y|X}\xi_i(y, x) - E^*\xi_i^*(y, x)) \right| \right).$$

Finally, by using a reasoning as the same in (4.22) and taking into account Lemma 4.2 and the fact of $C(y \mid x) > 0$ for $(y, x) \in [a, b] \times I$, we arrive at $(4.47) = O((\frac{\ln n}{nh})^{1/2})$ a.s. (P). This and (4.50) lead to $s_n^{*2}(y \mid x) - s_n^2(y \mid x) = O((\frac{\ln n}{nh})^{1/2})$ a.s. (P).

b) We will use Theorem 3 in Härdle and Marron (1991). To apply this result note that

$$b_n(y \mid x) = (nh)^{1/2}(1 - F(y \mid x))[E^{Y|X}\hat{\Phi}_h(x) - \Phi(x)]$$

and

$$b_n^*(y \mid x) = (nh)^{1/2}(1 - F(y \mid x))\left[\sum_{i=1}^{n} B_{hi}(x)\hat{\Phi}_g(X_i) - \hat{\Phi}_g(x)\right]$$

where $\Phi(u)$ is defined in (4.44) and $\hat{\Phi}_h(u) = \sum_{i=1}^n B_{hi}(u)\xi(Z_i, T_i, \delta_i, y, x)$ denotes the kernel estimator of $\Phi(u)$ with bandwidth h. Furthermore, the hypotheses in this part of the present lemma allows for the application of the mentioned result. Thus, we have the convergence to 1 in probability of

$$\frac{E^{Y|X}[(nh)^{-1}(1 - F(y \mid x))^{-2}(b_n^*(y \mid x) - b_n(y \mid x))^2]}{[h^4(C_1 n^{-1} g^{-5} + C_2 g^4)]} \tag{4.51}$$

where $C_1 = d_K^2 V(x)(\int K''(u)^2 du)/4m^{\#}(x)$ and $C_2 = d_K^4((m^{\#}(x)\Phi(x))^{IV} - (m^{\#''}(x)\Phi(x))'')^2/16m^{\#}(x)^2$ with $V(u) = \mathrm{Var}(\xi(Z, T, \delta, y, x) \mid T \leq Z, X = u)$ and $d_K = \int u^2 K(u)du$.

Reasoning as in the proof of Cao Abad's (1990) Theorem 3.18 (see also Cao Abad (1991)), one can see that (4.51) leads to $b_n^*(y \mid x) - b_n(y \mid x) = O_P(nh^5(C_1 n^{-1} g^{-5} + C_2 g^4))^{1/2}$, which is $O_P((nh^5 g^4)^{1/2})$ because the assumptions about h and g.

LEMMA 4.9. *Assume* (H1)–(H9), (H12), (H14)–(H15) *and* $h = Cn^{-1/5}$. *Then, for* $x \in I$ *and* $y \in [a, b]$, *it follows that*

a) $$b_n(y \mid x) - b(y \mid x) = O\left(\left(\frac{\ln n}{nh}\right)^{1/2}\right) = O\left(\frac{(\ln n)^{1/2}}{n^{1/5}}\right) \quad a.s.\ (P)$$

b) $$s_n^2(y \mid x) - s^2(y \mid x) = O\left(\left(\frac{\ln n}{(nh)^{1/2}}\right)^{1/2}\right) = O\left(\frac{(\ln n)^{1/2}}{n^{2/5}}\right) \quad a.s.\ (P).$$

PROOF. a) We can write

$$b_n(y \mid x) = \tilde{b}_n(y \mid x) + O\left(\tilde{b}_n(y \mid x)\left(\frac{\ln n}{nh}\right)^{1/2}\right) \quad \text{a.s. } (P) \tag{4.52}$$

where $\tilde{b}_n(y \mid x) = (nh)^{-1/2}(1 - F(y \mid x))m^{\#-1}(x)\sum_{i=1}^n K(\frac{x - X_i}{h})\Phi(X_i)$ and $\Phi(u)$ is defined in (4.44), because the properties about the estimator $\hat{m}_h^{\#}(x)$ of $m^{\#}(x)$. Moreover, using standard calculations to obtain the bias of a kernel estimator with Nadaraya Watson weights, we have $E\tilde{b}_n(y \mid x) = b(y \mid x) + O(n^{-1/5})$, which implies

$$\tilde{b}_n(y \mid x) = b(y \mid x) + \tilde{b}_n(y \mid x) - E\tilde{b}_n(y \mid x) + O(n^{-1/5}). \tag{4.53}$$

Finally, $|\tilde{b}_n(y \mid x) - E\tilde{b}_n(y \mid x)|$ is $O((\frac{\ln n}{n^{2/5}})^{1/2})$ a.s. (P), using Bernstein inequality on the variables $W_{ni} - EW_{ni}$, where $W_{ni} = 1 - F(y \mid x)(nh)^{-1/2}K(\frac{x-X_i}{h})\Phi(X_i)/m^{\#}(x)$. This, (4.52) and (4.53) lead to part a).

b) The proof parallels completely that of part a).

LEMMA 4.10. *Assume* (H1)–(H15) *and* $h = Cn^{-1/5}$. *Then, for* $x \in I$ *and* $y \in [a, b]$, *it follows that*

$$\sup_{t\in\mathbb{R}} \left| P^*[(nh)^{1/2}(\hat{F}_h^*(y \mid x) - \hat{F}_g(y \mid x)) \leq t] - \Phi_N\left(\frac{t - b(y \mid x)}{s(y \mid x)}\right) \right| \xrightarrow{P} 0.$$

PROOF. Due to Theorem 4.1 it suffices to prove that

$$\sup_{t\in\mathbb{R}} \Bigg| P^* \Bigg[\Bigg((nh)^{1/2}(1 - F(y \mid x)) \sum_{i=1}^{n} (B_{hi}(x)\xi(Z_i^*, T_i^*, \delta_i^*, y, x) - B_{gi}(x)\xi(Z_i, T_i, \delta_i, y, x)) \Bigg) \leq t \Bigg] - \Phi_N\left(\frac{t - b(y \mid x)}{s(y \mid x)}\right) \Bigg| \xrightarrow{P} 0.$$

This convergence is obtained as a consequence of Lemmas 4.7, 4.8 and 4.9, applied jointly with the triangular inequality and with the following bound (see Lemma 2.4 in Cao Abad (1990) or Cao Abad (1991)): $\sup_{z\in\mathbb{R}} |\Phi_N(\frac{z-\mu_2}{\sigma_2}) - \Phi_N(\frac{z-\mu_1}{\sigma_1})| \leq \|t\phi_N(t)\|_\infty(\sigma_1\sigma_2)^{-1} \times |\sigma_2 - \sigma_1| \max(\sigma_1, \sigma_2) + \|\phi_N\|_\infty\{(\sigma_1\sigma_2)^{-1}|\sigma_2 - \sigma_1|(|\mu_2| + \sigma_1^{-1}|\mu_1| \max(\sigma_1, \sigma_2)) + \sigma_1^{-1}|\mu_2 - \mu_1|\}$, where μ_1, μ_2, $\sigma_1 > 0$, $\sigma_2 > 0$ are real numbers, ϕ_N is the density function of a standard normal r.v. and, for any real function f, $\|f\|_\infty = \sup_x |f(x)|$.

PROOF OF THEOREM 4.2. It is immediate through Lemmas 4.6 and 4.10.

Remark 5. Observe that we obtain weak instead of strong consistency in Theorem 4.2 due, only, to the influence of $b_n^*(y \mid x) - b_n(y \mid x) \xrightarrow{P} 0$.

If we had obtained a strong consistency result, instead of a weak consistency result, Theorem 4.2 would generalize the strong consistency results of bootstrap method as defined by Van Keilegom and Veraverbeke (1997) in case of absence of truncation, and as defined by Lo and Singh (1986) in case of absence of truncation and covariables.

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