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EXACT LIKELIHOOD INFERENCE BASED ON TYPE-I AND TYPE-II HYBRID CENSORED SAMPLES FROM THE EXPONENTIAL DISTRIBUTION

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Abstract. Chen and Bhattacharyya (1988, *Comm. Statist. Theory Methods*, **17**, 1857–1870) derived the exact distribution of the maximum likelihood estimator of the mean of an exponential distribution and an exact lower confidence bound for the mean based on a hybrid censored sample. In this paper, an alternative simple form for the distribution is obtained and is shown to be equivalent to that of Chen and Bhattacharyya (1988). Noting that this scheme, which would guarantee the experiment to terminate by a fixed time T, may result in few failures, we propose a new hybrid censoring scheme which guarantees at least a fixed number of failures in a life testing experiment. The exact distribution of the MLE as well as an exact lower confidence bound for the mean is also obtained for this case. Finally, three examples are presented to illustrate all the results developed here.

Key words and phrases: Type-I and Type-II hybrid censoring, exponential distribution, order statistics, confidence bound, life testing.

1. Introduction

Consider a life testing experiment in which n units are put on test and successive failure times are recorded. Assume that the lifetimes are independent and identically distributed exponential random variables with probability density function (pdf)

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x \geq 0, \quad \theta > 0. \tag{1.1}$$

Let the ordered lifetimes of these items be denoted by $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Various interesting properties of these exponential order statistics are known in the literature; see, for example, David (1981), Arnold *et al.* (1992), and Balakrishnan and Basu (1995).

Epstein (1954) considered a hybrid censoring sampling scheme in which the life testing experiment is terminated at a random time $T_1^* = \min\{X_{r:n}, T\}$, where $1 \leq r \leq n$ and $T \in (0,\infty)$ are fixed in advance. As with a conventional Type-I censoring scheme, the termination point here is at most T, and so we will refer to this scheme as a *Type-I hybrid censoring scheme* (Type-I HCS). Motivated by the works of Bartholomew (1963) and Barlow *et al.* (1968), Chen and Bhattacharyya (1988) derived the distribution of the maximum likelihood estimator (MLE) of θ and also an exact lower confidence bound under such a Type-I HCS. As mentioned by these authors, Type-I HCS is used as a reliability acceptance test in MIL-STD-781 C (1977). For some more recent results on Type-I hybrid censoring, one may refer to Ebrahimi (1992), Jeong *et al.* (1996), and Gupta and Kundu (1998).

Like conventional Type-I censoring, the disadvantage of Type-I HCS is that the inference results are obtained under the condition that the number of observed failures is at least one, and in addition there may be very few failures occurring up to the prefixed time T. We, therefore, propose an alternative hybrid censoring scheme that would terminate the experiment at the random time $T_2^* = \max\{X_{r:n}, T\}$, where again $1 \leq r \leq n$ and $T \in (0, \infty)$ are fixed in advance. This scheme, which we shall call a *Type-II hybrid censoring scheme* (Type-II HCS), has the advantage of guaranteeing that at least r failures are observed. Such a censoring scheme may arise in a situation when the experimenter determines that at least r failures must be observed, and has prepaid for the use of the testing facility for T units of time. If the r failures occur before time T, then the experiment can continue up to time T to make full use of the testing facility. If the r-th failure does not occur before time T then he/she will naturally choose to continue until the r-th failure. Thus, in direct comparison between this Type-II HCS and Type-I HCS, there are the following advantages and disadvantages for both of them:

Table 1. Comparison of Type-I and Type-II hybrid censoring schemes*.

$\theta = 1$			Type-I HCS		Type-II HCS	
n	r	T	Expected Length of Life Test	Expected No. of Failures	Expected Length of Life Test	Expected No. of Failures
10	4	0.5	.393	3.366	.586	4.569
		0.75	.455	3.825	.774	5.452
		1.5	.479	3.998	1.500	7.77
	6	0.5	.483	3.872	.863	6.063
		0.75	.659	4.960	.937	6.316
		1.5	.833	5.936	1.512	7.832
	8	0.5	.499	3.933	1.430	8.002
		0.75	.740	5.258	1.439	8.018
		1.5	1.245	7.379	1.684	8.390
20	10	0.5	.480	7.679	.689	10.190
		0.75	.615	9.369	.804	11.184
		1.5	.669	9.997	1.500	15.540
	13	0.5	.499	7.862	1.006	13.007
		0.75	.728	10.409	1.027	13.143
		1.5	.995	12.914	1.510	15.623
	16	0.5	.500	7.869	1.514	16.000
		0.75	.749	10.549	1.515	16.003
		1.5	1.344	15.027	1.671	16.510

*Note that the same results will hold for any θ when we replace T by θT and expected length by θ (expected length).

• In the case of Type-I HCS, the termination time is fixed by the experimenter which is a clear advantage. However, if θ (the unknown mean lifetime) is not small compared to T (the pre-fixed termination time), then with a high probability the experimentation would terminate at T. In addition, there is a disadvantage that far fewer than r failures may be observed which may have an adverse effect on the efficiency of the inferential procedure based on Type-I HCS.

• In the case of Type-II HCS, the termination time is unknown to the experimenter which is a disadvantage. In the case when θ is not small compared to T, with a high probability the experimentation would terminate at $X_{r:n}$ thus resulting in a longer life-test. However, there is a clear advantage that more than r failures may be observed which will result in efficient inferential procedures based on Type-II HCS.

In Table 1, we present a comparison of these two hybrid censoring schemes for some selected choices of the parameters. The values of the expected length and expected number of failures for the two hybrid censoring schemes in the table give supportive evidence to the advantages and disadvantages of the Type-I and Type-II hybrid censoring schemes described above.

In this paper we present, in Section 2, an alternative simpler derivation and expression for the distribution of the MLE of θ and the resulting lower confidence bound under Type-I HCS. The expression obtained, which is shown to be algebraically equivalent to the one given by Chen and Bhattacharyya (1988), is computationally simpler and is expressed in terms of the incomplete gamma function. In Section 3, we derive the corresponding results for a Type-II HCS. Some illustrative examples are given in Section 4 wherein the effects of considering a hybrid censored sample as a conventional Type-II censored sample are examined in terms of confidence coefficients.

2. Simplified results for Type-I hybrid censoring

Under the Type-I HCS, the MLE of θ is given by

$$\widehat{\theta} = \begin{cases} \dfrac{1}{D}\left\{\displaystyle\sum_{i=1}^{D} X_{i:n} + (n-D)T\right\} & \text{if} \quad T < X_{r:n} \\ \dfrac{1}{r}\left\{\displaystyle\sum_{i=1}^{r} X_{i:n} + (n-r)X_{r:n}\right\} & \text{if} \quad X_{r:n} \le T, \end{cases}$$

where D denotes the number of observed failures that occur before time T. To derive the moment generating function of $\widehat{\theta}$, Chen and Bhattacharyya (1988) considered the following expression:

$$(2.1) \quad E_\theta(e^{w\hat{\theta}}) = \sum_{d=1}^{r-1} E_\theta(e^{w\hat{\theta}} \mid D=d) P_{\theta,c}(D=d) + \sum_{d=r}^{n} E_\theta(e^{w\hat{\theta}} \mid D=d) P_{\theta,c}(D=d),$$

where $P_{\theta,c}(D=d)$ is the conditional probability that $D=d$ given that $D \ge 1$. They obtained a rather complicated expression for the second term in (2.1) involving a triple summation. The values of D in the first sum correspond to all possible values when $T < X_{r:n}$. We can simplify their results for the second sum by simply conditioning on the event $X_{r:n} \le T$, i.e., we replace the second sum by $E_\theta(e^{w\hat{\theta}} \mid X_{r:n} \le T) P_{\theta,c}(X_{r:n} \le T)$. Given the event $\{X_{r:n} \le T\}$, the joint density function of $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ is given by

$$f(x_1, x_2, \dots, x_r \mid X_{r:n} \le T) = \frac{n!}{(n-r)! P_\theta(X_{r:n} \le T)} \prod_{j=1}^{r} f(x_j)\{1 - F(x_r)\}^{n-r}$$
$$0 < x_1 < \cdots < x_r < T.$$

Hence, we get

$$\begin{aligned}
&E_\theta(e^{w\hat{\theta}} \mid X_{r:n} \le T) P_{\theta,c}(X_{r:n} \le T) \\
&= (1 - e^{-nT/\theta})^{-1} \frac{n!}{(n-r)!\theta^r} \\
&\quad \cdot \int_0^T \int_0^{x_r} \cdots \int_0^{x_2} e^{-1/\theta(1-\theta w/r)\{\Sigma_{i=1}^{r-1} x_i + (n-r+1)x_r\}} dx_1 \cdots dx_r \\
&= (1 - e^{-nT/\theta})^{-1} \frac{n!(1-\theta w/r)^{-(r-1)}}{(n-r)!(r-1)!\theta} \\
&\quad \cdot \int_0^T \{1 - e^{-1/\theta(1-\theta w/r)x_r}\}^{r-1} e^{-1/\theta(1-\theta w/r)(n-r+1)x_r} dx_r \\
&= (1 - e^{-nT/\theta})^{-1} r \binom{n}{r} (1-\theta w/r)^{-r} \\
&\quad \cdot \sum_{k=0}^{r-1} \frac{(-1)^k}{n-r+k+1} \binom{r-1}{k} \{1 - e^{-(1-\theta w/r)(n-r+k+1)T/\theta}\},
\end{aligned}$$

where the second equality is obtained from the identity

$$\int_0^{x_r} \int_0^{x_{r-1}} \cdots \int_0^{x_2} e^{-a\Sigma_{i=1}^{r-1} x_i} dx_1 \cdots dx_{r-1} = \frac{(1 - e^{-ax_r})^{r-1}}{a^{r-1}(r-1)!},$$

and the last equality is obtained by expanding the first term in the last integral binomially. Combining this last expression with the results of Chen and Bhattacharyya (1988) for the first term in (2.1) gives the following theorem.

THEOREM 2.1. *Conditional on $D \ge 1$, the moment generating function of $\widehat{\theta}$ is given by*

$$\begin{aligned}
(2.2) \quad M_{\hat{\theta}}(w) = (1 - q^n)^{-1} \Bigg[&\sum_{d=1}^{r-1} \binom{n}{d} \frac{q^{(n-d)(1-\theta w/d)}}{(1 - w\theta/d)^d} (1 - q^{(1-\theta w/d)})^d \\
&+ r \binom{n}{r} (1 - \theta w/r)^{-r} \\
&\cdot \sum_{k=0}^{r-1} \frac{(-1)^k}{n-r+k+1} \binom{r-1}{k} \{1 - q^{(1-\theta w/r)(n-r+k+1)}\} \Bigg],
\end{aligned}$$
$$w < \frac{1}{\theta},$$

where $q = e^{-T/\theta}$.

Remark 2.1. The expression in (2.2) is equivalent to the expression of Chen and Bhattacharyya (1988), and the details of this equivalence are given in the Appendix.

Remark 2.2. Another way of viewing the difference in approach leading to the simplified expression in Theorem 2.1 is as follows. Chen and Bhattacharyya (1988) obtained the expression in (2.1) by viewing the random variable D as a binomial random variable with pmf $P_\theta(D=d)=\binom{n}{d}\{F(T)\}^d\{1-F(T)\}^{n-d}$, $d=0,1,\ldots,n$. However, under the proposed censoring scheme, D can really only take on values from 0 to r, since the experiment is terminated (at the latest) immediately following the r-th failure. Therefore, our approach is equivalent to viewing the random variable D as having the following pmf:

$$P_\theta(D=d)=\begin{cases}\binom{n}{d}\{F(T)\}^d\{1-F(T)\}^{n-d} & \text{for} \quad d=0,1,\ldots,r-1\\ \sum_{d=r}^{n}\binom{n}{d}\{F(T)\}^d\{1-F(T)\}^{n-d} & \text{for} \quad d=r.\end{cases}$$

THEOREM 2.2. *Conditional on $D\geq 1$, the pdf of $\widehat{\theta}$ is given by*

$$f_{\hat\theta}(x)=(1-q^n)^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d}C_{k,d}\,g\left(x-T^*_{k,d};\frac{d}{\theta},d\right)+g\left(x;\frac{r}{\theta},r\right)\right.$$
$$\left.+r\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k q^{n-r+k}}{n-r+k}\binom{r-1}{k-1}g\left(x-T^*_{k,r};\frac{r}{\theta},r\right)\right],\qquad 0<x<nT, \tag{2.3}$$

where $C_{k,d}=(-1)^k\binom{n}{d}\binom{d}{k}q^{n-d+k}$, $T^*_{k,d}=(n-d+k)T/d$, *and*

$$g(x;\alpha,p)=\begin{cases}\dfrac{\alpha^p}{\Gamma(p)}x^{p-1}e^{-\alpha x} & x>0\\ 0 & \textit{otherwise.}\end{cases}$$

PROOF. By expanding $(1-q^{(1-\theta w/d)})^d$ binomially, splitting the second term in (2.2) through $1-q^{(1-\theta w/r)(n-r+k+1)}$, and using the identity $\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{1}{k+n-r+1}=\frac{1}{r\binom{n}{r}}$, we get

$$M_{\hat\theta}(w)=(1-q^n)^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d}C_{k,d}e^{w(n-d+k)T/d}(1-\theta w/d)^{-d}+(1-\theta w/r)^{-r}\right.$$
$$\left.+r\binom{n}{r}\sum_{k=1}^{r}(-1)^k\binom{r-1}{k-1}\frac{q^{n-r+k}}{n-r+k}e^{w(n-r+k)T/r}(1-\theta w/r)^{-r}\right].$$

The proof is completed upon noting that $e^{wA}(1-w/\alpha)^{-p}$ is the moment generating function of the random variable $Y+A$, where Y has the pdf $g(x;\alpha,p)$.

Remark 2.3. The simplified representation of the pdf of $\widehat{\theta}$ given in Theorem 2.2 allows one to see how the results for Type-I hybrid censoring reduce to the results for conventional Type-II censoring. Indeed, if we let $T \to \infty$, then $f_{\widehat{\theta}}(x) = g(x; \frac{r}{\theta}, r)$, and we get the well-known result in this case that $\frac{2r\widehat{\theta}}{\theta}$ has a chi-square distribution with $2r$ degrees of freedom.

COROLLARY 2.1. *We have*

$$E_\theta(\widehat{\theta}) = (1-q^n)^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d} C_{k,d}(\theta + T^*_{k,d}) + \theta + r\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k q^{n-r+k}}{n-r+k}\binom{r-1}{k-1}(\theta + T^*_{k,r})\right],$$

and

$$E_\theta(\widehat{\theta}^2) = (1-q^n)^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d} C_{k,d}\left\{\frac{\theta^2}{d}(1+d) + 2T^*_{k,d}\theta + (T^*_{k,d})^2\right\} + \frac{\theta^2}{r}(1+r) + r\binom{n}{r} \cdot \sum_{k=1}^{r}\frac{(-1)^k q^{n-r+k}}{n-r+k}\binom{r-1}{k-1}\left\{\frac{\theta^2}{r}(1+r) + 2T^*_{k,r}\theta + (T^*_{k,r})^2\right\}\right].$$

By integrating the density function in (2.3), we obtain the following expression for $P_\theta(\widehat{\theta} > b)$ which can be used to obtain a lower confidence bound for θ, as illustrated in Section 4:

COROLLARY 2.2.

$$P_\theta(\widehat{\theta} > b) = (1-q^n)^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d}\frac{C_{k,d}}{(d-1)!}\Gamma(d, A_d(T^*_{k,d})) + \frac{\Gamma(r, rb/\theta)}{(r-1)!} + \frac{r}{(r-1)!}\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k q^{n-r+k}}{n-r+k}\binom{r-1}{k-1}\Gamma(r, A_r(T^*_{k,r}))\right],$$

where $A_k(a) = \frac{k}{\theta}\langle b-a\rangle$, $\langle x\rangle = \max\{x, 0\}$, *and* $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ *is the incomplete gamma function.*

Remark 2.4. One can also express $P_\theta(\widehat{\theta} > b)$ in terms of chi-square integrals as in the case of conventional Type-I censoring (Bartholomew (1963)).

3. Results for Type-II hybrid censoring

In this section, we derive analogous results for Type-II HCS, wherein the experiment is terminated at the random time $T_2^* = \max\{X_{r:n}, T\}$. As in Section 2, let D be the number of failures up to time T. Then the likelihood function is given by

$$L(\theta \mid \boldsymbol{x}) = \begin{cases} \dfrac{n!}{(n-r)!} e^{-1/\theta\{\sum_{i=1}^{r} x_i + (n-r)x_r\}} & \text{if} \quad D = 0, 1, \dots, r-1 \\ \dfrac{n!}{(n-D)!} e^{-1/\theta\{\sum_{i=1}^{D} x_i + (n-D)T\}} & \text{if} \quad D = r, r+1, \dots, n \end{cases}$$

and the MLE of θ in this case is

$$\widehat{\theta} = \begin{cases} \dfrac{1}{r}\left\{\displaystyle\sum_{i=1}^{r} x_i + (n-r)x_r\right\} & \text{if} \quad D = 0, 1, \dots, r-1 \\ \dfrac{1}{D}\left\{\displaystyle\sum_{i=1}^{D} x_i + (n-D)T\right\} & \text{if} \quad D = r, r+1, \dots, n. \end{cases}$$

THEOREM 3.1. *The moment generating function of $\widehat{\theta}$ is given by*

$$(3.1) \qquad M_{\hat{\theta}}(w) = (1-\theta w/r)^{-r} \sum_{d=0}^{r-1} \binom{n}{d} q^{(n-d)(1-\theta w/r)} \{1 - q^{(1-\theta w/r)}\}^d + \sum_{d=r}^{n} \binom{n}{d} q^{(n-d)(1-\theta w/d)} \{1 - q^{(1-\theta w/d)}\}^d (1-\theta w/d)^{-d}, \qquad w < \frac{r}{\theta}.$$

PROOF. In this case, D is binomial and takes on values from 0 to n. Again, we condition on the values of D, but this time the precise values of D must be specified in both cases,

$$M_{\hat{\theta}}(w) = \sum_{d=0}^{r-1} E_\theta(e^{w\hat{\theta}} \mid D = d) P_\theta(D = d) + \sum_{d=r}^{n} E_\theta(e^{w\hat{\theta}} \mid D = d) P_\theta(D = d).$$

Then, we note that for $d = 0, 1, \dots, r-1$, the conditional density function of $X_{1:n}, X_{2:n}, \dots, X_{r:n}$ given $D = d$ is given by

$$f(x_1, x_2, \dots, x_r \mid D = d) = \frac{n!}{(n-r)! P_\theta(D = d)} \prod_{j=1}^{r} f(x_j) \{1 - F(x_r)\}^{n-r}, \qquad 0 < x_1 < \cdots < x_d < T < x_{d+1} < \cdots < x_r < \infty;$$

also, for $d = r, r+1, \dots, n$, the conditional density function of $X_{1:n}, X_{2:n}, \dots, X_{d:n}$ given $D = d$ is given by

$$f(x_1, x_2, \dots, x_d \mid D = d) = \frac{n!}{(n-d)! P_\theta(D = d)} \prod_{j=1}^{d} f(x_j) \{1 - F(T)\}^{n-d}, \qquad 0 < x_1 < \cdots < x_d < T.$$

Hence, we have

$$M_{\hat{\theta}}(w)$$

$$= \frac{n!}{(n-r)!\theta^r} \sum_{d=0}^{r-1}$$

$$\cdot \int_T^{\infty} \cdots \int_{x_{r-1}}^{\infty} \int_0^T \cdots \int_0^{x_2} e^{-1/\theta(1-\theta w/r)\{\Sigma_{i=1}^{r} x_i+(n-r)x_r\}} dx_1 \cdots dx_d dx_r \cdots dx_{d+1}$$

$$+ \sum_{d=r}^{n} \frac{n!}{(n-d)!\theta^d} \int_0^T \int_0^{x_d} \cdots \int_0^{x_2} e^{-1/\theta(1-\theta w/d)\{\Sigma_{i=1}^{d} x_i+(n-d)T\}} dx_1 dx_2 \cdots dx_d.$$

Straightforward integration then completes the proof.

By proceeding along lines similar to that of Theorem 2.2, we obtain the following density function for $\widehat{\theta}$.

THEOREM 3.2. *The pdf of* $\widehat{\theta}$ *is given by*

$$f_{\hat{\theta}}(x) = q^n g\left(x - nT/r; \frac{r}{\theta}, r\right)$$
$$+ \sum_{d=1}^{r-1} \sum_{k=0}^{d} C_{k,d}\, g\left(x - a_{k,d}; \frac{r}{\theta}, r\right) + \sum_{d=r}^{n} \sum_{k=0}^{d} C_{k,d}\, g\left(x - a_{k,d}; \frac{d}{\theta}, d\right),$$

where $C_{k,d}$ *and* g *are as defined earlier, and*

$$a_{k,d} = \begin{cases} (n-d+k)T/r & \text{if} \quad d = 0, 1, \ldots, r-1 \\ (n-d+k)T/d & \text{if} \quad d = r, r+1, \ldots, n; k = 0, 1, 2, \ldots, d. \end{cases}$$

Remark 3.1. As with Type-I hybrid censoring, the results in Theorem 3.2 also reduce to the case of conventional Type-II censoring. This time, we let $T \to 0$ to get $f_{\hat{\theta}}(x) = g(x; \frac{r}{\theta}, r)$, and hence the well-known result that $\frac{2r\hat{\theta}}{\theta}$ has a chi-square distribution with $2r$ degrees of freedom.

COROLLARY 3.1. *The mean and variance of* $\widehat{\theta}$ *are given by*

$$E_\theta(\widehat{\theta}) = \theta + B,$$
$$V_\theta(\widehat{\theta}) = \theta^2 \left[\frac{q^n}{r} + \frac{1}{r} \sum_{d=1}^{r-1} \sum_{k=0}^{d} C_{k,d} + \sum_{d=r}^{n} \sum_{k=0}^{d} C_{k,d} \frac{1}{d}\right] + \sum_{d=0}^{n} \sum_{k=0}^{d} C_{k,d} a_{k,d}^2 - B^2,$$

where

$$B = \sum_{d=0}^{n} \sum_{k=0}^{d} C_{k,d} a_{k,d}.$$

Remark 3.2. From the above corollary, we see that, unlike with conventional Type-II censoring, $\widehat{\theta}$ is not unbiased in the case of Type-II hybrid censoring. Furthermore, if we let $T \to 0$ in the above expressions, then we get the well-known results for conventional Type-II censoring, i.e., $E_\theta(\widehat{\theta}) = \theta$ and $V(\widehat{\theta}) = \theta^2/r$.

Using the same notation as in the previous section, we get

COROLLARY 3.2.

$$P_\theta(\widehat{\theta} > b) = \sum_{d=0}^{r-1}\sum_{k=0}^{d} \frac{C_{k,d}}{(r-1)!}\Gamma(r, A_r(a_{k,d})) + \sum_{d=r}^{n}\sum_{k=0}^{d} \frac{C_{k,d}}{(d-1)!}\Gamma(d, A_d(a_{k,d})).$$

4. Illustrative examples

Assuming that $P_\theta(\widehat{\theta} > b)$ is a monotone increasing function of θ, a $100(1-\alpha)\%$ lower confidence bound for θ is obtained by solving the equation $\alpha = P_{\theta_L}(\widehat{\theta} > \widehat{\theta}_{\text{obs}})$ for θ_L. Due to the complex form of $P_\theta(\widehat{\theta} > b)$, Chen and Bhattacharyya (1988) were unable to analytically prove monotonicity. Although we have presented a simpler form of the function, we are also unable to establish the required monotonicity.

Illustration 1. Tables 2 and 3 give numerical values of $p_{\text{I}} = P_\theta(\widehat{\theta} > b)$ for Type-I hybrid censoring, and $p_{\text{II}} = P_\theta(\widehat{\theta} > b)$ for Type-II hybrid censoring which support the conjecture that the function is indeed increasing for both types of hybrid censoring scheme.

Table 2. Values of $p = P_\theta(\widehat{\theta} > b)$ with $b = 3.0$, $T = 2.0$, $n = 10$ and $r = 5$.

θ	p_{I}	p_{II}
1	.0011	.0011
2	.1416	.1523
3	.4537	.4780
4	.6864	.7111
5	.8194	.8397
6	.8922	.9079
7	.9328	.9448
8	.9564	.9656
9	.9707	.9778

Table 3. Values of $p = P_\theta(\widehat{\theta} > b)$ with $b = 4.0$, $T = 4.0$, $n = 15$ and $r = 10$.

θ	p_{I}	p_{II}
1	≈ 0	≈ 0
2	.0072	.0053
3	.1662	.1503
4	.4846	.4681
5	.7340	.7256
6	.8714	.8686
7	.9383	.9378
8	.9698	.9701
9	.9848	.9853

Note that values of p_{I} were calculated by Chen and Bhattacharyya (1988) for the same values of b, T, n, and r given in Tables 2, and 3. However, possibly due to the algebraic complexity of their expression for $P_\theta(\widehat{\theta} > b)$, it appears that many of their numbers are incorrect. It is for this reason that we have included these calculations here.

Illustration 2. Chen and Bhattacharyya (1988) considered data from Barlow *et al.* (1968) which was based on $n = 10$ items put on test in a time censored experiment with $T = 50$, resulting in the following observations: $4, 9, 11, 18, 27, 38$. By taking $r = 4$, $T = 50$; $r = 6$, $T = 50$; and $r = 8$, $T = 50$, they used this data set to obtain the corresponding Type-I hybrid censored data and resulting lower confidence bounds for θ. Again, we have repeated the calculations in Table 4 below because it appears that some of their numbers are incorrect. In addition, we have included the standard errors of the MLE's calculated using Corollary 2.1.

Table 4. Lower confidence bounds for θ.

r	$\widehat{\theta}_{\text{obs}}$	s.e.	$\alpha = .05$	$\alpha = .1$
4	37.50	19.78	19.35	22.45
6	43.17	23.64	24.64	27.93
8	51.17	31.11	28.46	32.12

What happens if a hybrid censored sample is treated as an ordinary Type-II censored sample? Since the analysis based on Type-II censoring is easier, this is often done; see, for example, Cohen (1991, 1995). In this case, a $100(1-\alpha)\%$ lower confidence bound for θ is given by $\frac{2r\hat{\theta}}{\chi^2_{2r,\alpha}}$. For the data used in Table 4, $r = 6$, $\widehat{\theta} = 43.167$, the lower 95% confidence bound is 24.636, and the 90% lower confidence bound is 27.925 which agree to 3 decimals with the confidence bounds in Table 4. But suppose that we treat the case $r = 8$ as a conventional Type-II censored sample. Then the resulting lower confidence bounds are the same as for $r = 6$ (i.e., 24.636 and 27.925). To obtain the true (exact) confidence coefficient assuming that the data came from a Type-I hybrid censored sample with $r = 8$, we simply calculate $P_a(\widehat{\theta} > \widehat{\theta}_{\text{obs}})$, where $\widehat{\theta}_{\text{obs}}$ is the hybrid censored MLE, and a is the proposed lower confidence bound. In this case, since $P_{24.636}(\widehat{\theta} > 51.17) = .0178$, the 95% lower confidence bound is in fact a 98.22% lower confidence bound, and the 90% lower confidence bound is in fact a 95.6% lower confidence bound.

As noted earlier, as $T \to \infty$ the results for Type-I hybrid censoring reduce to results for conventional Type-II censoring. Therefore, when T is large, or equivalently when there is a high probability that $X_{r:n} < T$, we would expect the results for Type-I hybrid censoring to agree closely with the results for conventional Type-II censoring. Indeed, when $r = 6$ in the above example,

$$P(X_{6:10} < 50) = \sum_{i=6}^{10} \binom{10}{i} (1 - e^{-50/\theta})^i e^{-(10-i)50/\theta}.$$

By taking $\theta = \widehat{\theta} = 43.167$, we get $P(X_{6:10} < 50) \approx 82.5\%$. Since this probability is high, it is reasonable that the results agree quite closely with conventional Type-II censoring in this case. But when $r = 8$, $P(X_{8:10} < 50) \approx 20.8\%$. This low probability explains why the lower confidence bounds obtained by regarding the sample as a conventional Type-II censored sample gives a confidence level which differs from the exact confidence level. In this case, if you disregard the hybridness of the sample, then the resulting confidence interval becomes too liberal.

Illustration 3. For an example using a Type-II hybrid censoring scheme, we consider the data given by Bartholomew (1963) consisting of $n = 20$ items being put on a life test for a prefixed time of 150 hours resulting in the following observed failure times:

3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, 138. In order to illustrate results for both cases of Type-II hybrid censoring, we suppose that a censoring time of $T = 50$ was used, and we calculate a lower confidence bound for the cases $r = 7$, and $r = 15$. When $r = 7$, the experiment is continued until time $T = 50$ which would result in the first 9 failure times being observed. If $r = 15$, then the experiment is terminated at the 15th failure. The resulting MLE's along with their standard errors calculated using Corollary 3.1, and the lower confidence bounds are presented in Table 5.

Table 5. Lower confidence bounds for θ.

r	$\widehat{\theta}_{\text{obs}}$	s.e.	$\alpha = .05$	$\alpha = .1$
7	89.89	30.96	53.56	59.54
15	101.8	26.28	69.77	75.86

In this case, suppose that we treat the Type-II hybrid censored sample as a conventional Type-II censored sample. When $r = 7$, since 9 failure times would be observed, the 95% lower confidence bound for θ is given by $\frac{2\times 9\times 89.89}{28.8693} = 56.046$, and the 90% lower confidence bound is 62.256. Since $P_{56.046}(\widehat{\theta} > 89.89) = .0684$, and $P_{62.256}(\widehat{\theta} > 89.89) = .1288$, the resulting 95% lower confidence bound is in fact a 93.16% lower confidence bound, and the 90% lower confidence bound is a 87.12% lower confidence bound. When $r = 15$, the 95% and 90% lower confidence bounds agree to three decimals with the values given in Table 5.

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Appendix

To evaluate the second term in (2.1), Chen and Bhattacharyya ((1988), p. 1868) used the following integral expression:

$$\begin{aligned}
&E_\theta(e^{w_r S_r} \mid D = d) \\
&\quad= \frac{C(r,d)}{p^d(1 - w_r\theta)^{(r-1)}} \\
&\qquad\times \int_0^T \frac{1}{\theta} e^{-\{1-w_r\theta(n-r+1)\}y/\theta}\{1 - e^{-(1-w_r\theta)y/\theta}\}^{r-1}\{e^{-y/\theta} - e^{-T/\theta}\}^{d-r} dy,
\end{aligned}$$

where $C(r,d) = \frac{d!}{(r-1)!(d-r)!}$, $p = 1 - q$, $w_r = w/r$. They first evaluated this integral by using two binomial expansions in the integrand, and then summing the resulting expression over d (and multiplying by $P_{\theta,c}(D = d)$), giving a triple summation expression for the second term in Theorem 2.1. However, if we calculate the second term in (2.1) by summing over d first (before evaluating the above integral), and then interchange the summation with integration, we obtain

$$\sum_{d=r}^{n} E_\theta(e^{w_r S_r} \mid D = d) P_{\theta,c}(D = d)$$
$$= (1 - q^n)^{-1} \int_0^T \sum_{d=r}^{n} \binom{n}{d} \frac{q^{n-d} C(r,d)}{(1 - w_r\theta)^{(r-1)}}$$
$$\times \frac{1}{\theta} e^{-\{1-w_r\theta(n-r+1)\}y/\theta} \{1 - e^{-(1-w_r\theta)y/\theta}\}^{r-1} \{e^{-y/\theta} - e^{-T/\theta}\}^{d-r} dy.$$

We then evaluate the sum over d, expand $\{1 - e^{-(1-w_r\theta)y/\theta}\}^{r-1}$ binomially and integrate the resulting expression to obtain the second term in Theorem 2.1.

An alternative way to establish the equivalence is to consider directly the triple summation occurring in the second term of Lemma 2.1 in Chen and Bhattacharyya (1988). Then we simply need to interchange the summation operators with respect to d and j, and simplify the resulting expression.

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