Ann. Inst. Statist. Math.
Vol. 55, No. 2, 243–250 (2003)

EXACT ONE-SIDED SIMULTANEOUS CONFIDENCE BANDS VIA UUSIPAIKKA'S METHOD

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(Received August 9, 2001; revised April 10, 2002)

Abstract. Computation of one-sided simultaneous confidence bands is detailed for a simple linear regression under interval restrictions on the predictor variable, using a method due to Uusipaikka (1983, *J. Amer. Statist. Assoc.*, **78**, 638–644). The case of a single interval restriction is emphasized. A WWW-based applet for computing the bands is described.

Key words and phrases: Simple linear regression, simultaneous confidence bounds, simultaneous inference, S-method, Web-based computing.

1. Introduction

In many experimental situations the response variable, Y, is observed along with a non-stochastic predictor variable, x. Often, the mean response is assumed linear in x: $E[Y \mid x] = \beta_0 + \beta_1 x$. A common model for this simple linear regression setting assumes that the observations are sampled from a normal distribution: $Y_i \sim \text{indep.}\, N(\beta_0 + \beta_1 x_i, \sigma^2)$, $i = 1, \dots, n$. In this case, least squares estimators $\hat{\boldsymbol{\beta}} = [\hat{\beta}_0\ \hat{\beta}_1]'$ of the unknown parameter vector $\boldsymbol{\beta} = [\beta_0\ \beta_1]'$ correspond to those achieved under maximum likelihood, and exact inferences on $E[Y \mid x]$ are readily available (Neter *et al.* (1996)).

Often, analysis of such data is restricted to a limited range of predictor x-values. Herein we direct attention at construction of simultaneous $1-\alpha$ *confidence bands* for the underlying mean $\beta_0 + \beta_1 x$ over all values of x in some relevant restriction set. We consider hyperbolic bands based on Scheffé's S-method (1953). (The hyperbolic shape derives by setting the band's width proportional to the standard error of the estimated linear predictor (Working and Hotelling (1929)).) Many authors have worked on building exact or conservative Scheffé-type bands with restrictions on the predictor variable, including Halperin *et al.* (1967), Halperin and Gurian (1968), Wynn and Bloomfield (1971), Casella and Strawderman (1980), and Naiman (1983), among others. Of particular note is an article by Uusipaikka (1983). He applied a novel geometric approach—which has since come to be known as *Uusipaikka's method*—to the determination of critical points for exact, two-sided, Scheffé-type bands when the restriction on x is taken as an arbitrary, finite union of intervals or points. (For the latter case, also see Lane and DuMouchel (1994).) This might be useful, e.g., when x is distance from a pollution source along a transect and Y is pollutant deposition in the soil along the transect line. In this case no soil pollutants would be recorded over bodies of water, so inferences on $E[Y \mid x]$ would be restricted to the disjoint intervals representing land/soil along the transect.

Less work has appeared for producing *one*-sided (lower or upper) simultaneous confidence bounds on $E[Y \mid x]$. With no restriction on x, Hochberg and Quade (1975) described a one-sided method for calculating Scheffé-type critical points in the multiple regression setting. Under a single interval restriction, say, $A \leq x \leq B$, Bohrer and Francis (1972) gave a method for computing critical points for the simple linear case we study here. To our knowledge, however, no method has appeared for deriving one-sided Scheffé-type bands under an Uusipaikka-type restriction (i.e., over an arbitrary, finite union of intervals). Herein we extend Uusipaikka's (1983) work to construct such an exact one-sided simultaneous band. We also detail use of the one-sided critical points for the special case of a single interval restriction, and we describe briefly a Java applet for computing one-sided or two-sided critical points on the World Wide Web (WWW).

2. Construction of the one-sided band

We base our construction on Uusipaikka's method. Begin by writing the simple linear regression model in matrix form: $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{y}$ is the $n \times 1$ response vector $[Y_1 \cdots Y_n]'$, $\boldsymbol{X}$ is a $n \times 2$ design matrix whose first column is all ones and whose second column is the recorded predictor values $[x_1 \cdots x_n]'$, and $\boldsymbol{\epsilon}$ is an n-variate multivariate normal disturbance vector with zero mean and covariance matrix $\sigma^2\boldsymbol{I}$. The variance parameter σ^2 is assumed unknown. Then, the least squares estimators are $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$ and the usual unbiased estimator of σ^2 is the mean squared error, $S^2 = (\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})/(n-2)$ (Neter *et al.* (1996)).

A one-sided (upper) Scheffé-type band on $\beta_0 + \beta_1 x$ corresponds to the set of parameters

$$\left\{\boldsymbol{\beta} : \beta_0 + \beta_1 x \leq \hat{\beta}_0 + \hat{\beta}_1 x + w_\alpha\sqrt{2}S\sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{S_{xx}}},\ \forall x \in \mathcal{B}\right\} \tag{2.1}$$

where $\mathcal{B}$ is a pertinent restriction set for x and $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2$. (A lower band is formed by replacing w_α with $-w_\alpha$.) Following Uusipaikka (1983), $\mathcal{B}$ can be any finite or infinite closed interval, a union of disjoint intervals, or a finite number of isolated points on the real line.

Of interest is computation of the critical point $w_\alpha > 0$ that gives (2.1) exact $1-\alpha$ coverage over all $x \in \mathcal{B}$. (The notation for the $1-\alpha$ critical point w_α is simplified here for presentation purposes. As we will see below, the point also depends on n, $\boldsymbol{X}$, and $\mathcal{B}$.) Let $\mathcal{C} = \{[1\ x]' : x \in \mathcal{B}\}$ and denote by $\boldsymbol{c}$ any element of $\mathcal{C}$. With this, express (2.1) as

$$\{\boldsymbol{\beta} : \boldsymbol{c}'\boldsymbol{\beta} \leq \boldsymbol{c}'\hat{\boldsymbol{\beta}} + w_\alpha\sqrt{2}S(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{1/2},\ \forall \boldsymbol{c} \in \mathcal{C}\}, \tag{2.2}$$

where $\boldsymbol{V} = (\boldsymbol{X}'\boldsymbol{X})^{-1}$. It is clear that

$$P[\boldsymbol{c}'\boldsymbol{\beta} \leq \boldsymbol{c}'\hat{\boldsymbol{\beta}} + w_\alpha\sqrt{2}S(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{1/2},\ \forall \boldsymbol{c} \in \mathcal{C}] = P\left[\sup_{\boldsymbol{c}\in\mathcal{C}} \frac{\boldsymbol{c}'(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})}{\sqrt{2}S(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{1/2}} \leq w_\alpha\right]$$

(Halperin and Gurian (1968); Uusipaikka (1983)). Thus (2.2) defines an exact $1-\alpha$ level confidence set if w_α is taken as the upper-α point from the c.d.f., $F_{W_\mathcal{C}}(\cdot)$, associated with the random variable

$$W_\mathcal{C} = \sup_{\boldsymbol{c}\in\mathcal{C}} \frac{\boldsymbol{c}'(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})}{\sqrt{2}S(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{1/2}}.$$

Formally, take $\mathcal{B} = \bigcup_{j=1}^{J}[A_j, B_j]$, where we assume $-\infty < A_1 \le B_1 \le A_2 \le B_2 \le \ldots \le A_J \le B_J < \infty$. For simplicity, define the vectors $\boldsymbol{a}_j = [1\ A_j]'$ and $\boldsymbol{b}_j = [1\ B_j]'$ and write

$$\rho_j = \frac{\boldsymbol{a}_{j+1}' \boldsymbol{V} \boldsymbol{b}_j}{\sqrt{\boldsymbol{a}_{j+1}' \boldsymbol{V} \boldsymbol{a}_{j+1} \boldsymbol{b}_j' \boldsymbol{V} \boldsymbol{b}_j}}, \tag{2.3}$$

for $j = 1, \ldots, J$ and $\boldsymbol{a}_{J+1} = \boldsymbol{a}_1$. Notice that each ρ_j-metameter is the correlation between $\boldsymbol{a}_{j+1}'\hat{\boldsymbol{\beta}}$ and $\boldsymbol{b}_j'\hat{\boldsymbol{\beta}}$. We can use Uusipaikka's method to manipulate $F_{W_C}(\cdot)$ into the expression

$$F_{W_C}(w) = F_{(2,n-2)}(w^2) + 2\int_w^\infty F_T\left(\frac{w}{q}\right) q f_{(2,n-2)}(q^2)dq, \tag{2.4}$$

when $w \ge 0$. Here, $F_{(2,n-2)}(\cdot)$ and $f_{(2,n-2)}(\cdot)$ are the c.d.f. and p.d.f., respectively, of an $F(2, n-2)$ random variable, and $F_T(\cdot)$ is the c.d.f. of a random variable, T, which is based on W_C. (It is the specification of T and of its c.d.f. that lies at the heart of Uusipaikka's elegant derivation. For our one-sided setting, we give corresponding details in the Appendix.) Note that when $w < 0$ we find

$$F_{W_C}(w) = 2\int_{|w|}^\infty F_T\left(\frac{w}{q}\right) q f_{(2,n-2)}(q^2)dq, \tag{2.5}$$

but this expression will not be used since we require $w > 0$ in (2.1). Indeed, forcing $w > 0$ restricts our implementation of the confidence band, since it obligates us to require $\alpha \le 1 - F_{W_C}(0) = 1 - F_T(0)$. In practice, however, this is not a hindrance, as we illustrate in the next section.

The results in the Appendix show that we can write $F_T(t)$ as a function solely of the ρ-metameters:

$$F_T(t) = \frac{1}{2\pi}\left\{2\pi + \left(\sum_{j=1}^{M_t} \cos^{-1}\{\rho_{(j)}\}\right) - 2[M_t + 1]\cos^{-1}(t) - \cos^{-1}(\rho_J)\right\}, \tag{2.6}$$

where $\rho_{(1)} \le \rho_{(2)} \le \cdots \le \rho_{(J-1)}$ are the ordered values of $\rho_1, \ldots, \rho_{J-1}$, and M_t is an index satisfying

$$\begin{aligned} \cos^{-1}\{\rho_{(1)}\} \ge \cos^{-1}\{\rho_{(2)}\} \ge \cdots \ge \cos^{-1}\{\rho_{(M_t)}\} \\ \ge 2\cos^{-1}(t) \ge \cos^{-1}\{\rho_{(1+M_t)}\} \ge \cdots \ge \cos^{-1}\{\rho_{(J-1)}\} \end{aligned} \tag{2.7}$$

for any argument $t > 0$. The critical point for the exact $1 - \alpha$ upper confidence band over $\mathcal{B}$ is then the upper-α point of $F_{W_C}(\cdot)$ from (2.4).

Uusipaikka (1983) notes that his method can also be used in the multiple linear regression setting, provided that the restriction subset $\mathcal{C}$ remains two-dimensional.

3. Single-interval restriction ($J = 1$) and WWW access

One important special case—also emphasized by Uusipaikka (1983)—occurs when $J = 1$, i.e., $\mathcal{B} = [A, B]$. For this setting, (2.3) simplifies to the single metameter

$$\rho_1 = \frac{v_{11} + (A + B)v_{12} + ABv_{22}}{\sqrt{(v_{11} + 2Av_{12} + A^2v_{22})(v_{11} + 2Bv_{12} + B^2v_{22})}},$$

where v_{ij} is the (i,j)-th element of $\boldsymbol{V}$. Note that ρ_1 is not the correlation between x_i and Y_i, although it does rely upon the predictor variables through its dependence on the v_{ij}s.

The expression for $F_{W_C}(\cdot)$ in (2.4) remains valid, with $F_T(t)$ from (2.6) simplifying to

$$F_T(t) = \frac{1}{2\pi}\{2\pi - 2\cos^{-1}(t) - \cos^{-1}(\rho_1)\}. \tag{3.1}$$

Using (3.1) in (2.4) and setting $F_{W_C}(w) = 1-\alpha$ yields a solution, w_α, which allows us to write (2.1) as an exact $1-\alpha$ upper band $\forall x \in [A,B]$. Of course, we must still require $1-\alpha \geq F_{W_C}(0) = F_T(0)$. Using (3.1), we find this is equivalent to $1-\alpha \geq \frac{1}{2}(1-\cos^{-1}\{\rho_1\})$. Table 1 gives values of this lower bound for $1-\alpha$ as a function of the single metameter ρ_1. As can be seen, the bounds are always below 0.5, and hence do not impose any practical hindrance to implementation of the one-sided bands.

To find upper-α critical points w_α for this single-interval setting, we have constructed WWW-based software that is available over the Internet. The software was created by linking together a front-end Java applet that serves as a user interface, and a back-end FORTRAN program that calculates w_α from information supplied by the user. The applet was constructed using Java classes from the *WebStat* analysis program (West and Ogden (1997)).

The applet is accessible via any Java-compatible Internet browser, and requires the user to enter the data in two columns (first x, then Y; a cut-and-paste option is provided), and also to supply the required band construction parameters α, A, and B. The correlation metameter ρ_1 is calculated directly from the data. If desired, the user can resort to the default values $\alpha = 0.05$, $A = \min\{x_1,\ldots,x_n\}$, and $B = \max\{x_1,\ldots,x_n\}$. Access the WWW applet at the URL `http://www.stat.sc.edu/rsrch/gasp/bands/`

The critical points for one-sided bands available from our WWW applet may be compared to earlier critical points given for this special case by Bohrer and Francis ((1972), Table 1). We performed this comparison, and found that in general our values

Table 1. Minimum values, $F_T(0)$, of confidence coefficient, $1-\alpha$, for one-sided Scheffé-type bands under a single interval restriction, as a function of the ρ_1 metameter.

ρ_1	$F_T(0)$
−1.0	0.0000
−0.8	0.1024
−0.6	0.1476
−0.4	0.1845
−0.2	0.2180
0.00	0.2500
0.20	0.2820
0.40	0.3155
0.60	0.3524
0.80	0.3976
0.90	0.4282
0.95	0.4495
0.99	0.4775
1.00	0.5000

agree to the 10^{-2} accuracy found in Bohrer and Francis' table. The only discrepancies we encountered were in the case of $\nu = 1$ (i.e., $n = 3$); there, our critical points were slightly but consistently higher than the previously-tabled values. To check this, we evaluated the actual coverage of the previous points and of our own; we found that the points in our tables exhibited correct, nominal coverage. For the previous points at $n = 3$, however, we found that coverage generally dropped below the nominal level by a small amount, and was only correct in the single point ($A = B$) case. (At $\alpha = 0.10$, the average coverage over a range of input values using the previous points was 0.887; at $\alpha = 0.05$ it was 0.948; and at $\alpha = 0.01$ it was 0.989. Obviously these are not major discrepancies, but we do caution that the previous tables may be slightly undervalued when $\nu = 1$.)

4. Example

To illustrate use of these one-sided bands consider the following data, discussed by Dalgård *et al.* (1994). Their study concerned mercury (Hg) toxicity in pregnant Faroe islanders, where potentially high mercury body burdens occur due to the islanders' large consumption of pilot whale meat. The response was taken as Hg concentration in the woman's umbilical cord blood (in μmol/l, recorded immediately after giving birth), viewed as a function of average daily Hg ingestion (in μg). The study involved $n = 12$ observations with an average Hg ingestion of $\bar{x} = 195.867$ μg, and with $S_{xx} = 366{,}189.947$ μg^2. The data appear in Table 2.

Mercury exposure to the women's gestating fetuses or neonates may present a risk of malformation or other toxic damage to the offspring, so public health officials are often interested in assessing the nature of the maternal Hg ingestion/exposure (Mahaffey (2000); Shipp *et al.* (2000)). Regulatory interest only concerns the severity of the Hg outcome, thus for purposes of risk assessment only upper confidence statements are required. This leads to use of only an upper simultaneous bound on $E[Y \mid x]$. Here, we take $J = 1$ and set the restriction interval as the range between zero and slightly past

Table 2. Data on mercury concentration in umbilical cords (Y) in postpartum women after ingestion of mercury (x) in the diet. Source: Dalgård *et al.* (1994).

Daily mercury ingestion (μg)	Maternal cord mercury (μmol/l)
1.4	0.007
49	0.23
90	0.43
96	0.46
108	0.52
125	0.60
146	0.70
153	0.73
233	1.12
324	1.56
354	1.70
671	3.22

the highest ingestion level, $0 \le x \le 675$.

Fitting a simple linear model to these data yields the least squares estimates $\hat{\beta}_0 = -0.0012$ and $\hat{\beta}_1 = 0.0048$, with root mean square error $S = 0.0027$. At $\alpha = 0.05$, the corresponding one-sided critical point is $w_{0.05} = 1.718$. This is used in constructing a simultaneous upper confidence band for these data over $0 \le x \le 675$. From (2.1), we find

$$\beta_0 + \beta_1 x \le -0.0012 + 0.0048x + 0.0066\sqrt{\frac{1}{12} + \frac{(x - 195.867)^2}{366189.947}}.$$

This suggests, e.g., that at very low ingestion levels—say, $x = 0.5$ μg—mean cord blood concentrations go no higher than about 0.004 μmol/l, while at very high levels—say, $x = 650$ μg—the concentration can reach up to 3.124 μmol/l. Since these inferences are derived from simultaneous upper bands they both hold with 95% confidence, as would any other upper confidence statement(s) made for ingestion levels between $0 \le x \le 675$ μg.

Acknowledgements

Thanks are due to Drs. Ralph L. Kodell, Obaid M. Al-Saidy, and an anonymous referee for their helpful comments during the preparation of this work. The research was supported by funding under grant #R01-CA76031 from the U.S. National Cancer Institute. Its contents are solely the responsibility of the authors and do not necessarily reflect the official views of the National Cancer Institute.

Appendix: Derivation of the distribution of $W_{\mathcal{C}}$

Our proof of the construction in (2.4) and (2.6) mimics that given by Uusipaikka for the two-sided case. Similar to that work, we outline the derivation by dividing it into two parts. First, we show that the c.d.f. of $W_{\mathcal{C}}$ for $w \ge 0$ has the integral representation given in (2.4). Then, we find that the c.d.f. $F_T(\cdot)$ has the explicit expression given in (2.6).

To begin, take any nonsingular matrix $\boldsymbol{B}$ such that the covariance matrix $\boldsymbol{V}$ is $\boldsymbol{V} = \boldsymbol{B}'\boldsymbol{B}$. With this, define $\boldsymbol{z} = \sigma^{-1}(\boldsymbol{B}')^{-1}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$, which is bivariate normal with zero mean vector and $\mathrm{Var}(\boldsymbol{z}) = \boldsymbol{I}$. Next, express $\boldsymbol{z}$ in polar coordinates: $\boldsymbol{z} = R\boldsymbol{u}$ where $\boldsymbol{u} = [\cos\Theta \ \ \sin\Theta]'$. It is well-known that $R^2 = z_1^2 + z_2^2 \sim \chi^2(2)$ is independent of $\Theta \sim \mathrm{Unif.}[0, 2\pi]$. Thus $\boldsymbol{u}$ is also independent of R^2. We can write $W_{\mathcal{C}}$ in the form

$$W_{\mathcal{C}} = \sup_{\boldsymbol{c}\in\mathcal{C}} \frac{(\boldsymbol{B}\boldsymbol{c})'\boldsymbol{z}}{\sqrt{2}\sigma^{-1}S(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{1/2}} = \left\{\frac{\frac{1}{2}R^2}{\frac{1}{\sigma^2}S^2}\right\}^{1/2} \sup_{\boldsymbol{c}\in\mathcal{C}^*}\{\boldsymbol{c}'\boldsymbol{u}\}, \tag{A.1}$$

where $\mathcal{C}^* = \{(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{-1/2}\boldsymbol{B}\boldsymbol{c} : \boldsymbol{c} \in \mathcal{C}\}$ is a subset of the unit circle (a finite union of J consecutive, disjoint arcs) in Euclidean two-space; we detail $\mathcal{C}^*$ more fully below. Notice that $Q^2 = \{(R^2/2)/(S^2/\sigma^2)\} \sim F(2, n-2)$ and that Q is independent of $T = \sup_{\boldsymbol{c}\in\mathcal{C}^*}\{\boldsymbol{c}'\boldsymbol{u}\}$.

We will see below that since $\mathcal{C}^*$ is a subset of the unit circle, it can be expressed as $\mathcal{C}^* = \{[\cos\phi \ \ \sin\phi]' : \phi \in \Gamma^*\}$, where Γ^* corresponds to a finite union of angles swept out by the finite union of disjoint arcs defining $\mathcal{C}^*$; again, we detail Γ^* more fully below.

There, we will write $T = \sup_{c \in \mathcal{C}^*}\{\boldsymbol{c}'\boldsymbol{u}\}$ as $T = \sup_{\phi \in \Gamma^*}\{\cos(\phi)\cos(\Theta) + \sin(\phi)\sin(\Theta)\} = \sup_{\phi \in \Gamma^*}\{\cos(\phi - \Theta)\}$; recall that $\Theta \sim$ Unif.$[0, 2\pi]$.

Now, since $W_{\mathcal{C}} = QT$, from (A.1) we have

$$\text{(A.2)} \qquad \begin{aligned} F_{W_{\mathcal{C}}}(w) &= P[QT \le w] = \int_0^\infty P\left[T \le \frac{w}{q}\right] f_Q(q)dq \\ &= 2\int_0^\infty F_T\left(\frac{w}{q}\right) q f_{(2,n-2)}(q^2)dq. \end{aligned}$$

Notice that $P[-1 \le T \le 1] = 1$. Thus, whenever $q \le |w|$, $F_T(w/q) = 1$ if $w \ge 0$, and $F_T(w/q) = 0$ if $w < 0$. Hence for $w \ge 0$, (A.2) corresponds to (2.4), while for $w < 0$, (A.2) corresponds to (2.5), as desired.

Next, consider the c.d.f. $F_T(\cdot)$. Recall that $\mathcal{B} = \bigcup_{j=1}^J [A_j, B_j]$, and with this, define the vectors $\boldsymbol{a}_j = [1\ A_j]'$ and $\boldsymbol{b}_j = [1\ B_j]'$. Translating $\mathcal{B}$ (and thus $\mathcal{C}$) to the polar-transformed space is equivalent to using the J disjoint intervals in $\mathcal{B}$ to define J consecutive, disjoint arcs on the unit circle, each with endpoints $\boldsymbol{a}_j^* = (\boldsymbol{a}_j'\boldsymbol{V}\boldsymbol{a}_j)^{-1/2}\boldsymbol{B}\boldsymbol{a}_j$ and $\boldsymbol{b}_j^* = (\boldsymbol{b}_j'\boldsymbol{V}\boldsymbol{b}_j)^{-1/2}\boldsymbol{B}\boldsymbol{b}_j$, $j = 1, \ldots, J$. (Note in particular that $\boldsymbol{a}_1$ maps to $\boldsymbol{a}_1^* = [1\ 0]'$.) These define the set $\mathcal{C}^*$. Thus we can express $\mathcal{C}^*$ as $\mathcal{C}^* = \{[\cos\phi\ \sin\phi]' : \phi \in \Gamma^*\}$, where Γ^* is the set of corresponding angles defining each arc's endpoint in polar space. That is, $\Gamma^* = \bigcup_{j=1}^J [\psi_j, \zeta_j]$ for $\boldsymbol{a}_j^* = [\cos(\psi_j)\ \sin(\psi_j)]'$ and $\boldsymbol{b}_j^* = [\cos(\zeta_j)\ \sin(\zeta_j)]'$, where $0 = \psi_1 \le \zeta_1 \le \psi_2 \le \zeta_2 \le \cdots \le \psi_J \le \zeta_J \le \pi$.

This construction allows us to write T as $T = \sup_{c \in \mathcal{C}^*}\{\boldsymbol{c}'\boldsymbol{u}\} = \sup_{\phi \in \Gamma^*}\{\cos(\phi)\cos(\Theta) + \sin(\phi)\sin(\Theta)\} = \sup_{\phi \in \Gamma^*}\{\cos(\phi - \Theta)\}$. Now, the event $\{T \le t\}$ may be expressed as $\{\cos(\phi - \Theta) \le t, \forall \phi \in \Gamma^*\}$; but, notice that for all $\phi \in \Gamma^*$, when $t \ge \cos\phi$, $0 \le \Theta \le \phi - \cos^{-1}(t)$ or $\phi + \cos^{-1}(t) \le \Theta \le 2\pi$, while when $t < \cos\phi$, $\phi + \cos^{-1}(t) \le \Theta \le 2\pi + \phi - \cos^{-1}(\mathrm{t})$. Thus $\{T \le t\}$ is equivalent to

$$\begin{cases} \Theta \notin (0, \phi + \cos^{-1}\{t\}) \cup (2\pi - \cos^{-1}\{t\}, 2\pi - \phi) & \text{if} \quad \phi < \cos^{-1}(t) \\ \Theta \notin (\phi - \cos^{-1}\{t\}, \phi + \cos^{-1}\{t\}) & \text{if} \quad \phi \ge \cos^{-1}(t). \end{cases}$$

Now, define $\psi_{J+1} = 2\pi$, so that if $\zeta_j + \cos^{-1}(t) \le \psi_{j+1} - \cos^{-1}(t)$, the event $\{T \le t\}$ can be written as $\{\Theta \in \bigcup_{j=1}^J [\zeta_j + \cos^{-1}\{t\}, \psi_{j+1} - \cos^{-1}\{t\}]\}$. Notice that $\cos(\psi_{j+1} - \zeta_j)$ is simply ρ_j as given in (2.3). If we also define

$$\rho_J = \frac{\boldsymbol{a}_1'\boldsymbol{V}\boldsymbol{b}_J}{\sqrt{\boldsymbol{a}_1'\boldsymbol{V}\boldsymbol{a}_1\boldsymbol{b}_J'\boldsymbol{V}\boldsymbol{b}_J}},$$

and $\tau_j = \cos\{(\psi_{j+1} - \zeta_j)/2\}$, $j = 1, \ldots, J$, then the c.d.f. of T can be written as

$$\begin{aligned} F_T(t) &= P[T \le t] = \frac{1}{2\pi}\sum_{\tau_j \le t}\{\psi_{j+1} - \zeta_j + 2\cos^{-1}(t)\} \\ &= \frac{1}{2\pi}\left(2\pi - \cos^{-1}\rho_J - 2\cos^{-1}t + \sum_{\substack{j=1 \\ \cos^{-1}\rho_j \ge 2\cos^{-1}t}}^{J-1}\{\cos^{-1}(\rho_j) - 2\cos^{-1}(t)\}\right) \end{aligned}$$

(or zero, whichever is greater). If we denote the ordered values of $\rho_1, \ldots, \rho_{J-1}$ by $\rho_{(1)} \le \rho_{(2)} \le \cdots \le \rho_{(J-1)}$, and let M_t be the index satisfying (2.7) for any $t > 0$, then we can simplify $F_T(t)$ into

$$F_T(t) = \frac{\left(\sum_{j=1}^{M_t} \cos^{-1}\{\rho_{(j)}\}\right) - 2M_t \cos^{-1}(t)}{2\pi} + \frac{2\pi - \cos^{-1}\rho_J - 2\cos^{-1} t}{2\pi}$$

(or zero, whichever is greater). This corresponds to the expression given in (2.6).

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