Ann. Inst. Statist. Math.
Vol. 55, No. 1, 153–167 (2003)

GENERALIZED BINOMIAL AND NEGATIVE BINOMIAL DISTRIBUTIONS OF ORDER k BY THE ℓ-OVERLAPPING ENUMERATION SCHEME

KIYOSHI INOUE* AND SIGEO AKI

Department of Informatics and Mathematical Science, Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama-cho, Toyonaka, Osaka 560-8531, Japan

(Received July 16, 2001; revised February 6, 2002)

Abstract. In this paper, we investigate the exact distribution of the waiting time for the r-th ℓ-overlapping occurrence of success-runs of a specified length in a sequence of two state Markov dependent trials. The probability generating functions are derived explicitly, and as asymptotic results, relationships of a negative binomial distribution of order k and an extended Poisson distribution of order k are discussed. We provide further insights into the run-related problems from the viewpoint of the ℓ-overlapping enumeration scheme. We also study the exact distribution of the number of ℓ-overlapping occurrences of success-runs in a fixed number of trials and derive the probability generating functions. The present work extends several properties of distributions of order k and leads us a new type of geneses of the discrete distributions.

Key words and phrases: Run, waiting time, binomial distribution, negative binomial distribution, Poisson distribution, double generating function, probability generating function, Markov chain, Markov chain imbedding method.

1. Introduction

Exact distributions on runs in independent trials go back as far as De Moivre's era (see Feller (1968)). For the last 20 years, exact distribution theory for so called discrete distributions of order k (see Philippou *et al.* (1983)) has been extensively developed by many authors in various situations and many works have appeared on the discrete distributions of order k (see Aki and Hirano (1988), Hirano *et al.* (1991) , Han and Aki (1998) and Uchida (1998)).

The relations between distributions of order k have been investigated by many authors. Hirano and Aki (1987) discussed relationships among the extended negative binomial, the extended Poisson and the extended logarithmic series distributions of order k. Philippou (1988) examined the interrelationships of multiparameter distributions of order k. Koutras (1997) considered negative binomial distributions of order k and showed that the limiting behavior is closely related to the class of distributions of the sum of Poisson number of iid random variables.

Furthermore, relations among distributions of different orders have been studied. Aki and Hirano (1994) investigated some properties of the geometric distributions of different orders. Several extensions and variations of their model were subsequently studied

*Now at The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan.

by Aki and Hirano (1995). Aki and Hirano (2000) pointed out that how to enumerate success-runs is also very important in order to obtain the corresponding distributional results in the case of the binomial distribution of order k. They proposed an enumeration scheme called ℓ-overlapping way of counting. In the case of $\ell = k - 1$, it corresponds to usual overlapping counting scheme (see Ling (1988)). For example, the sequence $SF(SSS)F(SS\{S)S[S\}SS]F(SSS)$ contains 5 (1-overlapping) success-runs of length 3. The ℓ-overlapping enumeration scheme derives the some interesting properties from the distribution of order k. We believe that this enumeration scheme plays an important role in the discrete distribution theory in future.

Recently, Han and Aki (2000) introduced the ℓ-overlapping counting method when ℓ is a negative integer, and considered the distribution of the number of ℓ-overlapping occurrences of success-runs of length k in a sequence of a fixed number of trials by a method based on the probability generating functions. When ℓ is a negative integer, it is intuitively recognized that the two runs of length k are $|\ell|$ apart from each other. For example, the sequence $SF(SSS)FS(SSS)SSSF(SSS)$ contains 3 ((-2)-overlapping) success- runs of length 3. Remark that when $\ell < 0$ there is a slight difference between our definition of ℓ-overlapping counting method in this paper and Han and Aki's (2000) definition.

Our aim of this paper is to provide the perspectives on the run-related problems from the viewpoint of the ℓ-overlapping enumeration scheme. We emphasize the importance of this enumeration scheme. The present paper is organized as follows. In Section 2, we study the waiting time distribution for the r-th ℓ-overlapping occurrence of success-run of length k in a sequence of $\{0,1\}$-valued Markov dependent trials, and derive the probability generating functions. We show that the corresponding variable is expressed as a sum of r independent variables. For this distribution, Koutras (1997) used the name *Markov Negative Binomial distribution of order k*. In Section 3, we investigate the limiting behavior of the distributions treated in Section 2 as $r \to \infty$, and show that the limiting behavior is closely related to an extended Poisson distribution of order k (see Aki (1985)). In Section 4, we consider the distributions of the number of ℓ-overlapping occurrences of success-runs of length k in a sequence of a fixed number of trials, and derive the probability generating functions. The ℓ-overlapping enumeration scheme leads us a new type of geneses of the distributions of order k.

The main tool for deriving the results in this paper is the Markov chain imbedding method introduced by Fu (1986) firstly, which has a great potential for extending to other problems (see Fu and Hu (1987), Chao and Fu (1989), chao (1991), Fu and Koutras (1994), Koutras (1996a), Koutras and Alexandrou (1997), Koutras *et al.* (1995) and Chadjiconstantinidis *et al.* (2000)).

2. The waiting time for the r-th occurrence

Let $X_0, X_1, X_2, \ldots$ be a time homogeneous $\{0,1\}$-valued Markov chain with transition probabilities,

$$p_{ij} = P(X_t = j \mid X_{t-1} = i), \tag{2.1}$$

for $t \geq 1$, $i,j = 0,1$ and initial probabilities $P(X_0 = 0) = p_0$, $P(X_0 = 1) = p_1$.

According to Koutras and Alexandou (1995), a non-negative integer random variable V_n is called Markov chain imbeddable variable of binomial type, if

(1) there exists a Markov chain $\{Y_t, t \geq 0\}$ defined on a state space Ω,

(2) there exists a partition $\{C_v : v \geq 0\}$ on the state space,

(3) for every v, $P(V_n = v) = P(Y_n \in C_v)$,

(4) $P(Y_t \in C_w \mid Y_{t-1} \in C_v) = 0$ for all $w \neq v, v+1$ and $t \geq 1$.

Assume first that the sets C_v of the partition $\{C_v, v \geq 0\}$ have the same cardinality $s = |C_v|$ for every v, more specifically $C_v = \{c_{v,0}, c_{v,1}, \dots, c_{v,s-1}\}$.

For the Markov chain $\{Y_t, t \geq 0\}$, we introduce the $s \times s$ transition probability matrices

$$A_t(v) = (P(Y_t = c_{v,j} \mid Y_{t-1} = c_{v,i}))_{s\times s},$$
$$B_t(v) = (P(Y_t = c_{v+1,j} \mid Y_{t-1} = c_{v,i}))_{s\times s},$$

the probability vectors of the t-step Y_t of the Markov chain

$$\boldsymbol{f}_t(v) = (P(Y_t = c_{v,0}), P(Y_t = c_{v,1}), \dots, P(Y_t = c_{v,s-1})), \qquad t \geq 0,$$

and the initial probabilities

$$\boldsymbol{\pi}_0 = (P(Y_0 = c_{v,0}), P(Y_0 = c_{v,1}), \dots, P(Y_0 = c_{v,s-1})).$$

Let now T_r, $r \geq 1$ be the waiting time for the r-th ℓ-overlapping occurrence of success-run of length k. Then the probability generating function and the double generating function of T_r are denoted by $H_r(z)$ and $H(z,w)$, respectively;

$$H_r(z) = E[z^{T_r}] = \sum_{n=0}^{\infty} \Pr[T_r = n] z^n,$$
$$H(z,w) = \sum_{r=0}^{\infty} H_r(z) w^r = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \Pr[T_r = n] z^n w^r.$$

For the homogeneous case (i.e. $A_t(v) = A$, $B_t(v) = B$ for all $t \geq 1$ and $v \geq 0$), the double generating function is

$$\begin{aligned} H(z,w) &= wz \sum_{i=1}^{s} \beta_i \left(\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \boldsymbol{f}_n(r) w^r z^n \right) \boldsymbol{e}_i', \\ &= wz\boldsymbol{\pi}_0 \sum_{i=1}^{s} \beta_i \left[I - z\left(A + wB\right)\right]^{-1} \boldsymbol{e}_i', \end{aligned}$$

where, $\beta_i = \boldsymbol{e}_i B \boldsymbol{1}'$, $1 \leq i \leq s$. We denote the i-th unit vector of R^s by $\boldsymbol{e}_i = (0, \dots, 1, \dots, 0)$.

The waiting time for the r-th ℓ-overlapping occurrence of success-run of length k are denoted by $T_r^{(+)}$ and $T_r^{(-)}$ with the superscript pointing out the enumeration scheme employed; $(+)$ indicates the case $0 < \ell \leq k-1$ and $(-)$ the case $\ell \leq 0$. In this paper, each one of the two enumeration schemes ($\ell \leq 0$, $0 < \ell \leq k-1$) is treated separately.

2.1 *Case* $0 < \ell \leq k-1$

We consider the partition $C_v = \{c_{v,0}, c_{v,1}, \dots, c_{v,k-1}, c_{v,\ell-k}, \dots, c_{v,-1}\}$, $v = 0, 1, \dots, [\frac{n-\ell}{k-\ell}]$, where,

$$c_{v,i} = \{(v,i)\}, \quad \ell - k \leq i \leq k-1, \quad v = 0, 1, \dots, \left[\frac{n-\ell}{k-\ell}\right], \quad s = |C_v| = 2k - \ell.$$

To introduce a proper Markov chain $\{Y_t : t \geq 0\}$, we define $Y_t \in c_{v,i}$ (or equivalently $Y_t = (v, i)$) as follows. For any sequence of outcomes of length t, say $SFS\cdots F\overbrace{SS\cdots S}^{m}$, let m be the number of trailing successes, and let v be the number of ℓ-overlapping occurrences of success-runs of length k. We define $Y_t = (v, m)$ if $m \leq k-1$ and $Y_t = (v, \ell - k + y)$ if $m \geq k$, where $m - k = y \pmod{k - \ell}$.

We have

$$A + wB = \begin{array}{c} \begin{array}{ccccccccccc} (,0) & (,1) & (,2) & \cdots & (,k-2) & (,k-1) & (,\ell-k) & (,\ell-k+1) & \cdots & (,-2) & (,-1) \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ p_{10} & 0 & p_{11} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{10} & 0 & 0 & \cdots & p_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ p_{10} & 0 & 0 & \cdots & 0 & p_{11} & 0 & 0 & \cdots & 0 & 0 \\ p_{10} & 0 & 0 & \cdots & 0 & 0 & wp_{11} & 0 & \cdots & 0 & 0 \\ p_{10} & 0 & 0 & \cdots & 0 & 0 & 0 & p_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{10} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & p_{11} & 0 \\ p_{10} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & p_{11} \\ p_{10} & 0 & 0 & \cdots & 0 & 0 & wp_{11} & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}.$$

The manipulation of partitioned matrix enables us to calculate the inverse matrix $[I - z(A + wB)]^{-1}$ easily (see, for example, Zhang (1999)). We should make use of the following symmetric partition.

$$I - z(A + wB) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}_{(2k-\ell)\times(2k-\ell)},$$

where K and N are k- and $(k-\ell)$-square matrices, respectively. Then, the inverse matrix is given by

$$[I - z(A + wB)]^{-1} = \begin{pmatrix} K^{-1} + XZ^{-1}Y & -XZ^{-1} \\ -Z^{-1}Y & Z^{-1} \end{pmatrix}_{(2k-\ell)\times(2k-\ell)},$$

where $X = K^{-1}L$, $Y = MK^{-1}$ and $Z = N - MK^{-1}L$. Since $\boldsymbol{\pi}_0 = (p_0, p_1, 0, \ldots, 0)$ and

$$\beta_i = \boldsymbol{e}_i B \boldsymbol{1}' = \begin{cases} p_{11}, & \text{if} \quad i = k \text{ or } i = 2k - \ell, \\ 0, & \text{otherwise,} \end{cases}$$

by algebraic manipulations, we get

$$H^{(+)}(z, w) = \frac{wP(z)(p_{11}z)^{k-1}}{[1 - w(p_{11}z)^{k-\ell}]Q(z) - wp_{01}p_{10}p_{11}^{k-1}z^{k+1}R_{k-\ell-1}(p_{11}z)}, \tag{2.2}$$

where,

$$P(z) = p_1 + (p_0 p_{01} - p_1 p_{00})z, \tag{2.3}$$

$$Q(z) = 1 - p_{00}z - p_{01}p_{10}z^2 \sum_{i=2}^{k} (p_{11}z)^{i-2}, \tag{2.4}$$

$$R_x(z) = \begin{cases} 1 + z + z^2 + \cdots + z^x, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Expanding (2.2) in a Taylor series around $w = 0$ and considering the coefficient of w^r, we obtain the probability generating function of $T_r^{(+)}$.

PROPOSITION 1. *The probability generating function of* $T_r^{(+)}$ *is*

$$H_r^{(+)}(z) = \frac{(p_{11}z)^{k-1}P(z)}{Q(z)} \left[(p_{11}z)^{k-\ell} + \frac{p_{01}p_{10}p_{11}^{k-1}z^{k+1}R_{k-\ell-1}(p_{11}z)}{Q(z)} \right]^{r-1} \tag{2.6}$$
$$r \geq 1.$$

The probability generating function of the random variable $T = T_1$ for the first occurrence of success-run of length k is given by

$$H(z) = \left[\frac{1}{w} H^{(+)}(z, w) \right]_{w=0} = \frac{P(z)(p_{11}z)^{k-1}}{Q(z)}. \tag{2.7}$$

Let T^* be the waiting time for the first occurrence of success-run of length k in a sequence of Markov dependent trials with transition probabilities (2.1) and initial conditions $P(X_0 = 0) = 1$, $P(X_0 = 1) = 0$. Its probability generating function is derived from (2.7) (setting $p_0 = 1, p_1 = 0$) as

$$H^*(z) = \frac{(p_{01}z)(p_{11}z)^{k-1}}{Q(z)}. \tag{2.8}$$

We show that $T_r^{(+)}$ $(r \geq 2)$ can be decomposed as a sum of r independent waiting time random variables.

THEOREM 2.1. *For* $r \geq 2$, *let* T_j^*, $1 \leq j \leq r-1$ *be independent duplicates of* T^* *(with probability generating function (2.8)) which are also independent of* T *(with probability generating function (2.7)). Let* W_j, $1 \leq j \leq r-1$ *be* $\{1, 2, \dots, k-\ell+1\}$*-valued iid multi-state variables which take "1" with probability* $p_{11}^{k-\ell}$ *and "*i*" with probability* $p_{10}p_{11}^{i-2}$ $(i = 2, 3, \dots, k-\ell+1)$,

$$W_j^* = \begin{cases} k - \ell, & \text{if } W_j = 1, \\ 1 + T_j^*, & \text{if } W_j = 2, \\ \quad\vdots & \qquad\vdots \\ i + T_j^*, & \text{if } W_j = i+1, \\ \quad\vdots & \qquad\vdots \\ k - \ell + T_j^*, & \text{if } W_j = k-\ell+1, \end{cases}$$

then $T, W_1^*, \dots, W_{r-1}^*$ *are independent and*

$$T_r^{(+)} \overset{d}{=} T + \sum_{j=1}^{r-1} W_j^*.$$

PROOF. From the equation (2.6), the probability generating function of $T_r^{(+)}$ takes the form

$$H_r^{(+)}(z) = H(z)\left[(p_{11}z)^{k-\ell} + (p_{10}z)H^*(z)R_{k-\ell-1}(p_{11}z)\right]^{r-1}. \tag{2.9}$$

Due to the definitions, the probability generating function of W_j^* is

$$G^*(z) = (p_{11}z)^{k-\ell} + (p_{10}z)H^*(z)R_{k-\ell-1}(p_{11}z),$$

which implies the independency of W_j^*, $1 \le j \le r-1$ and T. Accordingly, the probability generating function of $T + \sum_{j=1}^{r-1} W_j^*$ coincides with $H_r^{(+)}(z)$. □

Remark 1. The results of this subsection for overlapping success-runs (i.e. $\ell = k-1$) reduce to the ones derived by Koutras (1997).

2.2 *Case* $\ell \le 0$

We treat the case of $\ell \le 0$. Suppose that the success-run of length k is observed. Then we should restart the counting the success-run after $|\ell|$ trials pass (restarting state). That is, if we have currently the success-run of length k, we must wait counting the success-run from the next trial until $|\ell|$ trials pass (waiting state). For example, consider the sequence $SF(SSS)FS(SSS)SSSF(SSS)$. When $k=3$, $\ell=-2$ and $r=3$, we have $T_3^{(-)} = 17$. We consider the partition

$$\begin{aligned}
&C_v = \{c_{v,0}, c_{v,1}, \dots, c_{v,k-1}, c^1_{v,\ell}, c^0_{v,\ell+1}, c^1_{v,\ell+1}, \dots, c^0_{v,-1}, c^1_{v,-1}, c^1_{v,0}\},\\
&v = 0,1,\dots,\left[\frac{n+|\ell|}{k+|\ell|}\right], \quad \text{where,}\\
&c_{v,i} = \{(v,i)\}, \quad 0 \le i \le k-1, \quad v = 0,1,\dots,\left[\frac{n+|\ell|}{k+|\ell|}\right],\\
&c^j_{v,\ell+i} = \{(v,\ell+i;j)\}, \quad 1 \le i \le |\ell|-1, \quad j=0,1, \quad v = 0,1,\dots,\left[\frac{n+|\ell|}{k+|\ell|}\right],\\
&c^1_{v,\ell} = (v,\ell;1), \quad c^1_{v,0} = (v,0;1), \quad v = 0,1,\dots,\left[\frac{n+|\ell|}{k+|\ell|}\right],\\
&s = |C_v| = k + 2|\ell|.
\end{aligned}$$

• Restarting state: (v,m), $v = 0,1,\dots,[\frac{n+|\ell|}{k+|\ell|}]$, $m = 0,\dots,k-1$. $Y_t = (v,m)$ means that there exist v (ℓ-overlapping) success-runs of length k, and m trailing S after waiting state.

• Waiting state: $(v,\ell+i;j)$, $v = 0,1,\dots,[\frac{n+|\ell|}{k+|\ell|}]$, $i = 0,\dots,|\ell|$, $j = 0,1$. $Y_t = (v,\ell+i;j)$ means that there exist v (ℓ-overlapping) success-runs of length k, the i trials pass after the occurrence of success-run of length k, and "j" (S or F) has just occurred at t-th trial. Remark that the state $(v,\ell;0)$ does not make any sense, and $(v,0;0) \equiv (v,0)$

Since $\boldsymbol{\pi}_0 = (p_0, p_1, 0, \dots, 0)$ and

$$\beta_i = \boldsymbol{e}_i B \mathbf{1}' = \begin{cases} p_{11}, & \text{if } \ i = k,\\ 0, & \text{otherwise,}\end{cases}$$

after some calculations, we obtain

$$H^{(-)}(z,w) = \frac{wH(z)}{1 - w(p_{10}^{(|\ell|)} z^{|\ell|} H^*(z) + p_{11}^{(|\ell|)} z^{|\ell|} H^{**}(z))}, \tag{2.10}$$

where $p_{1j}^{(|\ell|)}$ $(j = 0, 1)$ denotes the $|\ell|$-step transition probability from state 1 to state j,

$$(2.11) \qquad (p_{10}^{(|\ell|)}, p_{11}^{(|\ell|)}) = (0, 1)\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}^{|\ell|},$$

(Convention: $p_{10}^{(0)} = 0$, $p_{11}^{(0)} = 1$), and

$$H^{**}(z) = \frac{(p_{01}p_{10}z^2 + p_{11}z - p_{11}p_{00}z^2)(p_{11}z)^{k-1}}{Q(z)}.$$

Expanding (2.10) in a Taylor series around $w = 0$ and considering the coefficient of w^r, we obtain the probability generating function of $T_r^{(-)}$.

PROPOSITION 2. *The probability generating function of $T_r^{(-)}$ is*

$$(2.12) \qquad H_r^{(-)}(z) = H(z)[p_{10}^{(|\ell|)}z^{|\ell|}H^*(z) + p_{11}^{(|\ell|)}z^{|\ell|}H^{**}(z)]^{r-1}, \qquad r \geq 1.$$

In the case of $\ell = 0$, Antzoulakos (1999) has also given the probability generating function (2.12) with a slight differences due to the different set-up used there. Let T^{**} be the random variable with the probability generating function

$$(2.13) \qquad G^{**}(z) = p_{10}^{(|\ell|)}z^{|\ell|}H^*(z) + p_{11}^{(|\ell|)}z^{|\ell|}H^{**}(z).$$

We show that $T_r^{(-)}$ $(r \geq 2)$ can be decomposed as a sum of r independent waiting time random variables.

THEOREM 2.2. *For $r \geq 2$, let T_j^{**}, $1 \leq i \leq r-1$ be independent duplicates of T^{**} (with probability generating function (2.13)). Let T be as in Theorem 2.1. Then, T, $T_1^{**}, \dots, T_{r-1}^{**}$ are independent and*

$$(2.14) \qquad T_r^{(-)} \stackrel{d}{=} T + \sum_{j=1}^{r-1} T_j^{**}.$$

PROOF. The equation (2.12) implies the representation (2.14). The proof is completed. □

Remark 2. Theorems 2.1 and 2.2 can be used for obtaining some simple approximations to negative binomial distriutions of order k as $r \to \infty$. This is accomplished by employing the central limit theorem on the differences $T_r^{(+)} - T$, $T_r^{(-)} - T$, which can be approximated by proper Normal distribution. Note that

$$\begin{aligned} E[T_r^{(+)} - T] &= (r-1)E[W_j^*] = (r-1)\dot{G}^*(1), \\ &= (r-1)\frac{(1 - p_{11}^{k-\ell})(p_{10} + p_{01})}{p_{10}p_{01}p_{11}^{k-1}}, \\ E[T_r^{(-)} - T] &= (r-1)E[T_j^{**}] = (r-1)\dot{G}^{**}(1), \end{aligned}$$

$$= (r-1)\left[\dot{p}_{10}^{(|\ell|)} \cdot \frac{p_{10}+p_{01}-p_{01}p_{11}^{k-1}}{p_{10}p_{01}p_{11}^{k-1}} + \dot{p}_{11}^{(|\ell|)} \cdot \frac{(p_{10}+p_{01})(1-p_{11}^{k})}{p_{10}p_{01}p_{11}^{k-1}} + |\ell|\right],$$

where, "·" means the differentiation. Similarly, by making use of the derivatives of the probability generating functions up to the second order, the variances can be obtained. However, we omit them, since the expressions are rather cumbersome. Note also that the numerical evaluation of the distribution of T is acquired by the expansion of (2.7), which nowadays can be easily achieved by computer algebra systems.

3. The asymptotic behavior

In this section, the limiting behavior of the distributions treated in Section 2 as $r \to \infty$ are considered. We shall discuss the relationships between binomial distributions of order k and extended Poisson distributions of order k more generally.

Following Aki (1985) (see also Aki *et al.* (1984)), the probability generating function of extended Poisson distributions of order k with parameters $\lambda_1, \lambda_2, \dots, \lambda_k$ is,

$$\psi(z; \lambda_1, \lambda_2, \dots, \lambda_k) = \exp\left\{-\sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \lambda_i z^i\right\},$$

the probability function is,

$$g(n; \lambda_1, \lambda_2, \dots, \lambda_k) = \sum_{y_1+2y_2+\cdots+ky_k=n} \exp\left\{-\sum_{i=1}^{k} \lambda_i\right\} \frac{\prod_{j=1}^{k} \lambda_i^{y_j}}{\prod_{j=1}^{k} y_j!}.$$

3.1 *Case* $0 < \ell \le k-1$

THEOREM 3.1. *If* $\lim_{r\to\infty} rp_{00} = \lambda > 0$ *and* $\lim_{r\to\infty} rp_{10} = \mu > 0$, *then the asymptotic distribution of* $T_r^{(+)} - rk + \ell(r-1) + 1$ *is a mixture of an extended Poisson distribution of order k with parameters*

$$\lambda_i = \begin{cases} 0, & \text{if } 1 \le i \le \ell, \\ \mu, & \text{if } \ell+1 \le i \le k, \end{cases} \tag{3.1}$$

and a shifted duplicate of it, the mixing parameters being p_1 *and* p_0.

PROOF. Evidently,

$$\lim_{r\to\infty} P(z) = p_1 + p_0 z, \qquad \lim_{r\to\infty} Q(z) = 1,$$

$$\lim_{r\to\infty} R_{k-\ell-1}(p_{11}z) = \sum_{i=0}^{k-\ell-1} z^i,$$

and therefore,

$$\lim_{r\to\infty}\frac{H(z)}{z^{k-1}}=\lim_{r\to\infty}\frac{p_{11}^{k-1}P(z)}{Q(z)}=p_1+p_0z,\qquad \lim_{r\to\infty}H^*(z)=z^k,$$
$$\lim_{r\to\infty}r\frac{p_{10}zH^*(z)R_{k-\ell-1}(p_{11}z)}{(p_{11}z)^{k-\ell}}=\mu\sum_{i=\ell+1}^{k}z^i.$$

By virtue of (2.9), the probability generating function of the shifted random variable $T_r^{(+)}-rk+\ell(r-1)+1$ can be expressed as

$$z^{-rk+\ell(r-1)+1}H_r^{(+)}(z)=\frac{H(z)}{z^{k-1}}\cdot(p_{11})^{(k-\ell)(r-1)}\cdot\left[1+\frac{p_{10}zH^*(z)R_{k-\ell-1}(p_{11}z)}{(p_{11}z)^{k-\ell}}\right]^{r-1},$$

and taking the limit as $r\to\infty$, we get

$$\begin{aligned}\lim_{r\to\infty}z^{-rk+\ell(r-1)+1}H_r^{(+)}(z)&=(p_1+p_0z)\exp\left\{-\mu\,(k-\ell)+\mu\sum_{i=\ell+1}^{k}z^i\right\},\\&=(p_1+p_0z)\psi(z;0,\ldots,0,\mu,\ldots,\mu).\end{aligned}$$

This completes the proof. □

Theorem 3.1 states that the random variable $T_r^{(+)}-rk+\ell(r-1)+1$ converges in law to a mixture of an extended Poisson distribution of order k with the parameters given by (3.1).

$$\begin{aligned}&\lim_{r\to\infty}P(T_r^{(+)}-rk+\ell(r-1)+1=n)\\&=(p_1+p_0z)g(n;0,\ldots,0,\mu,\ldots,\mu),\\&=p_1g(n;0,\ldots,0,\mu,\ldots,\mu)+p_0g(n-1;0,\ldots,0,\mu,\ldots,\mu).\end{aligned}$$

3.2 *Case $\ell\le 0$*

THEOREM 3.2. *If $\lim_{r\to\infty}rp_{00}=\lambda>0$ and $\lim_{r\to\infty}rp_{10}=\mu>0$, then the asymptotic distribution of $T_r^{(-)}-rk-|\ell|(r-1)+1$ is a mixture of an extended Poisson distribution of order k with parameters*

$$\lambda_i=\mu,\qquad 1\le i\le k, \tag{3.2}$$

and a shifted duplicate of it, the mixing parameters being p_1 and p_0.

PROOF. By virtue of (2.12), the probability generating function of the shifted random variable $T_r^{(-)}-rk-|\ell|(r-1)+1$ can be expressed as

$$\begin{aligned}&z^{-rk-|\ell|(r-1)+1}H_r^{(-)}(z)\\&=\frac{zH(z)}{H^{**}(z)}\cdot\left(p_{11}^{(|\ell|)}\right)^{r-1}\cdot\left(\frac{H^{**}(z)}{z^k}\right)^r\cdot\left(1+\frac{p_{10}^{(|\ell|)}H^*(z)}{p_{11}^{(|\ell|)}H^{**}(z)}\right)^{r-1}.\end{aligned}\tag{3.3}$$

We investigate the limiting expression of (3.3) as $r \to \infty$. First, we consider the case of $\ell = 0$. Then, the equation (3.3) reduces

$$z^{-rk+1} H_r^{(-)}(z) = \frac{zH(z)}{H^{**}(z)} \cdot \left(\frac{H^{**}(z)}{z^k}\right)^r.$$

Evidently

$$\lim_{r \to \infty} r(1 - Q(z)) = \exp\left\{\lambda z + \mu \sum_{i=2}^{k} z^i\right\},$$

therefore,

$$\begin{aligned} \lim_{r\to\infty} [z^{-k} H^{**}(z)]^r &= \lim_{r\to\infty} \frac{\left(1 + \frac{p_{01}p_{10}z}{p_{11}} - p_{00}z\right)^r [(1-p_{10})^r]^k}{Q^r(z)} \\ &= \exp\left\{-\mu k + \mu \sum_{i=1}^{k} z^i\right\}. \end{aligned}$$

From

$$\lim_{r\to\infty} \frac{zH(z)}{H^{**}(z)} = \lim_{r\to\infty} \frac{P(z)}{p_{01}p_{10}z + p_{11} - p_{11}p_{00}z} = p_1 + p_0 z,$$

we have the limiting expression

$$\begin{aligned} \lim_{r\to\infty} z^{-rk+1} H_r^{(-)}(z) &= (p_1 + p_0 z) \exp\left\{-\mu k + \mu \sum_{i=1}^{k} z^i\right\}, \\ &= (p_1 + p_0 z)\psi(z; \mu, \mu, \ldots, \mu). \end{aligned}$$

Next, we consider the case of $|\ell| \geq 1$. It is easy to check that $\lim_{r\to\infty} r p_{10}^{(|\ell|)} = \mu$ by induction with respect to $|\ell|$ ($|\ell| \geq 1$), and $\lim_{r\to\infty} H^*(z)/H^{**}(z) = 1$. Then

$$\lim_{r\to\infty} \left(1 + \frac{p_{10}^{(|\ell|)} H^*(z)}{p_{11}^{(|\ell|)} H^{**}(z)}\right)^{r-1} = e^{\mu}.$$

Therefore we have the limiting expression for $|\ell| \geq 1$

$$\begin{aligned} \lim_{r\to\infty} z^{-rk-|\ell|(r-1)+1} H_r^{(-)}(z) &= (p_1 + p_0 z) \exp\left\{-\mu k + \mu \sum_{i=1}^{k} z^i\right\}, \\ &= (p_1 + p_0 z)\psi(z; \mu, \mu, \ldots, \mu). \end{aligned}$$

The proof is completed. □

Theorem 3.2 states that the random variable $T_r^{(-)} - rk - |\ell|(r-1) + 1$ converges in law to a mixture of an extended Poisson distribution of order k with the parameters given by (3.2).

$$\begin{aligned} &\lim_{r\to\infty} P(T_r^{(-)} - rk - |\ell|(r-1) + 1 = n) \\ &\quad = (p_1 + p_0 z) g(n; \mu, \mu, \ldots, \mu), \\ &\quad = p_1 g(n; \mu, \mu, \ldots, \mu) + p_0 g(n-1; \mu, \mu, \ldots, \mu). \end{aligned}$$

Notice that the p.g.f. (2.12) for ℓ is different from the respective p.g.f. obtained by Koutras (1997) and the asymptotic result in Theorem 3.2 for $\ell = 0$ also differs from the respective one established by Koutras (1997). However, the p.g.f. (2.12) for $\ell = 0$ corresponds to the expressions (9.41) and (9.42) in Balakrishnan and Koutras (2002).

Remark 3. In the case of $\ell = 0$, $\ell = k-1$, Koutras (1997) investigated the asymptotic behavior of the negative binomial distribution of order k. In the special case of iid Bernoulli trials, many authors studied the asymptotic behavior. For example, the case of $\ell = 0$ was treated by Philippou *et al.* (1983) (see Koutras (1996*b*)), and the case of $\ell = k-1$ was tackled by Hirano *et al.* (1991). Theorems 3.1 and 3.2 show the relationships between the negative binomial distributions of order k and the extended Poisson distributions of order k more generally. Thus, ℓ-overlapping enumeration scheme provides further insight into the relationships among the distributions of order k.

4. The number of occurrences of success-runs

In this section, we consider the distribution of the number of ℓ-overlapping occurrences of success-runs of length k in the first n trials (n a fixed integer). Though the problem can be treated in Markov dependent sequence as Section 2, we deal with iid case only for lack of space in this section. Assume that $p_{01} = p_{11} = p$, $p_{00} = p_{10} = q$, $p_1 = 0$ and $p_0 = 1$.

Let $X_{n,k}$ be the number of ℓ-overlapping occurrences of success-runs of length k in $X_1, X_2, \ldots, X_n$. The probability generating function and the double generating function of $X_{n,k}$ are denoted by $\phi_n(z)$ and $\Phi(z,w)$, respectively;

$$\phi_n(z) = E[z^{X_{n,k}}] = \sum_{x=0}^{\infty} \Pr[X_{n,k} = x] z^x, \qquad n \geq 0,$$

$$\Phi(z,w) = \sum_{n=0}^{\infty} \phi_n(z) w^n = \sum_{n=0}^{\infty}\sum_{x=0}^{\infty} \Pr[X_{n,k} = x] z^x w^n.$$

For the homogeneous case (i.e. $A_t(v) = A$, $B_t(v) = B$ for all $t \geq 1$ and $v \geq 0$), the double generating function is

$$\begin{aligned} \Phi(z,w) &= \boldsymbol{\pi}_0 \left(\sum_{n=0}^{\infty}\sum_{x=0}^{\infty} \boldsymbol{f}_n(x) z^x w^n \right) \mathbf{1}', \\ &= \boldsymbol{\pi}_0 [I - w(A + zB)]^{-1} \mathbf{1}', \end{aligned} \tag{4.1}$$

where, we denote $\mathbf{1} = (1, 1, \ldots, 1)$ by the row vector of R^s with all its entries being 1.

Each one of the two enumeration schemes ($\ell \leq 0$, $0 < \ell \leq k-1$) is treated separately. We use the superscript pointing out the enumeration scheme employed in a similar fashion as before.

4.1 *Case* $0 < \ell \leq k-1$

By setting $p_{01} = p_{11} = p$, $p_{00} = p_{10} = q$ in matrices A, B in Subsection 2.1, from the equation (4.1) with $\boldsymbol{\pi}_0 = (1, 0, \ldots, 0)$, we have

$$\boldsymbol{\pi}_0 [I - w(A + zB)]^{-1} = \left(\alpha, (pw)\alpha, \ldots, (pw)^{k-1}\alpha, z(pw)^k \beta, \ldots, z(pw)^{2k-\ell-1}\beta \right),$$

where,

$$\alpha = \frac{1 - z(pw)^{k-\ell}}{1 - qw\sum_{i=1}^{k}(pw)^{i-1} - z(pw)^{k-\ell} + zqw(pw)^{k-\ell}\sum_{i=1}^{\ell}(pw)^{i-1}},$$
$$\beta = \frac{1}{1 - qw\sum_{i=1}^{k}(pw)^{i-1} - z(pw)^{k-\ell} + zqw(pw)^{k-\ell}\sum_{i=1}^{\ell}(pw)^{i-1}}.$$

Therefore, the double generating function is

$$(4.2)\quad \Phi^{(+)}(z,w) = \frac{R_{k-\ell-1}(pw) + (1-z)(pw)^{k-\ell}R_{\ell-1}(pw)}{1 - qwR_{k-\ell-2}(pw) - (pw)^{k-\ell-1}(zpw + qw) - (1-z)qw(pw)^{k-\ell}R_{\ell-1}(pw)}.$$

Expanding (4.2) in a Taylor series around $w = 0$ and considering the coefficient of w^n, we obtain the probability generating function of $X_{n,k}^{(+)}$, $\phi_n^{(+)}(z)$ say.

PROPOSITION 3. *The probability generating function of $X_{n,k}^{(+)}$ is written explicitly as*:

$$\begin{aligned}\phi_n^{(+)}(z) = &\sum_{m=0}^{k-\ell-1}\sum_{n_1+2n_2+\cdots+kn_k=n-m}\begin{bmatrix} n_1+n_2+\cdots+n_k \\ n_1,n_2,\ldots,n_k\end{bmatrix} p^n\left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \\ &\times\left(1+\frac{p}{q}z\right)^{n_{k-\ell}}(1-z)^{n_{k-\ell+1}+\cdots+n_k} \\ &+\sum_{m=k-\ell}^{k-1}\sum_{n_1+2n_2+\cdots+kn_k=n-m}\begin{bmatrix} n_1+n_2+\cdots+n_k \\ n_1,n_2,\ldots,n_k\end{bmatrix} p^n\left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \\ &\times\left(1+\frac{p}{q}z\right)^{n_{k-\ell}}(1-z)^{n_{k-\ell+1}+\cdots+n_k+1}.\end{aligned}$$

Remark 4. In the case of $\ell = k-1$, the probability generating function was obtained by Hirano *et al.* (1991).

4.2 *Case* $\ell \leq 0$

In the case of iid Bernoulli trials, with a slight modification of the partition in Subsection 2.2, we can treat the problem easier. To begin with, the case of $\ell < 0$ is examined. we consider the partition $C_v = \{c_{v,0}, c_{v,1}, \ldots, c_{v,k-1}, c_{v,\ell}, \ldots, c_{v,-1}\}$, $v = 0, 1, \ldots, [\frac{n+|\ell|}{k+|\ell|}]$, where,

$$c_{v,i} = \{(v,i)\},\qquad \ell \leq i \leq k-1,\quad v = 0,1,\ldots,\left[\frac{n+|\ell|}{k+|\ell|}\right],\quad s = |C_v| = k+|\ell|.$$

- Restarting state: (v,m), $v = 0,1,\ldots,[\frac{n+|\ell|}{k+|\ell|}]$, $m = 0,\ldots,k-1$. $Y_t = (v,m)$ means that there exist v success-runs of length k by ℓ-overlapping counting, and m trailing S after waiting state.
- Waiting state: $(v,\ell+i)$, $v = 0,1,\ldots,[\frac{n+|\ell|}{k+|\ell|}]$, $i = 0,\ldots,|\ell|-1$. $Y_t = (v,\ell+i)$ means that there exist v runs of success-run of length k by ℓ-overlapping counting, and the i trials pass after the occurrence of success-run of length k.

From the equation (4.1) with $\boldsymbol{\pi}_0 = (1, 0, \dots, 0)$, we have

$$\boldsymbol{\pi}_0[I - w(A + zB)]^{-1} = (\gamma, (pw)\gamma, \dots, (pw)^{k-1}\gamma, \delta, w\delta, \dots, w^{|\ell|-1}\delta),$$

where,

$$\gamma = \frac{1}{1 - qw\sum_{i=1}^{k}(pw)^{i-1} - zw^{|\ell|}(pw)^k}, \qquad \delta = \frac{z(pw)^k}{1 - qw\sum_{i=1}^{k}(pw)^{i-1} - zw^{|\ell|}(pw)^k}.$$

Therefore, the double generating function is

$$\Phi^{(-)}(z, w) = \frac{R_{k-1}(pw) + z(pw)^k R_{|\ell|-1}(w)}{1 - qwR_{k-1}(pw) - zw^{|\ell|}(pw)^k}. \tag{4.3}$$

For $\ell = 0$, recall that $R_{-1}(w) = 0$ from (2.5), the equation (4.3) reduces to

$$\Phi^{(-)}(z, w) = \frac{R_{k-1}(pw)}{1 - qwR_{k-1}(pw) - z(pw)^k}. \tag{4.4}$$

Clearly, the equation (4.4) corresponds to the double generating function in the case of non-overlapping enumeration scheme (see Koutras and Alexandrou (1995)). Therefore, the equation (4.3) holds for $\ell \le 0$.

For $\ell < 0$, expanding (4.3) in a Taylor series around $w = 0$ and considering the coefficient of w^n, we can obtain the probability generating function, $\xi_n(z)$ say,

$$\begin{aligned}
\xi_n(z) = {} & \sum_{m=0}^{k-1} \sum_{n_1+2n_2+\cdots+kn_k+(k+|\ell|)n_{k+|\ell|}=n-m} \begin{bmatrix} n_1 + n_2 + \cdots + n_k + n_{k+|\ell|} \\ n_1, n_2, \dots, n_k, n_{k+|\ell|} \end{bmatrix} \\
& \times p^n \left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \left(\frac{z}{p^{|\ell|}}\right)^{n_{k+|\ell|}} \\
& + \sum_{m=k}^{k+|\ell|-1} \sum_{n_1+2n_2+\cdots+kn_k+(k+|\ell|)n_{k+|\ell|}=n-m} \begin{bmatrix} n_1 + n_2 + \cdots + n_k + n_{k+|\ell|} \\ n_1, n_2, \dots, n_k, n_{k+|\ell|} \end{bmatrix} \\
& \times zp^{k+n-m} \left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \left(\frac{z}{p^{|\ell|}}\right)^{n_{k+|\ell|}}.
\end{aligned} \tag{4.5}$$

Similarly, for $\ell = 0$, we can obtain the probability generating function, $\varphi_n(z)$ say,

$$\begin{aligned}
\varphi_n(z) = {} & \sum_{m=0}^{k-1} \sum_{n_1+2n_2+\cdots+kn_k=n-m} \begin{bmatrix} n_1 + n_2 + \cdots + n_k \\ n_1, n_2, \dots, n_k \end{bmatrix} \\
& \times p^n \left(\frac{q}{p}\right)^{n_1+\cdots+n_k} \left(1 + \frac{p}{q}z\right)^{n_k}.
\end{aligned} \tag{4.6}$$

Combining the equations (4.5) and (4.6), we can obtain the probability generating function of $X_{n,k}^{(-)}$, $\phi_n^{(-)}(z)$ say.

PROPOSITION 4. *For $\ell \le 0$, the probability generating function of $X_{n,k}^{(-)}$ is written as:*

$$\phi_n^{(-)}(z) = \delta(\ell)\varphi_n(z) + (1 - \delta(\ell))\xi_n(z),$$

where

$$\delta(\ell) = \begin{cases} 1, & \textit{if} \ \ \ell = 0, \\ 0, & \textit{otherwise}, \end{cases}$$

and where $\xi_n(z)$, $\varphi_n(z)$ *are as in* (4.5), (4.6) *respectively.*

Propositions 3 and 4 may be rather complex in that inner sum is subject to the condition. However, we think that they are very helpful for explaining the combinatorial meanings. Han and Aki (2000) studied the distributions of the number of ℓ-overlapping occurrences of success-runs of length k in the first n trials (n a fixed integer). They used a different technique: the method of conditional generating functions.

Acknowledgements

We wish to thank the editor and the referees for careful reading of our paper and helpful suggestions which led improved results.

References

Aki, S. (1985). Discrete distributions of order k on a binary sequence, *Ann. Inst. Statist. Math.*, **37**, 205–224.

Aki, S. and Hirano, K. (1988). Some characteristics of the binomial distribution of order k and related distributions, *Statistical Theory and Data Analysis II*, *Proceedings of the Second Pacific Area Statistical Conference* (eds. K. Matusita), 211–222, North-Holland, Amsterdam.

Aki, S. and Hirano, K. (1994). Distributions of numbers of failures and successes until the first consecutive k successes, *Ann. Inst. Statist. Math.*, **46**, 193–202.

Aki, S. and Hirano, K. (1995). Joint distributions of numbers of success- runs and failures until the first consecutive k successes, *Ann. Inst. Statist. Math.*, **47**, 225–235.

Aki, S. and Hirano, K. (2000). Numbers of success-runs of specified length until certain stopping time rules and generalized binomial distributions of order k, *Ann. Inst. Statist. Math.*, **52**, 767–777.

Aki, S., Kuboki, H. and Hirano, K. (1984). On discrete distributions of order k, *Ann. Inst. Statist. Math.*, **36**, 431–440.

Antzoulakos, D. L. (1999). On waiting time problems associated with runs in Markov dependent trials, *Ann. Inst. Statist. Math.*, **51**, 323–330.

Balakrishnan, N. and Koutras, M. V. (2002). *Runs and Scans with Applications*, Wiley, New York.

Chadjiconstantinidis, S., Antzoulakos, D. L. and Koutras, M. V. (2000). Joint distributions of successes, failures and patterns in enumeration problems, *Adv. in Appl. Probab.*, **32**, 866–884.

Chao, M. T. (1991). The reliability of large series systems under Markov structure, *Adv. in Appl. Probab.*, **23**, 894–908.

Chao, M. T. and Fu, J. C. (1989). A limit theorem of certain repairable systems, *Ann. Inst. Statist. Math.*, **41**, 809–818.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd ed., Wiley, New York.

Fu, J. C. (1986). Reliability of consecutive-k-out-of-n: F systems with $(k-1)$-step Markov dependence, *IEEE Transactions on Reliability*, **R35**, 602–606.

Fu, J. C. and Hu, B. (1987). On reliability of a large consecutive-k-out-of-n: F system with $(k-1)$-step Markov dependence, *IEEE Transactions on Reliability*, **R36**, 75–77.

Fu, J. C. and Koutras, M. V. (1994). Distribution theory of runs : A Markov chain approach, *J. Amer. Statist. Assoc.*, **89**, 1050–1058.

Han, Q. and Aki, S. (1998). Formulae and recursions for the joint distributions of success runs of several lengths in a two-state Markov chain, *Statist. Probab. Lett.*, **40**, 203–214.

Han, S. and Aki, S. (2000). A unified approach to binomial-type distributions of order k, *Comm. Statist. Theory Methods*, **29**(8), 1929–1943.

Hirano, K. and Aki, S. (1987). Properties of the extended distributions of order k, *Statist. Probab. Lett.*, **6**, 67–69.

Hirano, K., Aki, S., Kashiwagi, N. and Kuboki, H. (1991). On Ling's binomial and negative binomial distributions of order k, *Statist. Probab. Lett.*, **11**, 503–509.

Koutras, M. V. (1996*a*). On a Markov chain approach for the study of reliability structures, *J. Appl. Probab.*, **33**, 357–367.

Koutras, M. V. (1996*b*). On a waiting time distribution in a sequence of Bernoulli trials, *Ann. Inst. Statist. Math.*, **48**, 789–806.

Koutras, M. V. (1997). Waiting time distributions associated with runs of fixed length in two- state Markov chain, *Ann. Inst. Statist. Math.*, **49**, 123–139.

Koutras, M. V. and Alexandrou, V. A. (1995). Runs, scans and urn model distributions: A unified Markov chain approach, *Ann. Inst. Statist. Math.*, **47**, 743–766.

Koutras, M. V. and Alexandrou, V. A. (1997). Sooner waiting time problems in a sequence of trinary trials, *J. Appl. Probab.*, **34**, 593–609.

Koutras, M. V., Papadopoulos, G. K. and Papastavridis, S. G. (1995). Runs on a circle, *J. Appl. Probab.*, **32**, 396–404.

Ling, K. D. (1988). On binomial distributions of order k, *Statist. Probab. Lett.*, **6**, 247–250.

Philippou, A. N. (1988). On multiparameter distributions of order k, *Ann. Inst. Statist. Math.*, **40**, 467–475.

Philippou, A. N., Georghiou, C. and Philippou, G. N. (1983). A generalized geometric distribution and some of its properties, *Statist. Probab. Lett.*, **1**, 171–175.

Uchida, M. (1998). On number of occurrences of success runs of specified length in a higher-order two state Markov chain, *Ann. Inst. Statist. Math.*, **50**, 587–601.

Zhang, F. Z. (1999). *Matrix Theory*, Springer, New York.