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DENSITY ESTIMATION FOR A CLASS OF STATIONARY NONLINEAR PROCESSES

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Abstract. Let $\{X_t; t \in \mathbb{Z}\}$ be a strictly stationary nonlinear process of the form $X_t = \varepsilon_t + \sum_{r=1}^{\infty} W_{rt}$, where W_{rt} can be written as a function $g_r(\varepsilon_{t-1}, \dots, \varepsilon_{t-r-q})$, $\{\varepsilon_t; t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed (*i.i.d.*) random variables with $E|\varepsilon_1|^g < \infty$ for some $\gamma > 0$ and $q \geq 0$ is a fixed integer. Under certain mild regularity conditions on g_r and $\{\varepsilon_t\}$ we then show that X_1 has a density function f and that the standard kernel type estimator $\widehat{f}_n(x)$ based on a realization $\{X_1, \dots, X_n\}$ from $\{X_t\}$ is, asymptotically, normal and converges a.s. to $f(x)$ as $n \to \infty$.

Key words and phrases: Nonlinear process, kernel type density estimators, bilinear process, central limit theorem, almost sure convergence.

1. Introduction

Let $X_1, \dots, X_n$ be a set of identically distributed random variables (r.v.) with a common distribution function (d.f.) F and let us assume that F admits a probability density (p.d.) f at some point x. If $f(x)$ is not known, it can be estimated by using kernel type density estimators $\widehat{f}_n$. Several important properties of these estimators have been discussed in Devroy (1987), Ibragimov and Khasminskii (1982), Parzen (1962), Rosenblatt (1956, 1971), and Prakasa Rao (1983) among others for the case where the r.v.'s are mutually independent. Some attempts have been made to extend these results to sequences of dependent variables—as examples, we may mention the works which appear in Ahmad (1977), Bradley (1983), Delacroix (1977), Györfi *et al.* (1989), Masry (1983), Robinson (1983), and Rosenblatt (1970). Chanda (1983, 1995) Hall and Hart (1990), Hallin and Tran (1996), and Tran (1992) have established asymptotic normality and strong consistency for $\hat{f}_n$ in the case where the underlying process is linear.

The aim of the present article is to extend these results for the X_t's when they form a realization from a class of nonlinear processes which can be decomposed as follows,

$$X_t = \varepsilon_t + \sum_{r=1}^{\infty} W_{rt}, \tag{1.1}$$

where $\{\varepsilon_t; t \in \mathbb{Z}\}$ is an innovation process consisting of *i.i.d.* r.v.'s, Wrt can be written as a function $g_r(\varepsilon_{t-1}, \dots, \varepsilon_{t-r-q})$ where q is a fixed integer ≥ 0, and the convergence

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of the infinite sum on the right side of (1.1) is in some probabilistic sense. As we shall see later on that the ARMA and (most) of the bilinear and Volterra processes belong to the nonlinear type (1.1). The primary use of density estimation is possibly its application to discriminate analysis as applied to stationary processes. In such cases, it will be interesting if these estimates behave in a manner similar to those based on *i.i.d.* observations.

Although the present analysis deals with the one dimensional p.d. f, one can routinely (albeit with more complicated technical details) extend these details to the estimation of higher dimensional p.d.'s.

2. Probability density estimator and its asymptotic distribution

We define the kernel estimator $\widehat{f}_n(x)$ by

$$\widehat{f}_n(x) = n^{-1}\sum_{t=1}^{n}\phi(x - X_t; r_n), \tag{2.1}$$

where $\{r_n\}$ is a sequence of real numbers such that $r_n \to 0$, but $nr_n \to \infty$ as $n \to \infty$, $\phi(y; r_n) = r_n^{-1}\phi(y/r_n)(-\infty < y < \infty)$, and ϕ is a nonnegative Borel function. Assume the following conditions to hold.

(A) (i) For every real y, $\phi(y) \le M$ where M here and elsewhere in this article is used as a generic symbol which denotes a finite positive constant independent of n, but may vary from situation to situation, (ii) $\int_{-\infty}^{\infty}\phi(y)dy < \infty$, (iii) $y\phi(y) \to 0$ as $y \to \pm\infty$, and (iv) for every real a, $\int_{-\infty}^{\infty}|\phi(y+a) - \phi(y)|dy \le M|a|$.

(B) If φ_ε denotes the characteristic function (ch.f.) of ε_1 then

$$\int_{-\infty}^{\infty}|u|^s|\varphi_\varepsilon(u)|du < \infty \qquad (s = 0, 1).$$

(C) $E|Wrt|^\gamma \le Mh_r^\gamma$ for some $\gamma > 0$ and some $h_r > 0$ $(1 \le r < \infty)$ such that if we set $H_r := (\sum_{s=r}^{\infty}h_s^\gamma)^{1/\gamma}$ whenever $0 < \gamma \le 1$ and $H_r := \sum_{s=r}^{\infty}h_s$ $(r \ge 1)$ if $\gamma \ge 1$, then $\sum_{r=v}^{\infty}H_r^{\gamma/(2+\gamma)} = O(v^{-1})$ as $v \to \infty$.

Our main purpose in this section is to prove the following

THEOREM 2.1. *Let conditions* (A), (B), (C) *and relation* (1.1) *hold, with* $\{r_n\}$ *chosen as above. Then* $f(x) \le M$ *for every real* x *and as* $n \to \infty$

$$\mathcal{L}((nr_n)^{1/2}(\widehat{f}_n - f_n)) \to \mathcal{N}(0, \sigma^2), \tag{2.2}$$

where $\widehat{f}_n = \widehat{f}_n(x)$, $f_n = f_n(x) = E(\phi(x - X_1; r_n))$ *and* $\sigma^2 = f(x)\int_{-\infty}^{\infty}\phi^2(y)dy$.

The interesting aspect of (1.1) is that the entire class of ARMA processes and most of the bilinear and Volterra processes belong to the type (1.1), and condition (C) is not hard to check in any of these situations. We consider below some specific cases.

Example 1. Let $\{X_t\}$ be a linear process defined by

$$X_t = \sum_{r=0}^{\infty} g_r\varepsilon_{t-r} \tag{2.3}$$

where $\{\varepsilon_t\}$ is a sequence of *i.i.d.* random variables (r.v.'s), with $E|\varepsilon_1|^\gamma < \infty$ for some $\gamma > 0$ and $\{g_r\}$ is a sequence of real numbers such that $\sum_{r=v}^{\infty} r|g_r|^{\gamma/(2+\gamma)} = O(v^{-1})$ as $v \to \infty$. We can then set $h_r = |g_r| \geq 0$, $(g_0 = 1)$ and condition (C) will hold. If the process (2.3) is an ARMA process then $|g_r| < M\rho^r$ for some $\rho \in (0,1)$ and condition (C) will easily hold.

Example 2. Let $\{X_t\}$ be a bilinear process defined by

$$X_t = \varepsilon_t + \sum_{j=1}^{s} \theta_j \varepsilon_{t-j} + \sum_{i=1}^{\ell} \phi_i X_{t-i} + \sum_{i=1}^{P}\sum_{j=1}^{Q} \beta_{ij} X_{t-i}\varepsilon_{t-j}, \tag{2.4}$$

where $\phi_1, \ldots, \phi_\ell$, $\theta_1, \ldots, \theta_s$ and β_{ij} $(1 \leq i \leq P,\ 1 \leq j \leq Q)$ are unknown parameters and ℓ, P, Q, s are known integers ≥ 0. Set $p = \max(\ell, P)$, $q = \max(s, Q)$.

We can formally write (2.4) as an infinite sum

$$\boldsymbol{X}_t = \boldsymbol{\Theta}\boldsymbol{\varepsilon}_t + \sum_{r=1}^{\infty} \boldsymbol{V}_{rt} \tag{2.5}$$

where $\boldsymbol{V}_{rt} = \boldsymbol{\Pi}_{u=0}^{r-1}(\boldsymbol{A} + \sum_{0=1}^{q} \boldsymbol{B}_j \varepsilon_{t-u-j})\boldsymbol{\Theta}\boldsymbol{\varepsilon}_{t-r}$,

$$\boldsymbol{\eta}{=}[1,0,\ldots,0]^T, \quad \underset{p\times p}{\boldsymbol{A}}= \begin{bmatrix} -\phi_1 & -\phi_2 & \cdots & -\phi_{p-1} & -\phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\underset{p\times p}{\boldsymbol{B}_j}= \begin{pmatrix} \beta_{1j} & \beta_{2j} & \cdots & \beta_{pj} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 1 \leq j \leq q, \qquad \underset{p\times(q+1)}{\boldsymbol{\Theta}}= \begin{pmatrix} 1 & \theta_1 & \cdots & \theta_q \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$\boldsymbol{\varepsilon}_t = (\varepsilon_t, \ldots, \varepsilon_{t-q})^T$, and $\boldsymbol{X}_t = (\boldsymbol{X}_t, \ldots, \boldsymbol{X}_{t-p+1})^T$. It is understood that $\phi_i = 0$ if $i > \ell$, $\theta_j = 0$, if $j > q$ and $\beta_{ij} = 0$ whenever $i > P$ or $j > Q$. It is easy to see then that X_t as defined in (2.4) can be represented by

$$X_t = \varepsilon_t + \sum_{r=1}^{\infty} W_{rt} \tag{2.6}$$

where $W_{1t} = \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \boldsymbol{\eta}_t^T \boldsymbol{V}_{1t}$ and $W_{rt} = \boldsymbol{\eta}^t \boldsymbol{V}_{rt}$ $(r \geq 2)$. The conditions under which the right side of (2.6) is a.s. convergent are discussed in Chanda (1991) and Liu and Brockwell (1988). Note that W_{rt} as defined in (2.6) is a function of $\varepsilon_{t-1}, \ldots, \varepsilon_{t-r-q}$. Also if we assume that $\nu_{2kq} := E|\varepsilon_1|^{2kq} < \infty$, and $\lambda := \rho_0 + (\sum_{j=1}^{q} \delta_j)\nu_{2kq}^{1/2kq} < 1$ $(\rho_0, \delta_1, \ldots, \delta_q$ have been defined in (2.4) (Chanda (1991))), then $E|W_{rt}|^k \leq M\lambda^{rk}$ which immediately establishes the validity of condition (C) with $h_r = \lambda^r$.

In passing, we note that under certain regularity conditions the process represented by (2.6) satisfies the Markovian property and is strongly mixing (see Pham (1986)).

Example 3. Let $\{X_t\}$ be a Volterra process defined by

$$X_t = \varepsilon_t + \sum_{r=1}^{\infty} g_r \varepsilon_{t-r} + \sum_{r_1,r_2=1}^{\infty} g_{r_1,r_2}\varepsilon_{t-r_1}\varepsilon_{t-r_2} + \cdots + \sum_{r_1,\ldots,r_q=1}^{\infty} g_{r_1,r_2,\ldots,r_q}\varepsilon_{t-r_1}\varepsilon_{t-r_2}\cdots\varepsilon_{t-r_q}$$

where $\{\varepsilon_t\}$ is an *i.i.d.* sequence of r.v.'s with $E|\varepsilon_1|^{q\gamma} < \infty$. Then we can write $h_r^\delta = \sum_{j=1}^q \sum_{S_{rj}} |g_{u_1,\ldots,u_j}|^\delta (r \geq 1)$, where $S_{rj} = \{(r_1,\ldots,r_j) : 1 \leq r_1,\ldots,r_j < \infty, \max(r_1,\ldots,r_j) = r\}$, and $\delta = \gamma$ if $\gamma \leq 1$ and $\delta = 1$ if $\gamma \geq 1$.

Then it is easy to see that representation (1.1) obtains and condition (C) holds with $Wrt = \sum_{j=1}^q \sum_{S_{rj}} g_{u_1,\ldots,u_j}\varepsilon_{t-u_1}\ldots\varepsilon_{t-u_j}$, $r \geq 1$.

The proof of Theorem 2.1 will rest on a few lemmas which we state and prove below. We first set

$$Y_t := r_n^{1/2}\tau_t \qquad \tau_t := \phi_t - f_n \tag{2.7}$$

where $\phi_t := \phi(x - X_t; r_n)$. Note that both Y_t and τ_t will depend on n.

LEMMA 2.1. *Let the conditions of Theorem* 2.1 *hold. Then*

$$\sum_{v=1}^{\infty} |E(Y_1 Y_{1+v})| \leq M r_n^{\gamma/(1+\gamma)}. \tag{2.8}$$

PROOF. Observe that for any integer $v \geq 1$ and integers $a, b \geq 1$ $|E(\tau_1^a \tau_{1+v}^b)| \leq M(E(\phi_1^a\phi_{1+v}^b) + E(\phi_1^a) + E(\phi_{1+v}^b) + 1)$. Also, if we set Ω_t = σ-algebra generated by ε_s, $s \leq t$ and write $T_t = X_t - \varepsilon_t$ (so that T_t is independent of ε_t) then $E(\phi_{1+v}^b|\Omega_v) = \int \phi^b(x - y - T_{1+v}; r_n) f_\varepsilon(y)dy = r_n^{-b+1}\int \phi^b(w) f_\varepsilon(x - T_{1+v} - r_n w)dw \leq M r_n^{-b+1}$ where f_ε is the p.d. of ε_1 (because $f_\varepsilon(y) \leq M$ by condition (B)). Therefore, $|E(\tau_1^a\tau_{1+v}^b)| \leq M r_n^{-b+1} E\phi_1^a \leq M r_n^{2-a-b}$ and hence

$$|E(Y_1^a Y_{1+v}^b)| \leq M r_n^{2-(a+b)/2}, \tag{2.9}$$

from which we conclude that for every $v \geq 1$

$$|E(Y_1 Y_{1+v})| \leq M r_n. \tag{2.10}$$

Now set $R_t = \varepsilon_t + \sum_{r=1}^{t-q-2} W_{rt}$, $S_t = X_t - R_t$, for $t \geq q+3$, where q is as defined in (1.1). Since R_{1+v} involves $\varepsilon_{1+v}, \varepsilon_v, \ldots, \varepsilon_2$. Whenever $v \geq q+2$, it is distributed independently of X_1 if $v \geq q+2$. Consequently for the same values of v,

$$\begin{aligned} E(\tau_1\tau_{1+v}) &= E(\tau_1(\phi(x - X_{1+v}; r_n) - f_n)) \\ &= E(\tau_1(\phi(x - R_{1+v} - S_{1+v;}r_n) - f_n)) \\ &= E(\tau_1(\phi(x - R_{1+v} - S_{1+v}; r_n) - \phi(x - R_{1+v}; r_n))) \\ &\quad + E(\tau_1(\phi(x - R_{1+v}; r_n) - f_n)). \end{aligned} \tag{2.11}$$

Again since τ_1 involves X_1, X_1 is independent of R_{1+v} and $E(\tau_1) = 0$, the last term on the right side of (2.11) vanishes. Now write $J_0(r,s) := E(\phi(x-\varepsilon_{1+v}-r-s;r_n)-\phi(x-\varepsilon_{1+v}-r;r_n))$ for every real r and s and take expectation of the expression on the right side of (2.11) over ε_{1+v} (which is independent of X_1). It is then easy to conclude that

$$E(\tau_1\tau_{1+v}) = E(\tau_1 J_0(R^*_{1+v}, S_{1+v})). \tag{2.12}$$

Since for every real r and s, $|J_0(r,s)| = |\int \phi(w)(f_\varepsilon(x-r-s-r_n w)-f_\varepsilon(x-r-r_n w))dw| \leq M|s|$ and $\leq M$ (which follows from the fact that $|f'_\varepsilon(x)| \leq M$ for every real x by condition (B)) and $|\tau_1| \leq M r_n^{-1}$, it follows that for every choice of $\eta_n > 0$,

$$\begin{aligned} |E(\tau_1\tau_{1+v})| &= |E(\tau_1 J_0(R^*_{1+v}, S_{1+v}))| \\ &\leq M(\eta_n + r_n^{-1}Q_n), \end{aligned} \tag{2.13}$$

where $Q_n = P(|S_{1+v}| > \eta_n) \leq MH^{\gamma}_{v-g}/\eta_n^{\gamma}$ whenever $v \geq q+2$. Therefore, we conclude by choosing $\eta_n = r_n^{-1/(1+\gamma)}H_{v-q}^{\gamma/(1+\gamma)}$ in (2.12) that for every $v \geq q+2$,

$$|E(Y_1Y_{1+v})| \leq M r_n^{\gamma/(1+\gamma)} H_{v-q}^{\gamma/(1+\gamma)}. \tag{2.14}$$

If now we use (2.9) for $1 \leq v \leq q+1$, and the relation (2.13) for $v \geq q+2$ and note that since condition (C) implies that $\sum_{v=q+2}^{\infty} H_{v-q}^{\gamma/(1+\gamma)} < \infty$ (on account of the fact that $1+\gamma < 2+\gamma$), we immediately obtain (2.8).

LEMMA 2.2. *Let the conditions of Theorem* 2.1 *hold. Then $f(x)$ exists everywhere, $f(x) \leq M$, $|f'(x)| \leq M$ for every real x and the infinite sum on the right side of* (1.1) *converges a.s.*

PROOF. First note that if $\varphi(u)$ and $\varphi_1(u)$ denote respectively the ch.f. of X_1 and T_1 then $\varphi(u) = \varphi_\varepsilon(u)\varphi_1(u)$ and hence $\int_{-\infty}^{\infty}|u|^s|\varphi(u)|du \leq \int_{-\infty}^{\infty}|u|^s|\varphi_\varepsilon(u)|du < \infty$ $(s=0,1)$ by condition (B). Consequently, $|f^{(s)}(x)| \leq M$ for every real x, $s = 0,1$. Also for every $\eta > 0$

$$\begin{aligned} \sum_{m=N}^{\infty} P\left(\left|\sum_{r=m}^{\infty} W_{rt}\right| > \eta\right) &\leq \sum_{m=N}^{\infty} E\left|\sum_{r=m}^{\infty} W_{rt}\right|^{\gamma}\eta^{-\gamma} \\ &\leq M\sum_{m=N}^{\infty} H_m^{\gamma}\delta^{-\gamma} \to 0, \end{aligned} \tag{2.15}$$

as $N \to \infty$, by definition of H_m in condition (C) and the fact that $1/(2+\gamma) < 1$. The second part of the Lemma follows immediately.

It is easy to see that for every integer $m \geq 1$

$$
\begin{aligned}
(2.16)\quad E\left(\sum_{t=1}^{m} Y_t\right)^4 \leq M\Big(mE(Y_1^4) &+ m\sum_{v=1}^{m}[E(Y_1^2Y_{1+v}^2) + |E(Y_1^3Y_{1+v})| + |E(Y_1Y_{1+v}^3)|] \\
&+ m\sum_{v=1}^{m}\sum_{w=1}^{m}[|E(Y_1^2Y_{1+v}Y_{1+v+w})| \\
&\qquad + |E(Y_1Y_{1+v}^2Y_{1+v+w})| + |E(Y_1Y_{1+v}Y_{1+v+w}^2)|] \\
&+ m\sum_{v=1}^{m}\sum_{w=1}^{m}\sum_{y=1}^{m}|E(Y_1Y_{1+v}Y_{1+v+w}Y_{1+v+w+y})|.
\end{aligned}
$$

The results of the following lemmas can now be used to determine an upper bound to the sum of the expressions on the right side of (2.16).

LEMMA 2.3. *For every* $v \geq 1$

$$(2.17)\qquad |E(Y_1^aY_{1+v}^b)| \leq M,$$

whenever $a + b = 4$ $(a \geq 1,\ b \geq 1)$.

PROOF. The result follows directly from (2.9).

LEMMA 2.4. *For every* $v, w \geq 1$ *and* $(a,b) = (1,2)$ *or* $(a,b) = (2,1)$,

$$(2.18)\qquad |E(Y_1^aY_{1+v}^bY_{1+v+w})| \leq Mr_n.$$

PROOF. Let q be as defined in (1.1) and let (a,b) be as given in Lemma 2.5. If $w \geq q+2$, then using arguments similar to those leading to (2.11) we can show that

$$
\begin{aligned}
(2.19)\qquad E(\tau_1^a\tau_{1+v}^b\tau_{1+v+w}) &= E(\tau_1^a\tau_{1+v}^bJ_0(\widetilde{R}_{1+v+w}^*, \widetilde{S}_{1+v+w})) \\
&\quad + E(\tau_1^a\tau_{1+v}^b)E(\phi(x - \widetilde{R}_{1+v+w}; r_n) - f_n),
\end{aligned}
$$

where $\widetilde{R}_{1+v+w} = \varepsilon_{1+v+w} + \sum_{r=1}^{w-q-1} W_{r,1+v+w}$, $\widetilde{S}_{1+v+w} = X_{1+v+w} - \widetilde{R}_{1+v+w}$, $\widetilde{R}_{1+v+w}^* = \widetilde{R}_{1+v+w} - \varepsilon_{1+v+w}$ and J_0 is defined as in (2.12). (Note that $\widetilde{R}_{1+v+w}$ is independent of X_1 and X_{1+v}, and ε_{1+v+w} is independent of $\widetilde{R}_{1+v+w}^*$.) We now use Lemma 2.4 and details similar to those leading to (2.14), and conclude routinely that whenever $v \geq 1$, $w \geq q+2$ and $\eta_n^* > 0$ is arbitrary,

$$
\begin{aligned}
(2.20)\qquad |E(\tau_1^a\tau_{1+v}^b\tau_{1+v+w})| &\leq M(E(|\tau_1^a\tau_{1+v}^b|)\eta_n^* + E^{1/2}(\tau_1^{2a}\tau_{1+v}^{2b})Q_n^{*1/2}) \\
&\quad + |E(\tau_1^a\tau_{1+v}^b)|(\eta_n^* + Q_n^*) \\
&\leq M(r_n^{2-a-b}\eta_n^* + r_n^{1-a-b}Q_n^{*1/2} + r_n^{2-a-b}(\eta_n^* + Q_n^*)),
\end{aligned}
$$

where $Q_n^* = P(|\tilde{S}_{1+v+w}| > \eta_n^*) \le MH_{w-q}^{\gamma}/\eta_n^{*\gamma}$. Therefore, if $a = 1$, $b = 2$ or $a = 2$, $b = 1$ then by choosing $\eta_n^* = r_n^{-2/(2+\gamma)} H_{w-q}^{\gamma/(2+\gamma)}$ we have the inequality

$$|E(Y_1^a Y_{1+v}^b Y_{1+v+w})| \le M(r_n^{\gamma/(2+\gamma)} H_{w-q}^{\gamma/(2+\gamma)} + r_n^{(2+3\gamma)/(2+\gamma)} H_{w-q}^{2\gamma/(2+\gamma)}), \tag{2.21}$$

whenever $v \ge 1$ and $w \ge q+2$. For every $v \ge 1$ and $w \ge 1$ we can show, by using the fact

$$|E(\tau_{1+v+w}|\Omega_{v+w})| \le M(E(\phi_{1+v+w}|\Omega_{v+w}) + 1) \le M$$

where Ω_{v+w} is the σ-algebra generated by ε_t $(t \le v+w)$ that

$$|E(Y_1^a Y_{1+v}^b Y_{1+v+w})| \le Mr_n^{(5-a-b)/2} \le Mr_n, \tag{2.22}$$

if $(a,b) = (1,2)$ or (2.1). This proves Lemma 2.4.

LEMMA 2.5. *For every* $v, w \ge 1$

$$E(Y_1 Y_{1+v} Y_{1+v+w}^2) \le Mr_n. \tag{2.23}$$

PROOF. Let q be as defined in (1.1), and in what follows we shall assume that $v \ge 1$ and $w \ge q+2$. We can write

$$\begin{aligned} E(\tau_1\tau_{1+v}\tau_{1+v+w}^2) &= E(\tau_1\tau_{1+v}\phi_{1+v+w}^2) - 2f_n E(\tau_1\tau_{1+v}\phi_{1+v+w}) \\ &\quad + f_n^2 E(\tau_1\tau_{1+v}). \end{aligned} \tag{2.24}$$

Note that $\tilde{R}_{1+v+w}$ is independent of X_1 and X_{1+v}, and hence

$$\begin{aligned} E(\tau_1\tau_{1+v}\phi_{1+v+w}^2) &= E(\tau_1\tau_{1+v}(\phi^2(x - \tilde{R}_{1+v+w} - \tilde{S}_{1+v+w}; r_n) \\ &\quad - \phi^2(x - \tilde{R}_{1+v+w}; r_n))) + E(\tau_1\tau_v) \\ &\quad E(\phi^2(x - \tilde{R}_{1+v+w}; r_n)). \end{aligned} \tag{2.25}$$

It is not difficult to show that $E(\phi^2(x - \tilde{R}_{1+v+w}; r_n)|\Omega_{v+w}) \le Mr_n^{-1}$ where Ω_{v+w} is the σ-algebra generated by $\varepsilon_t(t \le v+w)$, and that $E((\phi^2(x - \tilde{R}_{1+v+w} - \tilde{S}_{1+v+w}; r_n) - \phi^2(x - \tilde{R}_{1+v+w}; r_n))|\Omega_{1+v+w}) = J_1(\tilde{R}_{1+v+w}^*, \tilde{S}_{1+v+w})$ where for every real r, s, $J_1(r,s) := E(\phi^2(x - \varepsilon_{1+v+w} - r - s; r_n) - \phi^2(x - \varepsilon_{1+v+w} - r; r_n))$. Since $|J_1(r,s)| \le Mr_n^{-1}|s|$ and also $\le Mr_n^{-1}$ for every r and s, and (2.9) holds, the absolute value of the first expression on the right side of (2.24) does not exceed

$$\begin{aligned} &E(|\tau_1\tau_{1+v}|r_n^{-1} I(|\tilde{S}_{1+v+w}| \le \eta_n^*)) + E(|\tau_1\tau_{1+v}|r_n^{-1} I(|\tilde{S}_{1+v+w}| \le \eta_n^*)) \\ &\quad \le E(|\tau_1\tau_{1+v}|)r_n^{-1}\eta_n^* + E^{1/2}(\tau_1^2\tau_{1+v}^2)r_n^{-1}E^{1/2}(I^2(\tilde{S}_{1+v+w}| > \eta_n^*)) \\ &\quad \le M(r_n^{-1}\eta_n^* + r_n^{-2}Q_n^{*1/2}) \end{aligned} \tag{2.26}$$

where η_n^* and Q_n^* are defined as in (2.20). If, now we use (2.13), (2.25) and (2.26) we can easily conclude that whenever $v \ge 1$, $w \ge q+2$

$$E(\tau_1\tau_{1+v}\phi_{1+v+w}^2) \le M(r_n^{-1}\eta_n + r_n^{-1}\eta_n^* + r_n^{-2}Q_n + r_n^{-2}Q_n^{*1/2}). \tag{2.27}$$

Similar analysis will show that if $v \geq 1$ and $w \geq q+2$

$$E(\tau_1 \tau_{1+v} \phi_{1+v+w}) \leq M(\eta_n + \eta_n^* + r_n^{-1} Q_n + r_n^{-1} Q_n^{*1/2}). \tag{2.28}$$

We now combine (2.13), (2.23), (2.27), (2.28), choose η_n and η_n^* as in (2.13) and (2.21) respectively, and note that $f_n \leq M$. The following inequality will then emerge.

$$|E(Y_1 Y_{1+v} Y_{1+v+w}^2)| \leq M(r_n^{\gamma/(1+\gamma)} H_{v-q}^{\gamma/(1+\gamma)} + r_n^{\gamma/(2+\gamma)} H_{w-q}^{\gamma/(2+\gamma)}), \tag{2.29}$$

whenever $v \geq 1$ and $w \geq q+2$. For any $v, w \geq 1$, we can use the relations (2.9), (2.21), the facts that $f_n \leq M$, and $E(\phi_{1+v+w}^a | \Omega_{v+w}) = r_n^{-a+1} \int \phi^a(z) f_\varepsilon(x - T_{1+v+w} - r_n z) dz \leq M r_n^{-a+1} (a = 1, 2)(T_t = X_t - \varepsilon_t)$, to conclude that (2.23) holds. This proves Lemma 2.5.

LEMMA 2.6. *For every $v, w, y \geq 1$*

$$|E(Y_1 Y_{1+v} Y_{1+v+w} Y_{1+v+w+y})| \leq M r_n^2. \tag{2.30}$$

PROOF. Let q be as defined as in (1.1) and assume that $v, w \geq 1$ and $y \geq q + 2$. Write $\bar{R}_{1+v+w+y} = \varepsilon_{1+v+w+y} + \sum_{r=1}^{y-q-1} W_{r,1+v+w+y}$, $\bar{S}_{1+v+w+y} = X_{1+v+w+y} - \bar{R}_{1+v+w+y}$. Since $\bar{R}_{1+v+w+y}$ is independent of X_1, X_{1+v} and X_{1+v+w},

$$\begin{aligned} E(\tau_1 \tau_{1+v} \tau_{1+v+w} \tau_{1+v+w+y}) &= E(\tau_1 \tau_{1+v} J_0(\bar{R}_{1+v+w+y}^*, \bar{S}_{1+v+w+y})) \\ &\quad + E(\tau_1 \tau_{1+v} \tau_{1+v+w}) E(\phi(x - R_{1+v+w+y}; r_n) - f_n), \end{aligned} \tag{2.31}$$

where $R_{1+v+w+y}^* = R_{1+v+w+y} - \varepsilon_{1+v+w+y}$. Using arguments similar to those leading to (2.19), (2.21) and (2.22) we eventually conclude that whenever $v, w \geq 1$, $y \geq q+2$ and $\eta_n^* > 0$ is arbitrary,

$$|E(Y_1 Y_{1+v} Y_{1+v+w} Y_{1+v+w+y})| \leq M(r_n^2 \eta_n^{**} + r_n^{1/2} Q_n^{**1/2} + r_n^2 Q_n^{**}) \tag{2.32}$$

where $Q_n^{**} = P(|\bar{S}_{1+v+w+y}| > \eta_n^{**})$, and $\bar{S}_{1+v+w+y} = \sum_{r=y-q}^{\infty} W_{r,1+v+w+y}$. We now apply the inequality. $Q_n^{**} \leq M H_{y-q}^{\gamma} / \eta_n^{**\gamma}$ followed by the choice $\eta_n^{**} = r_n^{-3/(2+\gamma)} H_{w-q}^{\gamma/(2+\gamma)}$ to (2.30) and conclude easily that

$$\begin{aligned} |E(Y_1 Y_{1+v} Y_{1+v+w} Y_{1+v+w+y})| \leq M(& r_n^{(1+2\gamma)/(2+\gamma)} H_{y-q}^{\gamma/(2+\gamma)} \\ &+ r_n^{(4+5\gamma)/(2+\gamma)} H_{y-q}^{2\gamma/(2+\gamma)}) \end{aligned} \tag{2.33}$$

whenever $v, w \geq 1$ and $y \geq q+2$. For every $v, w, y \geq 1$ we can use the inequalities $E(\phi_{1+v+w+y} | \Omega_{v+w+y}) \leq M$, and $E(|Y_1 Y_{1+v} Y_{1+v+w}|) \leq M r_n^2$ to conclude that for $v, w, y \geq 1$

$$|E(Y_1 Y_{1+v} Y_{1+v+w} Y_{1+v+w+y})| \leq M r_n^2. \tag{2.34}$$

This proves Lemma 2.6.

LEMMA 2.7. *Let the conditions of Theorem 2.1 hold and let $\{m_n\}$ be a sequence of positive integers such that $m_n \to \infty$ as $n \to \infty$. Then*

$$E\left(\sum_{t=1}^{m_n} Y_t\right)^4 \leq M(m_n r_n^{-1} + m_n^2 + m_n^3 r_n^{(1+2\gamma)/(2+\gamma)}). \tag{2.35}$$

PROOF. If we combine (2.16), (2.18), (2.23), (2.30), use the inequality $E(Y_1^4) \leq Mr_n^{-1}$, and condition (C) we obtain the relation

$$E\left(\sum_{t=1}^{m} Y_t\right)^4 \leq M(mr_n^{-1} + m^2 + m^2(r_n + r_n^{\gamma/(1+\gamma)} + r_n^{\gamma/(2+\gamma)})) + m^3(r_n^2 + r_n^{(1+2\gamma)/(2+\gamma)} + r_n^{(4+5\gamma)/(2+\gamma)}). \tag{2.36}$$

The result (2.35) of Lemma 2.7 follows immediately if we set $m = m_n$.

LEMMA 2.8. *Let $\{k_n\}$, $\{m_n\}$ and $\{t_n\}$ be sequences of positive integers such that $m_n \sim n^{1/2}r_n^{-1/8}$, $t_n \sim n^{1/2}r_n^{1/8}k_n = [n/(m_n + t_n)]$ and let conditions of Theorem 2.1 hold. Write*

$$\begin{aligned} U_j &= n^{-1/2}\sum_{t\in A_j} Y_t \quad 1 \leq j \leq k_n, \\ V_j &= n^{-1/2}\sum_{t\in B_j} Y_t \quad 1 \leq j \leq k_n, \\ W &= n^{-1/2}\sum_{t\in C} Y_t, \end{aligned} \tag{2.37}$$

where $A_j = \{\alpha_{j-1}+1, \ldots, \alpha_j - t_n\}$, $B_j = \{\alpha_j - t_n + 1, \ldots, \alpha_j\}$, $C = \{n - d_n + 1, \ldots, n\}$, $\alpha_j = j(m_n + t_n)$ and $d_n = n - k_n(m_n + t_n)$. Then as $n \to \infty$

$$\mathcal{L}\left(\sum_{j=1}^{k_n} U_j\right) \to \mathcal{N}(0, \sigma^2), \tag{2.38}$$

and

$$W + \sum_{j=1}^{k_n} V_j \to 0 \quad \textit{in probability,}$$

where σ^2 is as defined in (2.2).

PROOF. Let $\varphi^{(j)}$ denote the ch.f. of $U_1, \ldots, U_j$ and let φ_j be the ch.f. of U_j. Then we can write

$$\begin{aligned} \left|\varphi^{(k_n)}(u, \ldots, u) - \prod_{j=1}^{k_n}\varphi_j(u)\right| &\leq \sum_{j=2}^{k_n} |\varphi^{(j)}(u, \ldots, u) - \varphi_j(u)\varphi^{(j-1)}(u, \ldots, u)| \\ &= \sum_{j=2}^{k_n} |E(N_j P_j)|, \end{aligned} \tag{2.39}$$

where $N_j := \exp(iu\sum_{r=1}^{j-1} U_r) - \varphi^{(j-1)}(u, \ldots, u)$ and $P_j := \exp(iuU_j)$. For fixed j and n we set $g = \alpha_{j-1} - t_n$, $h = \alpha_{j-1} + 1$, $\ell = \alpha_j - t_n$ and write $\overset{\approx}{R}_t := \varepsilon_t + \sum_{r=1}^{t-g-q-1} W_{rt}$,

$\overset{\approx}{S}_t := X_t - \overset{\approx}{R}_t$ $(h \leq t \leq \ell)$. Note that $N_j \in \Omega_g$, whereas $\overset{\approx}{R}_t$ belongs to the σ-algebra generated by ε_s, $s \geq g+1$ whenever $h \leq t \leq \ell$.

Also, we can write $P_j = L_j M_j$ where

$$L_j := \exp\left[iun^{-1/2} r_n^{1/2} \sum_{t=h}^{\ell} (\phi(x - \overset{\approx}{R}_t; r_n) - f_n) \right], \qquad \text{and}$$

$$M_j := \exp\left[iun^{-1/2} r_n^{1/2} \sum_{t=h}^{\ell} (\phi(x - X_t; r_n) - \phi(x - \overset{\approx}{R}_t; r_n)) \right].$$

Since $\overset{\approx}{R}_t$ $(h \leq t \leq \ell)$ are distributed independently of N_j, L_j and N_j are mutually independent with $E(N_j) = 0$ and hence

$$\begin{aligned} |E(N_j P_j)| &= |E(N_j L_j M_j)| \\ &\leq |E(N_j L_j)| + |E(N_j L_j (M_j - 1))| \\ &\leq E|M_j - 1|, \end{aligned} \tag{2.40}$$

because $E(N_j L_j) = E(N_j) E(L_j) = 0$. Define $\delta = \gamma$ if $0 < \gamma \leq 1$ and $\delta = 1$ if $\gamma \geq 1$. Since $|\exp(i\alpha) - 1| \leq 2|\alpha/2|^\delta$ for every real α and δ, $0 < \delta \leq 1$, we have that

$$\begin{aligned} E|M_j - 1| &\leq M|u|^\delta (nr_n)^{-\delta/2} E \left| \sum_{t=h}^{\ell} (\phi((x - X_t)/r_n) - \phi((x - \overset{\approx}{R}_t)/r_n)) \right|^\delta \\ &\leq M|u|^\delta (nr_n)^{-\delta/2} \sum_{t=h}^{\ell} E|\phi((x - X_t)/r_n) - \phi((x - \overset{\approx}{R}_t)/r_n)|^\delta. \end{aligned} \tag{2.41}$$

Also since $X_t = \varepsilon_t + T_t = \varepsilon_t + \overset{\approx}{R}{}_t^* + \overset{\approx}{S}_t (\because \overset{\approx}{R}{}_t^* = \overset{\approx}{R}_t - \varepsilon_t)$ and for every r.v. X $E|X|^\delta \leq (E|X|)^\delta$ we deduce from condition (A) and (B) that

$$\begin{aligned} &E(|\phi((x - X_t)/r_n) - \phi((x - \overset{\approx}{R}_t)/r_n)|^\delta \mid \Omega_{t-1}) \\ &\leq (E|\phi((x - X_t)/r_n) - \phi((x - \overset{\approx}{R}_t)/r_n)| \mid \Omega_{t-1})^\delta \\ &= \left(r_n \int |\phi(z - \overset{\approx}{S}_t/r_n) - \phi(z)| f_\varepsilon(x - \overset{\approx}{R}_t - r_n z) dz \right)^\delta \\ &\leq M|\overset{\approx}{S}_t|^\delta. \end{aligned} \tag{2.42}$$

The results (2.38)–(2.42), the fact that $E|\overset{\approx}{S}_t|^\delta \leq MH^\delta_{t-g-q}(h \leq t \leq \ell)$, and condition (C) will then imply that for all large n,

$$\begin{aligned} &\left| \varphi^{(k_n)}(u, \ldots, u) - \prod_{j=1}^{k_n} \varphi_j(u) \right| \\ &\leq M|u|^\delta (nr_n)^{-\delta/2} k_n \sum_{r=t_n-q}^{\infty} H_r^\delta \leq M|u|^\delta (nr_n)^{-\delta/2} k_n / t_n \\ &\to 0 \quad \text{as} \quad n \to \infty (\because \delta > \gamma/(2+\gamma) \quad \text{and} \quad k_n \sim t_n). \end{aligned} \tag{2.43}$$

Again, for sufficiently large n

$$s_n^2 := E(U_j^2) = n^{-1}m_n\left(E(Y_1^2) + 2\sum_{v=1}^{m_n}(1 - v/m_n)E(Y_1Y_{1+v})\right) \tag{2.44}$$
$$\geq \sigma^2 n^{-1} m_n/2,$$

by Lemma 2.2, and the fact that $E(Y_1^2) = r_n E(\phi^2(x - X_1; r_n)) - r_n f_n^2 \to \sigma^2$ as $n \to \infty$. We now use (2.35), (2.37), (2.44) and the specification of m_n, t_n and conclude eventually that

$$\sum_{j=1}^{k_n} E(U_j^4) \Big/ \left(\sum_{j=1}^{k_n} E(U_j^2)\right)^2 \tag{2.45}$$
$$\leq Mk_nE(U_1^4)/(k_nm_n/n)^2$$
$$\leq Mk_nE\left(\sum_{t=1}^{m_n} Y_t\right)^4 /n^2$$
$$\leq M((nr_n)^{-1} + n^{-1}m_n + n^{-1}m_n^2 r_n^{(1+2\gamma)/(2+\gamma)})$$
$$\leq M((nr_n)^{-1} + k_n^{-1} + r_n^{(2+7\gamma)/2+\gamma}) \to 0, \qquad \text{as} \quad n \to \infty.$$

Therefore, the Liapounov condition for the central limit theorem holds and hence $\prod_{j=1}^{k_n} \varphi_j(u) \to \exp(-u^2\sigma^2/2)$ as $n \to \infty$. If we use this relation and our conclusion in (2.43) we immediately obtain (2.38). Now observe that since $k_nm_n/n \to 1$ as $n \to \infty$, $n^{-1}(k_nt_n + d_n) = 1 - k_nm_n/n \to 0$ as $n \to \infty$. Consequently,

$$E\left(\sum_{j=1}^{k_n} V_j + W\right)^2 \leq n^{-1}(k_nt_n + d_n)\left(E(Y_1^2) + 2\sum_{v=1}^{\infty}|E(Y_1Y_{1+v})|\right) \tag{2.46}$$
$$\leq Mn^{-1}(k_nt_n + d_n) \to 0, \qquad \text{as} \quad n \to \infty,$$

and the proof of Lemma 2.8 is complete.

Now since $(nr_n)^{1/2}(\widehat{f}_n(x) - f_n(x)) = \sum_{t=1}^n Y_t/n^{1/2} = \sum_{j=1}^{k_n} U_j + \sum_{j=1}^{k_n} V_j + W$ the result (2.2) is a direct consequence of Lemma 2.8.

Let $\psi(u) = \int_{-\infty}^{t} \exp(iuy)\phi(y)dy$ and let φ be the ch.f. of X_1. Then assume that the following condition holds.

(D) $\int_{-\infty}^{\infty} \phi(y)dy = 1$ and for some $q > 0$, $\lim_{u\to 0}(1 - \psi(u))/|u|^q = k_q$, $|k_q| < \infty$ and $|\int \exp(-iux)|u|^q\varphi(u)du| < \infty$.

Note that if $q \leq 1$ then the last part of (D) holds by condition (B).

THEOREM 2.2. *Let the conditions* (A)–(D) *hold and assume that* $\{r_n\}$ *is such that* $r_n \to 0$, $nr_n^{2q+1} \to 0$ *but* $nr_n \to \infty$. *Then as* $n \to \infty$

$$\mathcal{L}((nr_n)^{1/2}(\widehat{f}_n - f)) \to \mathcal{N}(0, \sigma^2). \tag{2.47}$$

PROOF. Note that as $n \to \infty$,

$$(2.48) \qquad (f_n - f)/r_n^q = (2\pi)^{-1}\int_{-\infty}^{x} \exp(-iux)(\psi(r_n u) - 1)/|r_n u|^q |u|^q \varphi(u)du$$
$$\to -k_q(2\pi)^{-1}\int_{-\infty}^{x} \exp(-iux)|u|^q \varphi(u)du,$$

which implies that $(nr_n)^{1/2}(f_n - f) \to 0$ as $n \to \infty$. (2.38) will then follow from (2.39).

3. Almost sure convergence

For notational convenience we write $r_n = r(n)$ for integer value of n in Theorem 3.1 that follows.

THEOREM 3.1. *Let conditions* (A)–(D) *hold, and assume that* (i) $r(n) \downarrow 0$ *and for some* $\alpha(0 < \alpha < 1/2)n^a r(n) \to \infty$ *as* $n \to \infty$, (ii) *for every* $p > 1$, $k(1 - r((k+1)^p)/r(k^p)) \to$ *a finite constant as* $k \to \infty$, *where for every real* x, $r(x)$ *is defined by linear interpolation between integer which sandwich* x *and* (iii) *if* $q(a) = \int_{-\infty}^{\infty} |\phi(w) - a\phi(aw)|dw(0 < a < 1)$, *then* $q(a)/(1-a) \to$ *a finite constant as* $a \to 1^-$. *Then as* $n \to \infty$

$$(3.1) \qquad \widehat{f}_n \to f \quad a.s.$$

PROOF. Note that since $f_n \to f$ as $n \to \infty$ (by Lemma 2.3) it is sufficient if we establish that $\widehat{f}_n - f \to 0$ a.s. as $n \to \infty$. We can write $\widehat{f}_n - f_n = S_{nn}/n$ where $S_{nm} = \sum_{t=1}^{n}(\phi(x - X_t; r_m) - f_m) = \sum_{t=1}^{n} Y_{tm}$. Let $n_k = [k^p]$, where p is any number $\in ((1-\alpha)^{-1}, \alpha^{-1})$, α being given as in (ii) above and $k = 1, 2, \ldots$. Then since $E(Y_{1n}^2) \leq M$, and (2.7) holds we have that $\operatorname{var}(S_{n_k n_k}/n_k) = E(\sum_{t=1}^{n_k} Y_{tn_k})^2/n_k^2 r_{n_k} \leq M(n_k r_{n_k})^{-1} \leq Mk^{-p(1-\alpha)}$ by (2.7) and condition (i) above. Therefore,

$$(3.2) \qquad S_{n_k n_k}/n_k \to 0 \quad \text{a.s.}$$

as $k \to \infty$. Let n be any integer, $n_k \leq n < n_{k+1}$ for some k and set $C_k = \max_{n_k \leq n < n_{k+1}} \cdot |S_{nn} - S_{n_k n}|$, $D_k = \max_{n_k \leq n < n_{k+1}} |S_{n_k n} - S_{n_k n_k}|$. Then

$$(3.3) \qquad |S_{nn}/n| \leq |S_{n_k n_k}/n_k| + C_k/n_k + D_k/n_k.$$

It is easy to show that $E(C_k^2/n_k^2) \leq E\sum_{n=n_k}^{n_{k+1}}(S_{nn} - S_{n_k n})^2/n_k^2 \leq M\sum_{n=n_k}^{n_{k+1}}(n - n_k)/n_k^2 r_n$. Again we can conclude from condition (ii) above that $r_n/r_{n_k} \to 1$ as $k \to \infty$. Therefore, for all sufficiently large k and hence n, $E(C_k^2/n_k^2) \leq M(n_{k+1} - n_k)^2/n_k^2 r_{n_k} \leq Mk^{p\alpha - 2}$. Since $p\alpha < 1$ it follows immediately that

$$(3.4) \qquad C_k/n_k \to 0 \quad \text{a.s.}$$

as k and hence $n \to \infty$. Similarly from (3.6) below we conclude that $E(D_k^2/n_k^2) \leq \sum_{n=n_k}^{n_{k+1}} E(S_{n_k n} - S_{n_k n_k})^2 \leq M(n_{k+1} - n_k)(r_{n_k} - r_{n_{k+1}})/n_k r_{n_k}^2 \leq Mk^{p\alpha-2}$. Consequently,

$$(3.5) \qquad D_k/n_k \to 0 \quad \text{a.s.}$$

as $k \to \infty$ and hence $n \to \infty$. (3.1) now will follow directly from (3.2)–(3.5).

LEMMA 3.1. *Let the conditions of Theorem* 3.1 *hold. Then*

$$E(S_{n_k n} - S_{n_k n_k})^2 \le M(r_{n_k} - r_{n_{k+1}})n_k / r_{n_k}^2. \tag{3.6}$$

PROOF. Note that $S_{n_k n} - S_{n_k n_k}$ is the sum of n_k terms. For sufficiently large k, the expectation of the sums of squares terms can easily be shown to be $\le n_k E(\phi(x - X_1; r_n) - \phi(x - X_1; r_{n_k}))^2 \le n_k r_n^{-1} \int_{-\infty}^{\infty} (\phi(w) - \gamma_k \phi(\gamma_k w))^2 f(x - r_n w) dw \le M n_k r_n^{-1} \int_{-\infty}^{\infty} |\phi(w) - \gamma_k \phi(\gamma_k w)| dw \le M n_k r_n^{-1}(1 - \gamma_k) \le M(r_{n_k} - r_{n_{k+1}}) n_k r_{n_k}^2$ where $1 \ge \gamma_k = r_n / r_{n_k} \ge r_{n_{k+1}} / r_{n_k} \to 1$ as $k \to \infty$ by virtue of conditions (B) and (i)–(iii) above. If now we replace $\phi(x - X_1; r_n)$, $\phi(x - X_{1+v}; r_n)$ and f_n in Lemma 2.1 by $\phi(x - X_1; r_n) - \phi(x - X_1; r_{n_k})$, $\phi(x - X_{1+v}; r_n) - \phi(x - X_{1+v}; r_{n_k})$ and $f_n - f_{n_k}$ respectively then by routine analysis and following the same sequence of arguments as led to (2.12) and eventually to (2.7) we can establish that the expectation of the sums of cross products in $E(S_{n_k n} - S_{n_k n_k})^2$ is numerically $\le M(r_{n_k} - r_{n_{k+1}}) n_k r_n^{\gamma/(1+\gamma)} / r_{n_k}^2$. The result (3.6) follows immediately.

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