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A COMPARISON OF RESTRICTED AND UNRESTRICTED ESTIMATORS IN ESTIMATING LINEAR FUNCTIONS OF ORDERED SCALE PARAMETERS OF TWO GAMMA DISTRIBUTIONS

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Abstract. The problem of estimating linear functions of ordered scale parameters of two Gamma distributions is considered. A necessary and sufficient condition on the ratio of two coefficients is given for the maximum likelihood estimator (MLE) to dominate the crude unbiased estimator (UE) in terms of mean square error. A modified MLE which satisfies the restriction is also suggested, and a necessary and sufficient condition is also given for it to dominate the admissible estimator based solely on one sample. The estimation of linear functions of variances in two sample problem and also of variance components in a one-way random effect model is mentioned.

Key words and phrases: MLE, unbiased estimator, admissible estimator, variance estimation.

1. Introduction

In this paper, we discuss the problem of estimating linear functions of scale parameters of $Gamma(\alpha_i, \lambda_i)$, $i = 1, 2$, when α_i, $i = 1, 2$ are known and the restriction $\lambda_1 \leq \lambda_2$ is given. We note that a special case of this general problem is given in two samples problem with different but ordered variances. Estimation of smaller or larger variance has been discussed by Kushary and Cohen (1989). Among the linear functions of variances estimation of those with positive coefficients is especially important since they are the variances of linear functions of two random variables.

Consider, for another example, a one-way random effects model given by

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \qquad i = 1, \dots, I, \quad j = 1, \dots, J,$$

where $\alpha_i \sim N(0, \sigma_A^2)$ and $\varepsilon_{ij} \sim N(0, \sigma_1^2)$. Letting $S_1 = \sum_i \sum_j (y_{ij} - \bar{y}_{i.})^2$ and $S_2 = J \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2$ for $\bar{y}_{i.} = J^{-1} \sum_j y_{ij}$ and $\bar{y}_{..} = (IJ)^{-1} \sum_i \sum_j y_{ij}$, one has that $S_i/\sigma_i^2 \sim \chi^2_{n_i}, i = 1, 2$, for $n_1 = I(J-1)$, $n_2 = I - 1$ and $\sigma_2^2 = \sigma_1^2 + J\sigma_A^2$. In this situation, of great interest is to estimate the between component of variance σ_A^2, being represented by $\sigma_A^2 = J^{-1}\ (\sigma_2^2 - \sigma_1^2)$, which is a linear function of two ordered Gamma scale parameters σ_1^2 and σ_2^2.

There has been considerable interest in the estimation of the parameters when there are linear restrictions among parameters. Typical types of the restrictions are positivity, simple ordering and simple tree ordering. See, for example, Barlow *et al.* (1972) and Robertson *et al.* (1988). Many papers focus on normal mean estimation and on comparing the maximum likelihood estimator (MLE) which satisfies the order restriction with the unbiased estimator (UE) coordinately (Lee (1981), Kelly (1989)). However, MLE does not always improve UE (Lee (1988)), and it is not always true that every linear function of MLE dominates the one of UE in terms of mean square error (MSE) (see also Hwang and Peddada (1994) and Fernández *et al.* (1999)). In recent years, Rueda and Salvador (1995) have considered the problem of estimating general linear function of normal means when two linear inequality constraints are given, and have shown that MLE gives an improvement for any coefficients. In estimating linear functions of positive normal means, Shinozaki and Chang (1999) have given a necessary and sufficient condition on the coefficients so that linear function of MLE dominates the one of UE in terms of MSE. Thus they show that MLE dominates UE for any choice of coefficients if and only if the number of means is less than 5. Independently, Fernández *et al.* (2000) have discussed the same problem under a symmetric unimodal location model. Other than normal distribution, there are also many papers dealing with the estimation of parameters under order restrictions. Kushary and Cohen (1991) considered the estimation of ordered Poisson parameters. Kaur and Singh (1991) considered the estimation of ordered means of two exponential population with the same sample sizes. They compared MLE with UE coordinately and showed that MLE dominates UE. This is a special case of the estimation problem of Gamma scale parameters when order restriction is given. See Hwang and Peddada (1994) and Kubokawa and Saleh (1994) for general scale parameter estimation under order restriction.

Here we first compare MLE with UE in estimating linear functions of ordered scale parameters of two Gamma distributions. To evaluate the difference of MSE of two estimators we give some useful lemmas in Section 2. We give a necessary and sufficient condition on the ratio of coefficients for MLE to dominate UE in terms of MSE. We also numerically obtain the upper bounds of the ratios for some typical values of $\alpha_i, i = 1, 2$. All these results are given in Section 3. Other than UE, there is another standard estimator of λ_i which we can obtain by replacing α_i by $\alpha_i + 1$ in UE. This estimator is an admissible one based solely on one sample under quadratic loss. In Section 4, we suggest a modified MLE which satisfies the restriction and give a necessary and sufficient condition on the ratio of coefficients for the modified MLE to dominate the unrestricted one. The lower bounds of the ratios are also given for some typical values of $\alpha_i, i = 1, 2$. We give some concluding remarks in Section 5.

2. Preliminaries

Let X_i, $i = 1, 2$ be independent $Gamma(\alpha_i, \lambda_i)$ random variables, having density

$$f_{\lambda_i}(x_i) = x_i^{\alpha_i - 1} \lambda_i^{-\alpha_i} e^{-x_i/\lambda_i} / \Gamma(\alpha_i), \qquad 0 < x_i < \infty \tag{2.1}$$

where $\alpha_i (> 0)$ is known and $\lambda_i (> 0)$ is unknown but satisfying $0 < \lambda_1 \leq \lambda_2 < \infty$. We note that even if we have more than one observations, we can reduce the case to the above one by considering the sufficient statistics which also follow Gamma distributions. The MLE of λ_i is given by

$$\hat{\lambda}_i = \frac{X_i}{\alpha_i} + (-1)^i \frac{(\alpha_2 X_1 - \alpha_1 X_2)^+}{\alpha_i(\alpha_1 + \alpha_2)}, \qquad i = 1, 2,$$

where $a^+ = \max(0, a)$ and X_i/α_i is the unbiased estimator (UE) of λ_i.

The best estimator of λ_i of the form cX_i under squared error loss is $X_i/(\alpha_i + 1)$, which is an admissible estimator of λ_i based solely on X_i. We also consider a modified MLE that satisfies the restriction $0 < \lambda_1 \leq \lambda_2 < \infty$ given by

$$\tilde{\lambda}_i = \frac{X_i}{\alpha_i + 1} + (-1)^i \frac{((\alpha_2 + 1)X_1 - (\alpha_1 + 1)X_2)^+}{(\alpha_i + 1)(\alpha_1 + \alpha_2 + 2)}, \qquad i = 1, 2,$$

which we can obtain by replacing α_i by $\alpha_i + 1$ in the MLE $\hat{\lambda}_i$. We note that Kubokawa and Saleh (1994) have proposed another improving estimator of λ_1 by their general argument.

Let c_1, c_2 be given constants and we want to estimate $c_1\lambda_1 + c_2\lambda_2$. We first compare two estimators, UE, $\sum_{i=1}^2 c_i X_i/\alpha_i$ and, MLE, $\sum_{i=1}^2 c_i \hat{\lambda}_i$ by their mean square error (MSE) and give a condition on c_1 and c_2 for MLE to dominate UE. We also compare $\sum_{i=1}^2 c_i X_i/(\alpha_i + 1)$ with modified MLE $\sum_{i=1}^2 c_i \tilde{\lambda}_i$, and give a condition on c_1 and c_2 for the modified MLE to dominate the competitor.

We should first mention that the domination depends only on the ratio c_2/c_1. This is generally true so far as we are concerned with estimation of linear functions $\sum_{i=1}^2 c_i \theta_i$ of parameters θ_1 and θ_2 and compare two estimators of the form $\sum_{i=1}^2 c_i \hat{\theta}_i$ by their MSE, since MSE is a quadratic function of c_1 and c_2.

To evaluate the difference of MSE between the estimators, we need the following lemmas. The following Lemma 2.1 is well known and we can show it by applying integration by parts (Berger (1980)).

LEMMA 2.1. *Let X be a Gamma(α, λ) random variable and assume that $g(x)$ is absolutely continuous on $(0, \infty)$ with $g'(x) = \frac{dg(x)}{dx}$ satisfying*

(i) $E[|Xg'(X)|] < \infty$ *and* $E[|g(X)|] < \infty$

(ii) $\lim_{x \to 0} g(x) x^\alpha e^{-x/\lambda} = \lim_{x \to \infty} g(x) x^\alpha e^{-x/\lambda} = 0$, *for* $\lambda > 0$.

Then

$$E[Xg(X)] = \lambda \{\alpha E[g(X)] + E[Xg'(X)]\}.$$

LEMMA 2.2. *Let X_i, $i = 1, 2$ be independent Gamma(α_i, λ_i) random variables having density* (2.1). *For any constant $b \geq 0$, $I_{x_1 \geq bx_2}$ denotes indicator function of the set $\{(x_1, x_2) \mid x_1 \geq bx_2\}$ and $\rho = b/(b+1)$. Then*

$$\frac{E[X_2 I_{X_1 \geq bX_2}]}{E[X_1 I_{X_1 \geq bX_2}]} \geq \frac{E_0[X_2 I_{X_1 \geq bX_2}]}{E_0[X_1 I_{X_1 \geq bX_2}]} = \frac{\alpha_1 + \alpha_2}{\alpha_1} \frac{1 - I_\rho(\alpha_1, \alpha_2)}{1 - I_\rho(\alpha_1 + 1, \alpha_2)} - 1,$$

where $E_0[\cdot]$ denotes the expectation when $\lambda_1 = \lambda_2$ and $I_x(\alpha, \beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1} du / B(\alpha, \beta)$, where $B(\alpha, \beta)$ is the beta function.

The proof is rather technical and we give it in Appendix A.1. We note that $E_0[X_2 I_{X_1 \geq bX_2}]/E_0[X_1 I_{X_1 \geq bX_2}]$ is independent of the common value of λ_1 and λ_2.

3. MSE reduction by MLE in estimating linear functions of Gamma scale parameters

Here we evaluate the difference of MSE between MLE and UE in estimating $c_1\lambda_1 + c_2\lambda_2$, where c_1, c_2 are constants. The difference of squared errors between MLE and UE is given by

$$(3.1)\quad \left\{\sum_{i=1}^{2} c_i\left(\frac{X_i}{\alpha_i}-\lambda_i\right)\right\}^2-\left\{\sum_{i=1}^{2} c_i\left(\frac{X_i}{\alpha_i}-\lambda_i\right)-\frac{(\alpha_2X_1-\alpha_1X_2)^+}{\alpha_1+\alpha_2}\left(\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right)\right\}^2$$
$$=\left(\frac{\tilde{c}_1-\tilde{c}_2}{\alpha_1+\alpha_2}\right)\left\{2\sum_{i=1}^{2}\tilde{c}_i(X_i-\alpha_i\lambda_i)(\alpha_2X_1-\alpha_1X_2)^+ -\left(\frac{\tilde{c}_1-\tilde{c}_2}{\alpha_1+\alpha_2}\right)\left[(\alpha_2X_1-\alpha_1X_2)^+\right]^2\right\},$$

where $\tilde{c}_i = c_i/\alpha_i$, $i = 1, 2$. Without loss of generality we assume that $\tilde{c}_1 \geq \tilde{c}_2$ and also for simplicity we denote $I_{\alpha_2X_1\geq\alpha_1X_2}$ by I, hereafter.

To evaluate the expected value of (3.1) we use Lemma 2.1 and have

$$E[X_1(\alpha_2X_1-\alpha_1X_2)^+] = \lambda_1\{\alpha_1E[(\alpha_2X_1-\alpha_1X_2)^+] + E[\alpha_2X_1I]\}$$

and

$$(3.2)\quad E[X_2(\alpha_2X_1-\alpha_1X_2)^+] = \lambda_2\{\alpha_2E[(\alpha_2X_1-\alpha_1X_2)^+] - E[\alpha_1X_2I]\}.$$

Thus we see that the expected value of the quantity in the braces of (3.1) is given by

$$(3.3)\quad 2\tilde{c}_1\lambda_1E[\alpha_2X_1I] - 2\tilde{c}_2\lambda_2E[\alpha_1X_2I] - \left(\frac{\tilde{c}_1-\tilde{c}_2}{\alpha_1+\alpha_2}\right)\{\alpha_1\alpha_2(\lambda_1-\lambda_2)E[(\alpha_2X_1-\alpha_1X_2)^+] + \alpha_2\lambda_1E[\alpha_2X_1I] + \alpha_1\lambda_2E[\alpha_1X_2I]\}.$$

We first show that (3.3) is negative for sufficiently large λ_2 if $\tilde{c}_2 > 0$. Since the third term in (3.3) is non-positive we see from Lemma 2.2 that (3.3) is less than or equal to

$$2E[\alpha_2X_1I]\left\{\tilde{c}_1\lambda_1 - \tilde{c}_2\lambda_2\frac{\alpha_1}{\alpha_2}\frac{E_0[X_2 \mid \alpha_2X_1 \geq \alpha_1X_2]}{E_0[X_1 \mid \alpha_2X_1 \geq \alpha_1X_2]}\right\},$$

which is negative for sufficiently large λ_2 if $\tilde{c}_2 > 0$. This means that MLE does not improve UE if $\tilde{c}_2 > 0$. Thus we see that $\tilde{c}_2$ must be non-positive when $\tilde{c}_1 \geq \tilde{c}_2$ in order for MLE to dominate UE. In addition to the condition $\tilde{c}_1 \geq \tilde{c}_2$ we assume that $\tilde{c}_2 \leq 0$ in the following and give a condition on c_1 and c_2 for MLE to dominate UE.

Since (3.2) is non-negative, we have

$$\alpha_2E[(\alpha_2X_1-\alpha_1X_2)^+] \geq E[\alpha_1X_2I],$$

and we see that (3.3) is greater than or equal to

$$(3.4)\quad 2\tilde{c}_1\lambda_1E[\alpha_2X_1I] - 2\tilde{c}_2\lambda_2E[\alpha_1X_2I] - \left(\frac{\tilde{c}_1-\tilde{c}_2}{\alpha_1+\alpha_2}\right)\{\alpha_1(\lambda_1-\lambda_2)E[\alpha_1X_2I] + \alpha_2\lambda_1E[\alpha_2X_1I] + \alpha_1\lambda_2E[\alpha_1X_2I]\}$$
$$= \lambda_1\left(2\tilde{c}_1 - \frac{(\tilde{c}_1-\tilde{c}_2)\alpha_2}{\alpha_1+\alpha_2}\right)E[\alpha_2X_1I]$$

$$-\left(\frac{\tilde{c}_1-\tilde{c}_2}{\alpha_1+\alpha_2}\alpha_1\lambda_1+2\tilde{c}_2\lambda_2\right)E[\alpha_1X_2I]$$
$$\geq\frac{\lambda_1}{\alpha_1+\alpha_2}\{(\tilde{c}_1(2\alpha_1+\alpha_2)+\tilde{c}_2\alpha_2)E[\alpha_2X_1I]$$
$$-(\tilde{c}_1\alpha_1+\tilde{c}_2(\alpha_1+2\alpha_2))E[\alpha_1X_2I]\},$$

since $\lambda_2 \geq \lambda_1$ and $\tilde{c}_2 \leq 0$. We can easily see that if $\tilde{c}_1(2\alpha_1+\alpha_2)+\tilde{c}_2\alpha_2 \geq 0$, then (3.4) is non-negative since $E[\alpha_2X_1I] \geq E[\alpha_1X_2I]$ and $\tilde{c}_1 > \tilde{c}_2$. Even if $\tilde{c}_1(2\alpha_1+\alpha_2)+\tilde{c}_2\alpha_2 < 0$, (3.4) is non-negative if

$$\text{(3.5)}\qquad \frac{E[X_2I]}{E[X_1I]} \geq \frac{\alpha_2}{\alpha_1}\frac{(c_1/c_2)(2+\alpha_2/\alpha_1)+1}{(c_1/c_2)+(2+\alpha_1/\alpha_2)}.$$

We note that for fixed α_1 and α_2, the right-hand side of (3.5) is an increasing function of c_1/c_2. Thus we see that for fixed α_1 and α_2 if some c_1 and c_2 satisfy (3.5) then any c_1' and c_2' such that $c_1/c_2 > c_1'/c_2'$ satisfy (3.5).

Putting $R = \{1-I_\rho(\alpha_1+1,\alpha_2)\}/\{1-I_\rho(\alpha_1,\alpha_2)\}$, we have from Lemma 2.2

$$\frac{E_0[X_2I]}{E_0[X_1I]} = \frac{\alpha_1+\alpha_2}{\alpha_1}\frac{1}{R}-1,$$

where $\rho = \frac{\alpha_1}{\alpha_1+\alpha_2}$. Thus the inequality (3.5) is true if

$$\frac{\alpha_1+\alpha_2}{\alpha_1}\frac{1}{R}-1 \geq \frac{\alpha_2}{\alpha_1}\frac{c_1(2+\alpha_2/\alpha_1)+c_2}{c_1+(2+\alpha_1/\alpha_2)c_2},$$

which is equivalent to

$$R \leq \frac{c_1(1-\rho)+c_2(2-\rho)}{c_1\dfrac{1-\rho}{\rho}+c_2}.$$

The above inequality is also equivalent to the one

$$\frac{c_1}{c_2} \leq \frac{\rho}{1-\rho}\frac{2-\rho-R}{R-\rho}.$$

It should be noted that $R \geq 1 > \rho$, since $I_\rho(\alpha_1+1,\alpha_2) < I_\rho(\alpha_1,\alpha_2)$.

Thus we have shown that MLE dominates UE if c_1 and c_2 satisfy $c_1/c_2 \leq \rho(2-\rho-R)/\{(1-\rho)(R-\rho)\}$. Conversely, we see that this conditions is also necessary for MLE to dominate UE by examining each step of the above evaluation for the case $\lambda_1 = \lambda_2$. If we denote the MSE of an estimator φ of $\sum_{i=1}^2 c_i\lambda_i$ by $MSE(\varphi)$, we have the following theorem.

THEOREM 3.1. *$MSE(\sum_{i=1}^2 c_iX_i/\alpha_i) \geq MSE(\sum_{i=1}^2 c_i\hat{\lambda}_i)$ for any $0 < \lambda_1 \leq \lambda_2 < \infty$ if and only if*

$$\text{(3.6)}\qquad \frac{c_1}{c_2} \leq \frac{\rho}{1-\rho}\frac{2-\rho-R}{R-\rho}$$

including the case $c_2 = 0$.

Table 1. Upper bounds of c_1/c_2.

$\alpha_1\backslash\alpha_2$	0.1	0.3	0.5	0.8	1	1.5	2	2.5	3	5	8	12	100
0.1	−0.277	−0.152	−0.104	−0.070	−0.057	−0.039	−0.030	−0.024	−0.020	−0.012	−0.008	−0.005	−0.001
0.3	−0.275	−0.188	−0.142	−0.103	−0.086	−0.062	−0.048	−0.039	−0.033	−0.021	−0.013	−0.009	−0.001
0.5	−0.208	−0.153	−0.120	−0.090	−0.077	−0.057	−0.045	−0.037	−0.031	−0.020	−0.013	−0.008	−0.001
0.8	−0.086	−0.066	−0.054	−0.042	−0.036	−0.027	−0.022	−0.018	−0.016	−0.010	−0.006	−0.004	−0.001
1	0	0	0	0	0	0	0	0	0	0	0	0	0
1.5	0.220	0.178	0.149	0.120	0.106	0.082	0.067	0.056	0.049	0.031	0.021	0.014	0.002
2	0.444	0.364	0.308	0.250	0.222	0.174	0.143	0.121	0.105	0.069	0.045	0.031	0.004
2.5	0.670	0.552	0.471	0.385	0.344	0.271	0.224	0.191	0.167	0.110	0.073	0.050	0.006
3	0.896	0.743	0.636	0.524	0.469	0.372	0.309	0.264	0.231	0.154	0.102	0.071	0.009
5	1.803	1.512	1.306	1.089	0.981	0.790	0.662	0.571	0.502	0.340	0.230	0.161	0.021
8	3.167	2.673	2.322	1.949	1.764	1.432	1.210	1.049	0.928	0.637	0.436	0.308	0.042
12	4.987	4.225	3.681	3.103	2.815	2.298	1.949	1.697	1.505	1.044	0.721	0.512	0.071
100	45.043	38.403	33.658	28.586	26.058	21.483	18.386	16.131	14.406	10.216	7.220	5.244	0.797

When $c_2 = 0 (c_1 = 0)$ and $\alpha_1 = \alpha_2$ is a positive integer, the above theorem reduces to Theorem 2.1. (a) (Theorem 2.2. (a)) due to Kaur and Singh (1991). See Kushary and Cohen (1989) for another improving estimator of smaller variance and also Hwang and Peddada (1994) for related results.

We have calculated the values of the right-hand side of (3.6) for some typical values of α_1 and α_2 and have given them in Table 1. We see that the range of the value of c_1/c_2 for which MLE dominates UE is rather small. Especially when we are concerned with the case with positive coefficients it is quite small. If $\alpha_1 = \alpha_2 = 2$, we need $c_1/c_2 \le 0.143$ and MLE does not dominate UE for most of the choice of coefficients with the same sign. We notice that the range of c_1/c_2 for which MLE dominates UE becomes larger if α_1 or α_2 gets larger. Rather than α_2, α_1 seems to be important to make the range larger.

The case when $c_1 = 0$ corresponds to the estimation of λ_2 and is of particular interest. From Table 1 it is almost obvious that MLE dominates UE for $c_1 = 0$ if and only if $\alpha_1 \ge 1$. We formally give it in the following corollary whose proof is given in Appendix A.2.

COROLLARY 3.1. *$MSE(\sum_{i=1}^2 c_i X_i/\alpha_i) \ge MSE(\sum_{i=1}^2 c_i \hat{\lambda}_i)$ for any $0 < \lambda_1 \le \lambda_2 < \infty$ and for any $c_1 \ge 0$ and $c_2 \le 0$ (and also for any $c_1 \le 0$ and $c_2 \ge 0$) if and only if $\alpha_1 \ge 1$.*

4. MSE reduction of an admissible estimator based solely on one sample

In this section, we compare two estimators of $c_1\lambda_1 + c_2\lambda_2$, $\sum_{i=1}^2 c_i X_i/(\alpha_i + 1)$ and $\sum_{i=1}^2 c_i \tilde{\lambda}_i$, by their mean square errors and give a condition on c_1 and c_2 for the latter to dominate the former.

The difference of squared errors between $\sum_{i=1}^2 c_i X_i/(\alpha_i+1)$ and $\sum_{i=1}^2 c_i \tilde{\lambda}_i$ is given by

$$\left(\frac{\tilde{c}_1' - \tilde{c}_2'}{\alpha_1+\alpha_2+2}\right)\left\{2\sum_{i=1}^2 \tilde{c}_i'(X_i - (\alpha_i+1)\lambda_i)[(\alpha_2+1)X_1 - (\alpha_1+1)X_2]^+ - \left(\frac{\tilde{c}_1' - \tilde{c}_2'}{\alpha_1+\alpha_2+2}\right)\{[(\alpha_2+1)X_1 - (\alpha_1+1)X_2]^+\}^2\right\}, \tag{4.1}$$

where $\tilde{c}'_i = c_i/(\alpha_i+1), i = 1, 2$. Without loss of generality we assume that $\tilde{c}'_1 \geq \tilde{c}'_2$, and also for simplicity we denote $I_{(\alpha_2+1)X_1 \geq (\alpha_1+1)X_2}$ by I', hereafter. By applying Lemma 2.1 we see that the expected value in the braces of (4.1) is given by

$$(4.2)\quad 2\tilde{c}'_1\lambda_1 E[(\alpha_1+1)X_2 I'] - 2\tilde{c}'_2\lambda_2 E[(\alpha_2+1)X_1 I'] - \left(\frac{\tilde{c}'_1 - \tilde{c}'_2}{\alpha_1+\alpha_2+2}\right)\{(\alpha_2+1)\lambda_1 E[(\alpha_1+1)X_2 I'] + (\alpha_1+1)\lambda_2 E[(\alpha_2+1)X_1 I'] + (\alpha_1+1)(\alpha_2+1)(\lambda_1-\lambda_2)E[\{(\alpha_2+1)X_1 - (\alpha_1+1)X_2\}^+]\}.$$

Here we notice that (4.2) is negative for sufficiently large λ_2 if $\tilde{c}'_2 > 0$, since the third term in (4.2) is non-positive and $E[(\alpha_2+1)X_1 I'] \geq E[(\alpha_1+1)X_2 I']$. This implies that $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ does not dominate $\sum_{i=1}^2 c_i X_i/(\alpha_i+1)$ if $\tilde{c}'_1 \geq \tilde{c}'_2$ and $\tilde{c}'_2 > 0$ (or $\tilde{c}'_1 \leq \tilde{c}'_2$ and $\tilde{c}'_2 < 0$). Therefore in the following we only consider the case where $\tilde{c}_1 \geq \tilde{c}_2$ and $\tilde{c}_2 \leq 0$ to find the conditions on $\tilde{c}'_1$ and $\tilde{c}'_2$ for $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ to dominate $\sum_{i=1}^2 c_i X_i/(\alpha_i+1)$.

We first show that (4.2) is non-negative if $(\alpha_1+1)\tilde{c}'_1 + (\alpha_1+2\alpha_2+3)\tilde{c}'_2 \leq 0$. Since $\tilde{c}'_1 \geq \tilde{c}'_2$ and $\lambda_1 \leq \lambda_2$, (4.2) is greater than or equal to

$$(4.3)\quad \left(\frac{(2\alpha_1+\alpha_2+3)\tilde{c}'_1 + (\alpha_2+1)\tilde{c}'_2}{\alpha_1+\alpha_2+2}\right)\lambda_1 E[(\alpha_1+1)X_2 I'] - \left(\frac{(\alpha_1+1)\tilde{c}'_1 + (\alpha_1+2\alpha_2+3)\tilde{c}'_2}{\alpha_1+\alpha_2+2}\right)\lambda_2 E[(\alpha_2+1)X_1 I'].$$

Since $\lambda_1 \leq \lambda_2$, $E[(\alpha_1+1)X_2 I'] \leq E[(\alpha_2+1)X_1 I']$ and $(2\alpha_1+\alpha_2+3)\tilde{c}'_1 + (\alpha_2+1)\tilde{c}'_2 \geq (\alpha_1+1)\tilde{c}'_1 + (\alpha_1+2\alpha_2+3)\tilde{c}'_2$, we see that (4.2) is non-negative if $(\alpha_1+1)\tilde{c}'_1 + (\alpha_1+2\alpha_2+3)\tilde{c}'_2 \leq 0$.

In the following we assume that $(\alpha_1+1)\tilde{c}'_1 + (\alpha_1+2\alpha_2+3)\tilde{c}'_2 > 0$. Using the inequality

$$E[\{(\alpha_2+1)X_1 - (\alpha_1+1)X_2\}^+] \geq E[X_1 I'],$$

we see that (4.2) is greater than or equal to

$$(4.4)\quad \frac{\lambda_1}{\alpha_1+\alpha_2+2}\{[(2\alpha_1+\alpha_2+3)\tilde{c}'_1 + (\alpha_2+1)\tilde{c}'_2]E[(\alpha_1+1)X_2 I'] - [(\alpha_1+1)\tilde{c}'_1 + (\alpha_1+2\alpha_2+3)\tilde{c}'_2]E[(\alpha_2+1)X_1 I']\}.$$

(4.4) is non-negative if and only if

$$(4.5)\quad \frac{E[X_2 I']}{E[X_1 I']} \geq \frac{\alpha_2+1}{\alpha_1+1}\frac{c_1 + \{2+(\alpha_1+1)/(\alpha_2+1)\}c_2}{\{2+(\alpha_2+1)/(\alpha_1+1)\}c_1 + c_2}.$$

Now we denote $\rho' = (\alpha_1+1)/(\alpha_1+\alpha_2+2)$ and $R' = (1 - I_{\rho'}(\alpha_1+1, \alpha_2))/(1 - I_{\rho'}(\alpha_1, \alpha_2))$. Then from Lemma 2.2, we see that the inequality (4.5) is true if

$$(4.6)\quad \frac{\alpha_1+\alpha_2}{\alpha_1 R'} \geq \frac{2(c_1+c_2)}{(\rho'+1)c_1 + \rho' c_2}.$$

Since the right-hand side of (4.6) is a decreasing function of c_1/c_2, we see that if $\alpha_1 R'/(\alpha_1+\alpha_2) \leq (\rho'+1)/2$, then $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ dominates $\sum_{i=1}^2 c_i X_i/(\alpha_i+1)$ for any c_1 and c_2 such that $-\infty < c_1/c_2 \leq (\alpha_1+1)/(\alpha_2+1)$ including the case $c_2 = 0$. Similarly

Table 2. Lower bounds of c_1/c_2. (A blank means that lower bound does not exist).

$\alpha_1 \backslash \alpha_2$	0.1	0.3	0.5	0.8	1	1.5	2	2.5
0.1	−3.610	−5.315	−9.856					
0.3	−3.784	−5.312	−8.806	−302.000				
0.5	−3.966	−5.384	−8.320	−43.602				
0.8	−4.251	−5.564	−8.037	−23.911				
1	−4.444	−5.714	−8.000	−20.000				
1.5	−4.938	−6.145	−8.159	−16.197	−47.793			
2	−5.438	−6.619	−8.500	−15.000	−31.000			
2.5	−5.943	−7.118	−8.929	−14.654	−25.934			
3	−6.450	−7.630	−9.407	−14.692	−23.800			
5	−8.490	−9.746	−11.541	−16.242	−22.594			
8	−11.565	−12.993	−14.972	−19.779	−25.500	−105.36		
12	−15.673	−17.363	−19.663	−25.021	−30.974	−83.649		
100	−106.18	−114.15	−124.71	−147.56	−170.11	−291.02	−1292.8	

in the case when $\alpha_1 R'/(\alpha_1+\alpha_2) > (\rho'+1)/2$, $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ dominates $\sum_{i=1}^2 c_i X_i/(\alpha_i+1)$ for any c_1 and c_2 such that $\{(\alpha_1+\alpha_2)\rho' - 2\alpha_1 R'\}/\{2\alpha_1 R' - (\alpha_1+\alpha_2)(\rho'+1)\} < c_1/c_2 \le (\alpha_1+1)/(\alpha_2+1)$.

By examining each step of the above evaluation for the case $\lambda_1 = \lambda_2$ we see that this condition is also necessary. Thus we have shown the following theorem.

THEOREM 4.1. *$MSE(\sum_{i=1}^2 c_i X_i/(\alpha_i+1)) \ge MSE(\sum_{i=1}^2 c_i\tilde{\lambda}_i)$ for any $0 < \lambda_1 \le \lambda_2 < \infty$ if and only if*

$$\frac{\rho' - 2\rho R'}{2\rho R' - (\rho'+1)} \le \frac{c_1}{c_2} \le \frac{\alpha_1+1}{\alpha_2+1} \quad \textit{when} \quad 2\rho R' > \rho' + 1$$

and

$$-\infty < \frac{c_1}{c_2} \le \frac{\alpha_1+1}{\alpha_2+1} \quad \textit{when} \quad 2\rho R' \le \rho' + 1$$

including the case $c_2 = 0$.

We have calculated the lower bounds of c_1/c_2 if they exist for some typical values of α_1 and α_2 and have given them in Table 2.

The case when $c_2 = 0$ corresponds to the estimation of λ_1 and $\tilde{\lambda}_1$ dominates $X_1/(\alpha_1+1)$ if and only if $2\rho R' \le \rho' + 1$. Although it seems clear from Table 2 for what values of α_1 and α_2 this condition is satisfied, we give the following analytical result which is not the best possible in any sense.

COROLLARY 4.1. *$MSE(X_1/(\alpha_1+1)) \ge MSE(\tilde{\lambda}_1)$ for any $0 < \lambda_1 \le \lambda_2 < \infty$ if $\alpha_1 \le \alpha_2$ and $\alpha_2 \ge 1$.*

The proof is given in Appendix A.3.

From Table 2 it seems that $\alpha_2 \ge 2.5$ is sufficient for $\tilde{\lambda}_1$ to dominate $X_1/(\alpha_1+1)$ for any α_1, although by Corollary 4.1 we show that $\tilde{\lambda}_1$ dominates $X_1/(\alpha_1+1)$ if $\alpha_1 \le \alpha_2$ and $\alpha_2 \ge 1$. The range of positive coefficients for which $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ dominates $\sum_{i=1}^2 c_i X_i/(\alpha_i+1)$ is completely determined by the ratio $(\alpha_1+1)/(\alpha_2+1)$. If $(\alpha_1+1)/(\alpha_2+1)$ gets larger, the range gets larger. Thus if α_1 is large compared with α_2 we can get the uniform improvement for wide range of positive coefficients.

5. Concluding remarks

A comparison of the results given by Theorems 3.1 and 4.1 (or Tables 1 and 2) may be in order. Although we cannot give clear explanation, we will also point out possible reason of the difference of the two regions of c_1/c_2.

(i) For any c_1 and c_2 with opposite sign both MLE and modified MLE give uniform improvement over their competitors except for the case when α_1 is quite small (in case of MLE) or α_2 is quite small (in case of modified MLE). This implies that we can use these estimators safely to estimate between component of variance in a one-way random effects model.

(ii) Both MLE and modified MLE have larger MSE than their competitors for larger c_1/c_2 ($c_1/c_2 > \alpha_1/\alpha_2$ in case of MLE and $c_1/c_2 > (\alpha_1+1)/(\alpha_2+1)$ in case of modified MLE) when λ_2/λ_1 is sufficient large.

(iii) MLE has larger MSE than UE for the case $\lambda_1 = \lambda_2$ if $\rho(2-\rho-R)/\{(1-\rho)(R-\rho)\} < c_1/c_2 < \alpha_1/\alpha_2$. We note that MLE expands UE in this case, but this does not explain the possible improvement for the case $c_1 = 0$.

(iv) Modified MLE has larger MSE than its competitor for the case $\lambda_1 = \lambda_2$ if $-\infty \le c_1/c_2 < (\rho' - 2\rho R')/\{2\rho R' - (\rho'+1)\}$ when $2\rho R' > \rho' + 1$. We note that modified MLE shrinks $\sum_{i=1}^2 c_i X_i/(\alpha+1)$ although $X_i/(\alpha_i+1)$ itself is a shrinkage of the UE X_i/α_i.

Next, we give some results on the comparison of the two estimators $\sum_{i=1}^2 c_i\hat{\lambda}_i$ and $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ without proof. We have restricted ourselves to the case $\alpha_1 = \alpha_2 = \alpha$ because of a technical difficulty in evaluating the risk difference by the same sort of calculations given in Sections 3 and 4.

(i) $MSE(\sum_{i=1}^2 c_i\hat{\lambda}_i) \ge MSE(\sum_{i=1}^2 c_i\tilde{\lambda}_i)$ for any $0 < \lambda_1 \le \lambda_2$ if $|c_1/c_2| \le 1$.

(ii) For $\lambda_1 = \lambda_2, MSE(\sum_{i=1}^2 c_i\hat{\lambda}_i) < MSE(\sum_{i=1}^2 c_i\tilde{\lambda}_i)$ if and only if

$$\{-(4\alpha^2+2\alpha-1)c_1^2 - 2(2\alpha-1)c_1c_2 + (4\alpha^2+6\alpha+5)c_2^2\} + \frac{E_0[X_2I]}{E_0[X_1I]}\{(4\alpha^2+6\alpha+5)c_1^2 - 2(2\alpha-1)c_1c_2 - (4\alpha^2+2\alpha-1)c_2^2\} < 0.$$

In particular $MSE(\hat{\lambda}_1) < MSE(\tilde{\lambda}_1)$ for $\lambda_1 = \lambda_2$ if and only if $E_0(X_2I)/E_0(X_1I) < (4\alpha^2+2\alpha-1)/(4\alpha^2+6\alpha+5)$. By numerical evaluation we have found that this inequality is satisfied for $\alpha_1 = \alpha_2 > 1$. Thus we see that $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ does not improve $\sum_{i=1}^2 c_i\hat{\lambda}_i$ if $|c_1/c_2|$ is sufficiently large and $\alpha_1 = \alpha_2$ is moderately large.

(iii) For any c_1 and c_2, $MSE(\sum_{i=1}^2 c_i\hat{\lambda}_i) > MSE(\sum_{i=1}^2 c_i\tilde{\lambda}_i)$ if λ_1/λ_2 is sufficiently small. Thus $\sum_{i=1}^2 c_i\hat{\lambda}_i$ does not improve $\sum_{i=1}^2 c_i\tilde{\lambda}_i$ for any c_1 and c_2.

Finally, we should mention the case of more than two populations. In case of two populations we have partitioned the sample space into two subregions and have given the expressions of the estimators. Even in case of three populations we have to partition the sample space into six subregions and the expressions of the estimators become much more complicated. Although we believe that the technique used in this paper will be useful, we have not succeeded in obtaining explicit results unfortunately.

Appendix

A.1. *Proof of Lemma* 2.2.

Let

$$W = \frac{X_1}{\lambda_1} + \frac{X_2}{\lambda_2} \quad \text{and} \quad Z = \frac{\frac{X_1}{\lambda_1}}{\frac{X_1}{\lambda_1} + \frac{X_2}{\lambda_2}}.$$

Then W and Z are independent random variables having $Gamma(\alpha_1+\alpha_2, 1)$ distribution and $Beta(\alpha_1, \alpha_2)$ one, respectively. The random variables X_1 and X_2 can be expressed as

$$X_1 = \lambda_1 WZ, \quad \text{and} \quad X_2 = \lambda_2 W(1-Z)$$

respectively.

We first note that $X_1 \geq bX_2$ if and only if $Z \geq b\lambda_2/(b\lambda_2 + \lambda_1)$. If we set $\gamma = b\lambda_2/(b\lambda_2 + \lambda_1)$, we see that $\lambda_1 \leq \lambda_2$ if and only if $\gamma \geq b/(b+1)$.

Thus we have

$$\begin{aligned} E[bX_2 \mid X_1 \geq bX_2] &= b\lambda_2 E[W(1-Z) \mid Z \geq \gamma] \\ &= (\alpha_1+\alpha_2)b\lambda_2 E[1-Z \mid Z \geq \gamma] \quad \text{and} \\ E[X_1 \mid X_1 \geq bX_2] &= \lambda_1 E[WZ \mid Z \geq \gamma] \\ &= (\alpha_1+\alpha_2)(b\lambda_2+\lambda_1)(1-\gamma)E[Z \mid Z \geq \gamma]. \end{aligned}$$

Therefore

$$\frac{E[bX_2 I_{X_1 \geq bX_2}]}{E[X_1 I_{X_1 \geq bX_2}]} = \frac{E[bX_2 \mid X_1 \geq bX_2]}{E[X_1 \mid X_1 \geq bX_2]} = \frac{\gamma}{1-\gamma}\frac{E[1-Z \mid Z \geq \gamma]}{E[Z \mid Z \geq \gamma]} \equiv T(\gamma).$$

Since we show that $T(\gamma)$ is an increasing function of γ it is minimal when $\gamma = b/(b+1)$ or $\lambda_1 = \lambda_2$ and

$$\frac{E[bX_2 \mid X_1 \geq bX_2]}{E[X_1 \mid X_1 \geq bX_2]} \geq \frac{E_0[bX_2 \mid X_1 \geq bX_2]}{E_0[X_1 \mid X_1 \geq bX_2]} = b\frac{E[1-Z \mid Z \geq \rho]}{E[Z \mid Z \geq \rho]}.$$

Since Z is random variable with Beta distribution $Beta(\alpha_1, \alpha_2)$, we have

$$\begin{aligned} E[Z \mid Z \geq \rho] &= \frac{\int_\rho^1 z^{\alpha_1}(1-z)^{\alpha_2-1}dz}{\int_\rho^1 z^{\alpha_1-1}(1-z)^{\alpha_2-1}dz} \\ &= \frac{B(\alpha_1+1,\alpha_2)}{B(\alpha_1,\alpha_2)} \frac{\frac{1}{B(\alpha_1+1,\alpha_2)}\int_\rho^1 z^{\alpha_1+1-1}(1-z)^{\alpha_2-1}dz}{\frac{1}{B(\alpha_1,\alpha_2)}\int_\rho^1 z^{\alpha_1-1}(1-z)^{\alpha_2-1}dz} \\ &= \frac{\alpha_1}{\alpha_1+\alpha_2}\frac{1-I_\rho(\alpha_1+1,\alpha_2)}{1-I_\rho(\alpha_1,\alpha_2)}. \end{aligned}$$

To show that $T(\gamma)$ is an increasing function of γ, we express it as

$$T(\gamma) = \frac{\displaystyle\int_\gamma^1 \frac{\gamma}{1-\gamma} z^{\alpha_1-1}(1-z)^{\alpha_2} dz}{\displaystyle\int_\gamma^1 z^{\alpha_1}(1-z)^{\alpha_2-1} dz}.$$

In both integrals we make the change of variable $v = (1-z)/(1-\gamma)$ and have

$$T(\gamma) = \frac{\displaystyle\int_0^1 \frac{\gamma v}{1-(1-\gamma)v}\{1-(1-\gamma)v\}^{\alpha_1} v^{\alpha_2-1} dv}{\displaystyle\int_0^1 \{1-(1-\gamma)v\}^{\alpha_1} v^{\alpha_2-1} dv}.$$

If we put

$$f(v;\gamma) = \frac{\{1-(1-\gamma)v\}^{\alpha_1} v^{\alpha_2-1}}{\displaystyle\int_0^1 \{1-(1-\gamma)v\}^{\alpha_1} v^{\alpha_2-1} dv}$$

then $f(v;\gamma)$ is a density function with parameter γ, and $T(\gamma)$ is the expected value of $\varphi(v;\gamma) = \frac{\gamma v}{1-(1-\gamma)v}$, and we denote it as $E_\gamma[\varphi(V;\gamma)]$. We show that $f(v;\gamma)$ has monotone likelihood ratio in v. Suppose that $\gamma > \gamma'$. Then

$$\frac{f(v;\gamma)}{f(v;\gamma')} \sim \left(\frac{1-(1-\gamma)v}{1-(1-\gamma')v}\right)^{\alpha_1}$$

is an increasing function of v. Furthermore, since $\varphi(v;\gamma)$ is an increasing function of γ, we have

$$T(\gamma) = E_\gamma[\varphi(V;\gamma)] \geq E_{\gamma'}[\varphi(V;\gamma)] > E_{\gamma'}[\varphi(V;\gamma')] = T(\gamma').$$

This completes the proof.

A.2. *Proof of Corollary* 3.1.

From Theorem 3.1 we see that it is enough for us to show that

$$\text{(A.1)} \qquad \frac{\rho}{1-\rho}\frac{2-\rho-R}{R-\rho} \geq 0$$

or $R \leq 2-\rho$ if and only if $\alpha_1 \geq 1$, where $\rho = \alpha_1/(\alpha_1+\alpha_2)$ and $R = \{1 - I_\rho(\alpha_1 + 1, \alpha_2)\}/\{1 - I_\rho(\alpha_1, \alpha_2)\}$.

By applying an integration by parts we can easily show that

$$I_\rho(\alpha_1+1,\alpha_2) = I_\rho(\alpha_1,\alpha_2) - \frac{\rho^{\alpha_1}(1-\rho)^{\alpha_2}}{(\alpha_1+\alpha_2)B(\alpha_1+1,\alpha_2)}.$$

Thus we see that (A.1) is equivalent to

$$\text{(A.2)} \qquad \frac{1}{1-I_\rho(\alpha_1,\alpha_2)}\frac{\rho^{\alpha_1}(1-\rho)^{\alpha_2}}{(\alpha_1+\alpha_2)B(\alpha_1+1,\alpha_2)} \leq \frac{\alpha_2}{\alpha_1+\alpha_2}.$$

We note that

$$\begin{aligned}(\alpha_1+\alpha_2)&B(\alpha_1+1,\alpha_2)\{1-I_\rho(\alpha_1,\alpha_2)\}\\ &=\alpha_1\int_\rho^1 x^{\alpha_1-1}(1-x)^{\alpha_2-1}dx\\ &=(\alpha_1+\alpha_2)\rho^{\alpha_1}(1-\rho)^{\alpha_2}\int_0^1\left(1+\frac{1-\rho}{\rho}u\right)^{\alpha_1-1}(1-u)^{\alpha_2-1}du,\end{aligned}$$

if we make the change of variable

$$\frac{x-\rho}{1-\rho}=u.$$

Therefore we see that (A.2) is equivalent to the condition

$$\text{(A.3)}\qquad \int_0^1\left(1+\frac{1-\rho}{\rho}u\right)^{\alpha_1-1}(1-u)^{\alpha_2-1}du\ge\frac{1}{\alpha_2}.$$

Since $(1+\frac{1-\rho}{\rho}u)^{\alpha_1-1}\ge 1$ if and only if $\alpha_1\ge 1$ and since

$$\int_0^1(1-u)^{\alpha_2-1}du=\frac{1}{\alpha_2},$$

we see that (A.3) is true if and only if $\alpha_1\ge 1$.

A.3. *Proof of Corollary* 4.1.

We need only to show that if $\alpha_2\ge\alpha_1$ and $\alpha_2\ge 1$, then

$$\text{(A.4)}\qquad \rho R'\le(\rho'+1)/2.$$

By the same argument given in Appendix A.2 we can show that the inequality (A.4) is equivalent to the one

$$\text{(A.5)}\qquad \int_0^1\left(1+\frac{1-\rho'}{\rho'}u\right)^{\alpha_1-1}(1-u)^{\alpha_2-1}du\ge\frac{2(\alpha_1+1)}{\alpha_2(\alpha_1+\alpha_2+3)-\alpha_1}.$$

If we express the left-hand side of (A.5) as $\frac{1}{\alpha_2}E[(1+\frac{1-\rho'}{\rho'}U)^{\alpha_1-1}]$, where U is a random variable having Beta distribution $Beta(1,\alpha_2)$, then we see that the inequality (A.5) is equivalent to the one

$$\text{(A.6)}\qquad E\left[\left(1+\frac{1-\rho'}{\rho'}U\right)^{\alpha_1-1}\right]\ge\frac{2\alpha_2(\alpha_1+1)}{\alpha_2(\alpha_1+\alpha_2+3)-\alpha_1}.$$

When $\alpha_1\ge 1$, the left-hand side of (A.6) is greater or equal to 1, and the right-hand side of (A.6) is less than or equal to 1, if $\alpha_2\ge\alpha_1$. When $\alpha_1<1$, we first note that $(1+\frac{1-\rho'}{\rho'}u)^{\alpha_1-1}$ is a decreasing function of u. Thus we see that for $\alpha_2\ge 1$ the left-hand side of (A.6) is minimized when $\alpha_2=1$. Since the right-hand side of (A.6) is a decreasing function of α_2, we need only to show the inequality (A.6) for the case $\alpha_2=1$. In this case it reduces to the one $\{(\alpha_1+3)/(\alpha_1+1)\}^{\alpha_1}\ge\alpha_1+1$ which is true for $0<\alpha_1\le 1$.

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