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INFORMATION INEQUALITIES FOR THE BAYES RISK FOR A FAMILY OF NON-REGULAR DISTRIBUTIONS

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Abstract. For a family of non-regular distributions with a location parameter including the uniform and truncated distributions, the stochastic expansion of the Bayes estimator is given and the asymptotic lower bound for the Bayes risk is obtained and shown to be sharp. Some examples are also given.

Key words and phrases: Loss function, Bayes risk, Bayes estimator, lower bound, uniform distribution, truncated distributions.

1. Introduction

In the non-regular case, the Cramér-Rao type inequality was discussed by Vincze (1979), Khatri (1980), Móri (1983) and others (see also Akahira and Takeuchi (1995)). For a family of uniform distributions on $[\theta-(1/2), \theta+(1/2)]$, the information inequality for the Bayes risk of any estimator of θ under the quadratic loss and the uniform prior distribution on an interval $[-\tau, \tau]$ is exactly given and shown to have the sharp bound by Ohyauchi and Akahira (2000, 2001), where $\tau > 1/2$. Akahira and Takeuchi (2001) also show that for any $\tau > 0$ the Bayes risk of any estimator in the interval of θ values of length 2τ and centered at θ_0 can not be smaller than that of the mid-range. And the lower bound for the limit inferior of the Bayes risk of any estimator of θ as $\tau \to \infty$ is attained by the mid-range, which involves the result for unbiased estimators of θ by Móri (1983). For a family of symmetrically truncated normal distributions with a location parameter, the Bayes estimator with respect to the quadratic loss and the uniform prior distribution on the interval $[-\tau, \tau]$ is obtained and its stochastic expansion is given up to the order $o_p(n^{-1})$ as the size n of sample tends to infinity, and also the attainment of the lower bound is asymptotically discussed by Ohyauchi (2002) and Ohyauchi and Akahira (2001).

In this paper, for a family of non-regular distributions with a location parameter θ including the uniform and truncated ones, in a similar way to Akahira (1988) the stochastic expansion of the Bayes estimator is given up to the order $o_p(n^{-1})$, and the information inequality for the Bayes risk of any estimator of θ is asymptotically obtained.

2. Information inequalities for the Bayes risk

Suppose that $X_1, X_2, \dots, X_n, \dots$ is a sequence of independent and identically distributed random variables with a density $p(x-\theta)$ w.r.t. a σ-finite measure μ, where θ is a real-valued parameter. We also assume the following conditions (A1) to (A3).

(A1) $p(x) > 0$ for $a < x < b$; $p(x) = 0$ otherwise, where a and b are constants with $a < b$.

(A2) p is twice continuously differentiable in the open interval (a, b), and

$$\lim_{x \to a+0} p(x) = \lim_{x \to b-0} p(x) =: c > 0,$$
$$\lim_{x \to b-0} p'(x) = -\lim_{x \to a+0} p'(x) =: h \leq 0.$$

(A3) $0 < I := -\int_a^b \frac{d^2 \log p(x)}{dx^2} p(x) d\mu(x) < \infty$.

Let $\underline{\theta} := \max_{1 \leq i \leq n} X_i - b$ and $\overline{\theta} := \min_{1 \leq i \leq n} X_i - a$, and $L(u)$ be a loss function on $\boldsymbol{R}^1$ which is nonnegative-valued, three times continuously differentiable and monotone increasing in $|u|$. Now, we consider a uniform distribution π on an interval $[-\tau, \tau]$ as a prior distribution of θ. Then the Bayes estimator w.r.t. L and π is the estimator $\hat{\theta}$ minimizing

$$\int_A^B L(\hat{\theta} - \eta) \prod_{i=1}^n p(x_i - \eta) d\eta$$

for *a.a.* $(x_1, \dots, x_n)$, where $A := \max\{-\tau, \underline{\theta}\}$, $B := \min\{\tau, \overline{\theta}\}$. And the Bayes risk of an estimator $\hat{\theta}$ of θ is defined by

$$r_\tau(\hat{\theta}) := \frac{1}{2\tau} \int_{-\tau}^{\tau} E_\theta[L(\hat{\theta} - \theta)] d\theta.$$

Let $l(x) := \log p(x)$ for $a < x < b$, $l^{(i)}(x) := (d^i/dx^i) l(x)$ $(i = 1, 2)$, and

$$Z_1(\theta) := -\frac{1}{\sqrt{n}} \sum_{i=1}^n l^{(1)}(X_i - \theta), \qquad Z_2(\theta) := \frac{1}{\sqrt{n}} \sum_{i=1}^n l^{(2)}(X_i - \theta) + \sqrt{n} I$$

for $A \leq \theta \leq B$. Then the Bayes estimator of θ is obtained as a solution of the equation

$$\int_A^B L^{(1)}(\hat{\theta} - \eta) \prod_{i=1}^n p(x_i - \eta) d\eta = 0, \tag{2.1}$$

where $L^{(1)}(u) := (\partial/\partial u) L(u)$. Let $t := \sqrt{n}(\hat{\theta} - \theta)$ and $u = \sqrt{n}(\eta - \theta)$. Since $\sqrt{n}(A - \theta) = \max\{\sqrt{n}(-\tau - \theta), \sqrt{n}(\underline{\theta} - \theta)\}$ and $\sqrt{n}(B - \theta) = \min\{\sqrt{n}(\tau - \theta), \sqrt{n}(\overline{\theta} - \theta)\}$, it follows that, for $-\tau \leq \theta \leq \tau$, $\sqrt{n}(A - \theta) = \sqrt{n}(\underline{\theta} - \theta)$ and $\sqrt{n}(B - \theta) = \sqrt{n}(\overline{\theta} - \theta)$. From (2.1) we have

$$\int_{\sqrt{n}(\underline{\theta} - \theta)}^{\sqrt{n}(\overline{\theta} - \theta)} L^{(1)}\left(\frac{1}{\sqrt{n}}(t - u)\right) \left[\exp\left\{\sum_{i=1}^n \log p\left(x_i - \theta - \frac{u}{\sqrt{n}}\right)\right\}\right] \frac{1}{\sqrt{n}} du = 0 \tag{2.2}$$

hence, in order to obtain the Bayes estimator, it is enough to get a solution of t of the equation (2.2). We also assume that $L^{(1)}(0) = 0$ and put $U := n(\overline{\theta} - \theta)$ and $V := n(\underline{\theta} - \theta)$. Then we have the following (see also Akahira (1988)).

THEOREM 1. *Under the conditions* (A1) *to* (A3), *the Bayes estimator* $\hat{\theta}_B$ *of* θ *w.r.t. the loss function* L *and the prior uniform distribution* π *on* $[-\tau, \tau]$ *has the stochastic* *expansion*

$$n(\hat{\theta}_B - \theta) = \frac{1}{2}(U+V) + \frac{1}{12\sqrt{n}}Z_1(\theta)(U-V)^2 - \frac{I}{24n}(U+V)(U-V)^2 - \frac{b_3}{24b_2 n}(U-V)^2 + o_p\left(\frac{1}{n}\right),$$

where $-\tau < \theta < \tau$, $b_i := (d^i/du^i)L(0)$ $(i = 2,3)$ *and* $b_2 \neq 0$.

PROOF. Since $L^{(1)}(0) = 0$, it follows that

$$L^{(1)}\left(\frac{1}{\sqrt{n}}(t-u)\right) = \frac{1}{\sqrt{n}}(t-u)L^{(2)}(0) + \frac{1}{2n}(t-u)^2 L^{(3)}(0) + o_p\left(\frac{1}{n^2}\right) = \frac{b_2}{\sqrt{n}}(t-u) + \frac{b_3}{2n}(t-u)^2 + o_p\left(\frac{1}{n^2}\right). \tag{2.3}$$

We also have

$$\begin{aligned}
&\exp\left\{\sum_{i=1}^n \log p\left(x_i - \theta - \frac{u}{\sqrt{n}}\right)\right\} \\
&= \exp\left\{\sum_{i=1}^n l(x_i-\theta) - \frac{u}{\sqrt{n}}\sum_{i=1}^n l^{(1)}(x_i-\theta) + \frac{u^2}{2n}\sum_{i=1}^n l^{(2)}(x_i-\theta) + O_p\left(\frac{1}{n}\right)\right\} \\
&= \prod_{i=1}^n p(x_i-\theta)\left[1 - \frac{u}{\sqrt{n}}\sum_{i=1}^n l^{(1)}(x_i-\theta) + \frac{u^2}{2n}\sum_{i=1}^n l^{(2)}(x_i-\theta) + \frac{u^2}{2n}\left\{\sum_{i=1}^n l^{(1)}(x_i-\theta)\right\}^2 + O_p\left(\frac{1}{n}\right)\right] \\
&= \prod_{i=1}^n p(x_i-\theta)\left[1 + uZ_1(\theta) + \frac{u^2}{2\sqrt{n}}\left\{Z_2(\theta) - \sqrt{n}I(\theta)\right\} + \frac{u^2}{2}Z_1^2(\theta) + O_p\left(\frac{1}{n}\right)\right].
\end{aligned} \tag{2.4}$$

From (2.2) to (2.4) we obtain

$$\begin{aligned}
0 &= \frac{1}{\sqrt{n}}\int_{\sqrt{n}(\underline{\theta}-\theta)}^{\sqrt{n}(\bar{\theta}-\theta)}\left\{\frac{b_2}{\sqrt{n}}(t-u) + \frac{b_3}{2n}(t-u)^2\right\} \cdot\left\{1 + uZ_1 + \frac{u^2}{2}(Z_1^2 - I)\right\}du + O_p\left(\frac{1}{n^{5/2}}\right) \\
&= \frac{1}{\sqrt{n}}\int_{\sqrt{n}(\underline{\theta}-\theta)}^{\sqrt{n}(\bar{\theta}-\theta)}\left\{b_2(t-u) + b_2 Z_1 u(t-u) + \frac{b_2}{2}(Z_1^2 - I)u^2(t-u) + \frac{b_3}{2\sqrt{n}}(t-u)^2 + \frac{b_3}{2\sqrt{n}}Z_1 u(t-u)^2 + \frac{b_3}{4\sqrt{n}}(Z_1^2 - I)u^2(t-u)^2\right\}du + O_p\left(\frac{1}{n^{5/2}}\right),
\end{aligned} \tag{2.5}$$

where $Z_1 = Z_1(\theta)$ and $I = I(\theta)$. Let $S := (U+V)/2$ and $T := (U-V)/2$ for $-\tau < \theta < \tau$. From (2.5) we have

$$\begin{aligned}
0 = {} & 2b_2T\left(t - \frac{S}{\sqrt{n}}\right) + 2b_2Z_1T\left\{\frac{t}{\sqrt{n}}S - \frac{1}{3n}(3S^2+T^2)\right\} \\
& + \frac{b_2}{2}(Z_1^2 - I)\left\{\frac{2t}{3n}T(3S^2+T^2) - \frac{2}{n\sqrt{n}}TS(S^2+T^2)\right\} \\
& + \frac{b_3}{\sqrt{n}}T\left\{t^2 - \frac{2}{\sqrt{n}}tS + \frac{1}{3n}(3S^2+T^2)\right\} + \frac{b_3}{n}t^2Z_1TS \\
& + \frac{b_3t^2}{6n\sqrt{n}}(Z_1^2 - I)T(3S^2+T^2) + O_p\left(\frac{1}{n^2}\right),
\end{aligned}$$

which implies

$$\begin{aligned}
& b_2\sqrt{n}t\left\{1 + \frac{1}{\sqrt{n}}Z_1S + \frac{b_3}{2b_2\sqrt{n}}t - \frac{b_3}{b_2n}\left(1 - \frac{t}{2}Z_1\right)S + \frac{1}{6n}(Z_1^2 - I)(3S^2+T^2)\right\} \\
& = b_2S + \frac{b_2}{3\sqrt{n}}Z_1(3S^2+T^2) + \frac{b_2}{2n}(Z_1^2 - I)S(S^2+T^2) \\
& \quad - \frac{b_3}{6n}(3S^2+T^2) + O_p\left(\frac{1}{n^{3/2}}\right).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
n(\hat{\theta} - \theta) & = \sqrt{n}t \\
& = \left\{S + \frac{1}{3\sqrt{n}}Z_1(3S^2+T^2) + \frac{1}{2n}(Z_1^2 - I)S(S^2+T^2)\right. \\
& \qquad \left. - \frac{b_3}{6b_2n}(3S^2+T^2)\right\} \\
& \quad \cdot\left\{1 - \frac{1}{\sqrt{n}}Z_1S - \frac{b_3}{2b_2\sqrt{n}}t + \frac{b_3}{b_2n}\left(1 + \frac{t}{2}Z_1\right)S\right. \\
& \qquad \left. - \frac{1}{6n}(Z_1^2 - I)(3S^2+T^2) + \frac{1}{n}Z_1^2S^2 + \frac{b_3^2}{4b_2^2n}t^2\right\} + o_p\left(\frac{1}{n}\right) \\
& = S + \frac{1}{3\sqrt{n}}Z_1T^2 - \frac{I}{3n}ST^2 - \frac{b_3}{6b_2n}T^2 + o_p\left(\frac{1}{n}\right) \\
& = \frac{1}{2}(U+V) + \frac{1}{12\sqrt{n}}Z_1(U-V)^2 - \frac{I}{24n}(U+V)(U-V)^2 \\
& \quad - \frac{b_3}{24b_2n}(U-V)^2 + o_p\left(\frac{1}{n}\right)
\end{aligned}$$

for $-\tau < \theta < \tau$. Thus we complete the proof.

Henceforth, we assume that μ is the Lebesgue measure. Letting $I_0 := \int_a^b \{p'(x)\}^2 / p(x)dx$, we have $I = -2h + I_0$. Let $I_0 > 0$. Then we also obtain the following from Theorem 1.

THEOREM 2. *Under the conditions* (A1) *to* (A3), *for any fixed* $\tau > 0$ *the information inequality for the Bayes risk w.r.t. the quadratic loss L and the uniform prior* π *on*

$[-\tau,\tau]$ *of any estimator* $\hat{\theta} = \hat{\theta}(X_1,\dots,X_n)$ *of* θ *is given by*

$$\varliminf_{n\to\infty} n\left\{n^2 r_\tau(\hat{\theta}) - \frac{1}{2c^2}\right\} \geq -\frac{1}{2c^4}(3c^2-4h) - \frac{5I_0}{6c^4} =: B_2(c). \tag{2.6}$$

PROOF. Letting $L(u) = u^2$, we have from Theorem 1

$$\begin{aligned} n(\hat{\theta}_B - \theta) &= \frac{1}{2}(U+V) + \frac{1}{12\sqrt{n}} Z_1 (U-V)^2 \\ &\quad -\frac{I}{24n}(U+V)(U-V)^2 + o_p\left(\frac{1}{n}\right) \end{aligned} \tag{2.7}$$

for $-\tau < \theta < \tau$, which implies

$$\begin{aligned} E[n^2(\hat{\theta}_B - \theta)^2] &= \frac{1}{4}E[(U+V)^2] + \frac{I_0}{144n}[(U-V)^4 Z^2] \\ &\quad + \frac{\sqrt{I_0}}{12\sqrt{n}} E[(U+V)(U-V)^2 Z] \\ &\quad -\frac{I}{24n}E[(U+V)^2(U-V)^2] + o\left(\frac{1}{n}\right). \end{aligned} \tag{2.8}$$

Now, the second order asymptotic joint density of (U,V) is given by

(2.9)

$$f_{U,V}(u,v) := \begin{cases} c^2 e^{-c(u-v)}\left[1 + \frac{1}{n}\left\{-1 + 2c(u-v) \right.\right. & \\ \qquad + \frac{h}{4}((u+v)^2 + (u-v)^2) - \frac{c^2}{2}(u-v)^2 & \\ \qquad \left.\left. - \frac{h}{c}(u-v)\right\}\right] + o\left(\frac{1}{n}\right) & \text{for } v < 0 < u, \\ 0 & \text{otherwise} \end{cases}$$

(see Akahira (1991), p. 191). Letting $Z(\theta) := Z_1(\theta)/\sqrt{I_0}$, we have the second order asymptotic conditional density given $U = u$ and $V = v$ as

$$f_Z(z \mid u,v) = \phi(z) - \frac{h(u+v)}{\sqrt{nI_0}} z\phi(z) - \frac{K}{6\sqrt{n}I_0^{3/2}}(z^3 - 3z)\phi(z) + O\left(\frac{1}{n}\right) \tag{2.10}$$

for $-\infty < z < \infty$, where $\phi(z) = (1/\sqrt{2\pi})e^{-z^2/2}$ and

$$K = -\int_a^b \{l^{(1)}(x)\}^3 p(x)dx$$

(see Akahira (1996), p. 358). Since, for nonnegative integer p and q

$$\int_0^\infty c u^p e^{-cu} du = \frac{p!}{c^p}, \qquad \int_{-\infty}^0 c v^q e^{cv} dv = (-1)^q \frac{q!}{c^q},$$

it follows from (2.9) that

$$E(U^2) = \frac{2}{c^2} - \frac{6}{c^2 n} + \frac{6h}{c^4 n} + o\left(\frac{1}{n}\right),$$
$$E(V^2) = \frac{2}{c^2} - \frac{6}{c^2 n} + \frac{6h}{c^4 n} + o\left(\frac{1}{n}\right),$$
$$E(UV) = -\frac{1}{c^2} + \frac{3}{c^2 n} - \frac{2h}{c^4 n} + o\left(\frac{1}{n}\right),$$

hence

$$E\left[(U+V)^2\right] = \frac{2}{c^2} - \frac{6}{c^2 n} + \frac{8h}{c^4 n} + o\left(\frac{1}{n}\right). \tag{2.11}$$

From (2.9) and (2.10) we have

$$\begin{aligned} E\left[(U-V)^4 Z^2\right] &= E\left[(U-V)^4 E(Z^2 \mid U,V)\right] \\ &= E\left[(U-V)^4\right] + O\left(\frac{1}{n}\right) \\ &= \frac{120}{c^4} + O\left(\frac{1}{n}\right), \end{aligned} \tag{2.12}$$

$$\begin{aligned} &E\left[(U+V)(U-V)^2 Z\right] \\ &= E\left[(U+V)(U-V)^2 E(Z \mid U,V)\right] \\ &= E\left[(U+V)(U-V)^2 \left\{-\frac{h(U+V)}{\sqrt{nI_0}} + O\left(\frac{1}{n}\right)\right\}\right] \\ &= -\frac{40h}{c^4\sqrt{nI_0}} + O\left(\frac{1}{n}\right), \end{aligned} \tag{2.13}$$

$$E[(U+V)(U-V)^2] = E[U^3 - U^2V - UV^2 + V^3] = o(1). \tag{2.14}$$

From (2.8) and (2.11) to (2.14), we obtain

$$E[n^2(\hat\theta_B - \theta)^2] = \frac{1}{2c^2} - \frac{1}{2c^4 n}(3c^2 - 4h) - \frac{5I_0}{6c^4 n} + o\left(\frac{1}{n}\right)$$

for $-\tau < \theta < \tau$, since $I = -2h + I_0$. Hence for any estimator $\hat\theta$ of θ and any fixed $\tau > 0$

$$\begin{aligned} &\varliminf_{n\to\infty} n\left\{n^2 r_\tau(\hat\theta) - \frac{1}{2c^2}\right\} \\ &\geq \varliminf_{n\to\infty} n\left\{n^2 r_\tau(\hat\theta_B) - \frac{1}{2c^2}\right\} \\ &\geq \varliminf_{n\to\infty} n\left\{\frac{1}{2\tau}\int_{-\tau}^{\tau} E[n^2(\hat\theta_B - \theta)^2]d\theta - \frac{1}{2c^2}\right\} \\ &= -\frac{1}{2c^4}(3c^2 - 4h) - \frac{5I_0}{6c^4}. \end{aligned}$$

Thus we complete the proof.

Remark. In the second order asymptotic lower bound $B_2(c)(< 0)$ in (2.6), the first term represents the information on the endpoints a and b of the density p through (U, V) and the second one means the information on the central part of the distribution with p through $Z_1(\theta)$. The Bayes estimator $\hat{\theta}_B$ is seen to attain the lower bound $B(c)$.

COROLLARY. *Under the conditions* (A1) *and* (A2), *for any fixed* $\tau > 0$ *the information inequality for the Bayes risk w.r.t. the quadratic loss* L *and the uniform prior* π *on* $[-\tau, \tau]$ *of any estimator* $\hat{\theta}$ *of* θ *is given by*

$$\varliminf_{n\to\infty} n^2 r_\tau(\hat{\theta}) \geq \frac{1}{2c^2} =: B_1(c). \tag{2.15}$$

The first order asymptotic lower bound $B_1(c)$ *is attained by the mid-range* $\hat{\theta}_0 := (\underline{\theta}+\overline{\theta})/2$.

The proof of the first half is straightforward from Theorem 2, and that of the latter half is given since from (2.7)

$$n(\hat{\theta}_B - \theta) = \frac{1}{2}(U + V) + o_p(1) = \frac{1}{2}(\hat{\theta}_0 - \theta) + o_p(1)$$

and from (2.9)

$$E[n^2(\hat{\theta}_B - \theta)^2] = \frac{1}{4}E\left[(U + V)^2\right] + o(1) = \frac{1}{2c^2} + o(1).$$

3. Examples

As the application of Theorem 2, we consider the uniform and truncated distribution cases.

Example 1. If p is a density of the uniform distribution on the interval $[-1/2, 1/2]$, then $c = 1$, hence, from (2.15) we have $B_1(c) = 1/2$ as the first order asymptotic lower bound which coincides with that of Ohyauchi and Akahira (2000). In this case, the mid-range $\hat{\theta}_0 = (\min_{1\leq i\leq n} X_i + \max_{1\leq i\leq n} X_i)/2$ attains the lower bound $B_1(c)$.

Example 2. If p is a symmetrically truncated normal density, i.e.

$$p(x) = \begin{cases} ke^{-x^2/2} & \text{for } |x| < 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

where k is some constant, then $c = ke^{-1/8}$, $h = -c/2$ and $I_0 = 1 - c$, hence, from (2.6) we have

$$B_2(c) = -\frac{3c + 2}{2c^3} - \frac{5(1 - c)}{6c^4}$$

as the second order asymptotic lower bound which coincides with that of Ohyauchi (2002) (see also Ohyauchi and Akahira (2001)). In this case, it is shown from (2.1) that the

Bayes estimator $\hat{\theta}_B$ w.r.t. the quadratic loss and the uniform prior π on the interval $[-\tau, \tau]$ is given by

$$\hat{\theta}_B(\boldsymbol{X}) = \int_A^B \theta \exp\left\{-\frac{n}{2}(\bar{X}-\theta)^2\right\} d\theta \bigg/ \int_A^B \exp\left\{-\frac{n}{2}(\bar{X}-\theta)^2\right\} d\theta,$$

where $A := \max\{-\tau, X_{(n)}-(1/2)\}$ and $B := \min\{\tau, X_{(1)}+(1/2)\}$ with $X_{(1)} := \min_{1\le i\le n} X_i$ and $X_{(n)} := \max_{1\le i\le n} X_i$ and $\bar{X} = (1/n)\sum_{i=1}^n X_i$. Letting ϕ be the density of the standard normal distribution, we have

$$\hat{\theta}_B(\boldsymbol{X}) = \int_A^B \theta\phi(\sqrt{n}(\theta-\bar{X}))d\theta \bigg/ \int_A^B \phi(\sqrt{n}(\theta-\bar{X}))d\theta,$$

which attains the above lower bound $B_2(c)$ (see Ohyauchi (2002) and Ohyauchi and Akahira (2001)).

In this case the maximum likelihood estimator (MLE) $\hat{\theta}_{ML}$ of θ is given by

$$\hat{\theta}_{ML} := \begin{cases} \underline{\theta} & \text{for } \overline{X} \le \underline{\theta}, \\ \overline{\theta} & \text{for } \overline{X} \ge \overline{\theta}, \\ \overline{X} & \text{for } \underline{\theta} < \overline{X} < \overline{\theta}, \end{cases}$$

where $\underline{\theta} = X_{(n)} - (1/2)$ and $\overline{\theta} = X_{(1)} + (1/2)$, which is asymptotically equivalent to

$$\tilde{\theta}_n := \begin{cases} \underline{\theta} & \text{with probability } 1/2 + o(1), \\ \overline{\theta} & \text{with probability } 1/2 + o(1) \end{cases}$$

as $n \to \infty$ (see Akahira and Takeuchi (1981), Ohyauchi and Akahira (2001) and Ohyauchi (2001)). Since the asymptotic density of $n(\hat{\theta}_{ML} - \theta)$ is

$$f_{\hat{\theta}_{ML}}(t) = \frac{c}{2}e^{-c|t|} \qquad (-\infty < t < \infty),$$

for any fixed $\tau > 0$ the Bayes risk of the MLE is given by

$$R_{ML}(c) := \lim_{n\to\infty} n^2 r_\tau(\hat{\theta}_{ML}) = \frac{2}{c^2} > B_1(c) = \frac{1}{2c^2}. \tag{3.1}$$

From (3.1) it is seen that the MLE does not attain the first order asymptotic lower bound $B_1(c)$ and the Bayes risk of the MLE has four times as large magnitude as the bound $B_1(c)$.

Next we consider the maximum probability estimator (MPE) $\hat{\theta}_{MP}^r$ which is defined as that value of d maximizing

$$\int_{d-(r/n)}^{d+(r/n)} \prod_{i=1}^n p(x_i - \theta)d\theta$$

for a fixed number $r > 0$ (Weiss and Wolfowitz (1974)). Then we have

$$\hat{\theta}_{MP}^r = \begin{cases} \frac{1}{2}(\underline{\theta} + \overline{\theta}) & \text{for} \quad \overline{\theta} - \underline{\theta} < 2r/n \\ \overline{\theta} - (r/n) & \text{for} \quad \overline{X} > \overline{\theta} - (r/n), \ \overline{\theta} - \underline{\theta} > 2r/n, \\ \underline{\theta} + (r/n) & \text{for} \quad \overline{X} < \underline{\theta} + (r/n), \ \overline{\theta} - \underline{\theta} > 2r/n, \\ \overline{X} & \text{for} \quad \underline{\theta} + (r/n) \le \overline{X} \le \overline{\theta} - (r/n) \end{cases}$$

(see Akahira and Takeuchi (1981)). Since the asymptotic density of $n(\hat{\theta}^r_{MP} - \theta)$ is given by

$$f_{\hat{\theta}^r_{MP}}(t) = \begin{cases} ce^{-2c|t|} & \text{for} \quad |t| < r, \\ \dfrac{1}{2}ce^{-c(|t|+r)} & \text{for} \quad |t| \geq r, \end{cases}$$

it follows that for any fixed $\tau > 0$

$$R_{MP}(r;c) := \lim_{n\to\infty} n^2 r_\tau(\hat{\theta}^r_{MP}) = \left(\frac{3}{2c^2} + \frac{r}{c}\right) e^{-2cr} + \frac{1}{2c^2} > B_1(c).$$

It is noted that the MPE $\hat{\theta}^r_{MP}$ tends to the mid-range $(\underline{\theta} + \overline{\theta})/2$ as $r \to \infty$ and the MLE $\hat{\theta}_{ML}$ as $r \to 0$ which corresponds to the fact

$$\lim_{r\to\infty} R_{MP}(r;c) = \frac{1}{2c^2} = B_1(c),$$
$$\lim_{r\to 0} R_{MP}(r;c) = \frac{2}{c^2} = R_{ML}(c).$$

Example 3. If p is a symmetrically truncated density of the form

$$p(x) = \begin{cases} ce^{(1-x^2)^3} & \text{for } |x| < 1, \\ 0 & \text{otherwise}, \end{cases}$$

where c is some positive constant, then $h = 0$ and

$$0 < I_0 = 72c \int_0^1 x^2(1-x^2)^4 e^{(1-x^2)^3} dx < \infty,$$

hence, from (2.6) we have

$$B_2(c) = -\frac{3}{2c^2} - \frac{5I_0}{6c^4}$$

as the second order asymptotic lower bound which is attained by the Bayes estimator $\hat{\theta}_B$ w.r.t. the quadratic loss and the uniform prior π on $[-\tau, \tau]$.

References

Akahira, M. (1988) Second order asymptotic properties of the generalized Bayes estimators for a family of non-regular distributions, *Statistical Theory and Data Analysis II*, 87–100, Elsevier Science Publ. B. V. (North-Holland), Amsterdam.

Akahira, M. (1991). The 3/2th and 2nd order asymptotic efficiency of maximum probability estimators in non-regular cases, *Ann. Inst. Statist. Math.*, **43**, 181–195.

Akahira, M. (1996). Loss of information of a statistic for a family of non-regular distributions, *Ann. Inst. Statist. Math.*, **48**, 349–364.

Akahira, M. and Takeuchi, K. (1981). *Asymptotic Efficiency of Statistical Estimators: Concepts and Higher Order Asymptotic Efficiency*, Lecture Notes in Statist., **7**, Springer, New York.

Akahira, M. and Takeuchi, K. (1995). *Non-Regular Statistical Estimation*, Lecture Notes in Statist., **107**, Springer, New York.

Akahira, M. and Takeuchi, K. (2001). Information inequalities in a family of uniform distributions, *Ann. Inst. Statist. Math.*, **53**, 427–435.

Khatri, C. G. (1980). Unified treatment of Cramér-Rao bound for the non-regular density functions, *J. Statist. Plann. Inference*, **4**, 75–79.

Móri, T. F. (1983). Note on the Cramér-Rao inequality in the non-regular case: The family of uniform distributions, *J. Statist. Plann. Inference*, **4**, 353–358.

Ohyauchi, N. (2002). Comparison of the Bayes risks of estimators for a family of truncated normal distributions, *Commun. Statist. Theory Methoths*, **31**, 699–718.

Ohyauchi, N. and Akahira, M. (2000). An information inequality for the Bayes risk in a family of uniform distributions (submitted).

Ohyauchi, N. and Akahira, M. (2001). On lower bounds for the Bayes risk of estimators in the uniform and truncated normal cases, *Proceedings of the Symposium, Research Institute for Mathematical Sciences, Kyoto University*, **1224**, 11–35.

Vincze, I. (1979). On the Cramér-Fréchet-Rao inequality in the non-regular case, *Contributions to Statistics. The J. Hájek Memorial Volume*, 253–262, Academia, Prague.

Weiss, L. and Wolfowitz, J. (1974). *Maximum Probability Estimators and Related Topics*, Lecture Notes in Mathematics, **424**, Springer, Berlin.