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ON A MULTIPARAMETER VERSION OF TUKEY'S LINEAR SENSITIVITY MEASURE AND ITS PROPERTIES

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Abstract. A multiparameter version of Tukey's (1965, *Proc. Nat. Acad. Sci. U.S.A.*, **53**, 127–134) linear sensitivity measure, as a measure of informativeness in the joint distribution of a given set of random variables, is proposed. The proposed sensitivity measure, under some conditions, is a matrix which is non-negative definite, weakly additive, monotone and convex. Its relation to Fisher information matrix and the best linear unbiased estimator (BLUE) are investigated. The results are applied to the location-scale model and it is observed that the dispersion matrix of the BLUE of the vector location-scale parameter is the inverse of the sensitivity measure. A similar property was established by Nagaraja (1994, *Ann. Inst. Statist. Math.*, **46**, 757–768) for the single parameter case when applied to the location and scale models. Two illustrative examples are included.

Key words and phrases: BLUE, Fisher information, location-scale model, multiparameter, sensitivity measure.

1. Introduction

Tukey (1965) proposed a linear sensitivity measure in a set of order statistics as a measure of informativeness about a real parameter. Nagaraja (1994) discussed this measure at length and investigated the merits and demerits in the light of the desirable properties expected of an information measure as given by Ferentinos and Papaioannou (1981); see also Gokhale and Kullback (1977).

It is well known that Fisher information measure is applicable only if the underlying distribution belongs to a regular family. The concept is well connected with exact as well as asymptotic theory of point estimation (Rao (1973) and Lehmann and Casella (1998)). For an introduction to linear sensitivity measure and its implications in statistical theory and practice, the reader is referred to Tukey (1965).

In this paper, we propose a multiparameter version of the linear sensitivity measure and study some of its properties. Further, its relation to optimal simultaneous estimation is obtained. The linear sensitivity measure, under some conditions, is a matrix which is non-negative definite, weakly additive, monotone and convex. Its relation to Fisher information contained in the distribution of a vector unbiased estimator and to the BLUE of a vector parametric function are discussed. In the case of a two-parameter location-scale model, it is observed that the dispersion matrix of the BLUE of the location-scale parameter vector is the inverse of the sensitivity measure. This suggests that one can propose measures of efficiency for a vector estimator, in non-regular cases also, based on the extended version of Tukey's sensitivity measure. Two illustrative examples are included.

The paper is organized as follows: Section 2 provides the notations and definitions required in the sequel. Section 3 establishes weak additivity and monotonicity of the proposed measure. The measure associated with a single observation is shown to be convex in the matrix sense. Section 4 deals with the relation to Fisher information matrix. The relation to BLUE is studied in Section 5. Section 6 deals with the BLUE of the vector location-scale parameter. Finally, the last section provides two illustrative examples.

2. Definitions and notations

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_r)'$ be a random vector with distribution function $F_{\boldsymbol{\theta}}$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)'$. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)'$ and Σ denote, respectively, the mean vector and the positive definite dispersion matrix of $\boldsymbol{Y}$. Assume that the partial derivatives

$$d_{ij} = \frac{\partial \mu_i}{\partial \theta_j}, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, k$$

exist and let $D = ((d_{ij}))$.

For $k = 1$, Tukey (1965) proposed the following linear sensitivity measure

$$S(\boldsymbol{Y}; \theta) = \boldsymbol{d}'\Sigma^{-1}\boldsymbol{d}, \tag{2.1}$$

where $\boldsymbol{d}$ is the column vector of derivatives of the means of $Y_1, \ldots, Y_r$ with respect to θ.

The following theorem provides a motivation for introducing the concept of linear sensitivity measure in the multiparameter situation.

THEOREM 2.1. *For any matrix A of order $k \times r$,*

$$D'\Sigma^{-1}D - D'A'(A\Sigma A')^{-1}AD$$

is non-negative definite for all $\boldsymbol{\theta}$. Further, the difference matrix is null if $A = CD'\Sigma^{-1}$, for some non-singular matrix C which may depend on $\boldsymbol{\theta}$.

PROOF. The dispersion matrix of

$$\begin{pmatrix} D'\Sigma^{-1}\boldsymbol{Y} \\ A\boldsymbol{Y} \end{pmatrix}$$

may be partitioned as

$$\begin{bmatrix} D'\Sigma^{-1}D & D'A' \\ AD & A\Sigma A' \end{bmatrix}.$$

Assuming $A\Sigma A'$ to be non-singular, we have the non-negative definiteness of $D'\Sigma^{-1}D - D'A'(A\Sigma A')^{-1}AD$. The nullity of the difference matrix is obvious, when $A = CD'\Sigma^{-1}$.

DEFINITION 2.1. The linear sensitivity measure contained in the distribution of $\boldsymbol{Y}$ about $\boldsymbol{\theta}$ is

$$S(\boldsymbol{Y}; \boldsymbol{\theta}) = \sup_A D'A'(A\Sigma A')^{-1}AD,$$

where supremum is taken over those A for which $A\Sigma A'$ is non-singular.

Recall that for any two square matrices E and F of same order, $E \geq F$ if $E - F$ is non-negative definite.

From Theorem 2.1, it follows that

$$\sup_A D'A'(A\Sigma A')^{-1}AD = D'\Sigma^{-1}D$$

and thus

$$S(\boldsymbol{Y};\boldsymbol{\theta}) = D'\Sigma^{-1}D. \tag{2.2}$$

Remark 2.1. If $k = 1$, the above linear sensitivity measure readily reduces to that of Tukey (1965) given in (2.1) in the context of order statistics and studied by Nagaraja (1994). If one is interested in a scalar sensitivity measure in the multiparameter case, the trace or the determinant of the above-mentioned matrix measure may be taken.

3. Weak additivity, monotonicity and convexity

It is obvious that the proposed matrix sensitivity measure is non-negative definite. In this section, we establish, under some conditions, weak additivity, monotonicity and convexity properties of the measure.

The following theorem establishes the weak additivity property.

THEOREM 3.1. *Let* $\boldsymbol{U} = (U_1, \ldots, U_n)'$ *and* $\boldsymbol{V} = (V_1, \ldots, V_r)'$ *be two uncorrelated random vectors whose distributions depend on* $\boldsymbol{\theta}$. *Then,*

$$S(\boldsymbol{U}, \boldsymbol{V};\boldsymbol{\theta}) = S(\boldsymbol{U};\boldsymbol{\theta}) + S(\boldsymbol{V};\boldsymbol{\theta}), \qquad \forall\boldsymbol{\theta}.$$

PROOF. Let $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$ denote the mean vectors of $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, and let Σ_1 and Σ_2 denote their dispersion matrices. Define

$$d_{ij}^{(1)} = \partial\mu_i^{(1)}/\partial\theta_j, \qquad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, k$$

and

$$d_{ij}^{(2)} = \partial\mu_i^{(2)}/\partial\theta_j, \qquad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, k.$$

Further, let

$$D_1 = ((d_{ij}^{(1)})) \qquad \text{and} \qquad D_2 = ((d_{ij}^{(2)})).$$

Since $\boldsymbol{U}$ and $\boldsymbol{V}$ are uncorrelated, we have

$$\begin{aligned} S(\boldsymbol{U}, \boldsymbol{V};\boldsymbol{\theta}) &= (D_1' D_2') \begin{bmatrix} \Sigma_1^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma_2^{-1} \end{bmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \\ &= D_1'\Sigma_1^{-1}D_1 + D_2'\Sigma_2^{-1}D_2 \\ &= S(\boldsymbol{U};\boldsymbol{\theta}) + S(\boldsymbol{V};\boldsymbol{\theta}), \qquad \forall\boldsymbol{\theta}. \end{aligned}$$

Hence, the measure is weakly additive.

In order to establish the monotonicity of S and Theorem 3.2, we introduce the following notations in addition to those of Section 2. For $\boldsymbol{Z} = (Y_1, \ldots, Y_r, Y_{r+1})'$, let

$$\begin{aligned} d_j &= \frac{\partial}{\partial \theta_j} E_\theta(Y_{r+1}), \qquad j = 1, 2, \ldots, k, \\ d &= (d_1, \ldots, d_k)', \\ \sigma_i &= \mathrm{cov}_\theta(Y_i, Y_{r+1}), \qquad i = 1, 2, \ldots, r, \\ \sigma &= (\sigma_1, \ldots, \sigma_r)', \end{aligned}$$

and

$$\lambda = V_\theta(Y_{r+1}).$$

THEOREM 3.2. *If* $\boldsymbol{Y} = (Y_1, \ldots, Y_r)'$ *and* $\boldsymbol{Z} = (Y_1, \ldots, Y_r, Y_{r+1})'$, *then*

$$S(\boldsymbol{Z}; \boldsymbol{\theta}) = S(\boldsymbol{Y}; \boldsymbol{\theta}) + (d - D'\Sigma^{-1}\sigma)(d - D'\Sigma^{-1}\sigma)'/(\lambda - \sigma'\Sigma^{-1}\sigma), \qquad \forall \boldsymbol{\theta}.$$

PROOF. The dispersion matrix of $\boldsymbol{Z}$ may be partitioned as

$$\Sigma_0 = \begin{bmatrix} \Sigma & \sigma \\ \sigma' & \lambda \end{bmatrix}.$$

Following Graybill (1983),

$$\Sigma_0^{-1} = \begin{bmatrix} (\Sigma - \sigma\sigma'/\lambda)^{-1} & -\Sigma^{-1}\sigma/(\lambda - \sigma'\Sigma^{-1}\sigma) \\ -\sigma'\Sigma^{-1}/(\lambda - \sigma'\Sigma^{-1}\sigma) & 1/(\lambda - \sigma'\Sigma^{-1}\sigma) \end{bmatrix}.$$

Thus,

$$\begin{aligned} S(\boldsymbol{Z}; \boldsymbol{\theta}) &= (D'd)\Sigma_0^{-1}\begin{pmatrix} D \\ d' \end{pmatrix} \\ &= D'(\Sigma - \sigma\sigma'/\lambda)^{-1}D - d\sigma'\Sigma^{-1}D/(\lambda - \sigma'\Sigma^{-1}\sigma) \\ &\quad -D'\Sigma^{-1}\sigma d'/(\lambda - \sigma'\Sigma^{-1}\sigma) + dd'/(\lambda - \sigma'\Sigma^{-1}\sigma). \end{aligned} \tag{3.1}$$

Recall that (Graybill (1983))

$$(A + cd')^{-1} = A^{-1} - \frac{(A^{-1}c)(d'A^{-1})}{1 + d'A^{-1}c}$$

for any square matrix A and two vectors c and d, provided the inverses exist. Now take $A = \Sigma$, $c = -\sigma/\lambda$ and $d = \sigma$. Then

$$(\Sigma - \sigma\sigma'/\lambda)^{-1} = \Sigma^{-1} + \frac{\Sigma^{-1}\sigma\sigma'\Sigma^{-1}}{\lambda - \sigma'\Sigma^{-1}\sigma}.$$

Thus, (3.1) gives

$$\begin{aligned} S(\boldsymbol{Z}; \boldsymbol{\theta}) &= D'\Sigma^{-1}D + (D'\Sigma^{-1}\sigma\sigma'\Sigma^{-1}D - d\sigma'\Sigma^{-1}D \\ &\qquad -D'\Sigma^{-1}\sigma d' + dd')/(\lambda - \sigma'\Sigma^{-1}\sigma) \\ &= S(\boldsymbol{Y}; \boldsymbol{\theta}) + (d - D'\Sigma^{-1}\sigma)(d - D'\Sigma^{-1}\sigma)'/(\lambda - \sigma'\Sigma^{-1}\sigma). \end{aligned} \tag{3.2}$$

Based on this theorem, we have the monotonicity property.

COROLLARY 3.1. *If* $\boldsymbol{Y} = (Y_1, \ldots, Y_r)'$ *and* $\boldsymbol{Z} = (Y_1, \ldots, Y_r, y_{r+1})'$, *then*

$$S(\boldsymbol{Y}; \boldsymbol{\theta}) \leq S(\boldsymbol{Z}; \boldsymbol{\theta}), \qquad \forall \boldsymbol{\theta}.$$

PROOF. The proof is an immediate consequence of (3.2) since the second term on the right side is a non-negative definite matrix.

Remark 3.1. The proof of Corollary 3.1 involves in considering Definition 2.1 for $S(\boldsymbol{Z}; \boldsymbol{\theta})$ and choosing the last column of A to be a null vector.

Remark 3.2. From (3.2), it is clear that the increase is a null matrix if and only if $d - D'\Sigma^{-1}\sigma = \mathbf{0}$. If the matrix Σ_0 does not depend on $\boldsymbol{\theta}$, following the arguments similar to those of Nagaraja (1994), it can be shown that the increase in the measure is null if and only if $E_\theta(Y_{r+1}) - E_\theta(\boldsymbol{Y}')\Sigma^{-1}\sigma$ does not depend on $\boldsymbol{\theta}$. Such a condition may be satisfied if $\boldsymbol{Z}$ has a multivariate normal distribution.

In the following theorem, we establish the convexity property of $S(Y; \boldsymbol{\theta})$ when $r = 1$, if some conditions are satisfied.

THEOREM 3.3. *The sensitivity measure, associated with a single observation, is convex.*

PROOF. Since $r = 1$,

$$S(Y; \boldsymbol{\theta}) = D'D/V_{\boldsymbol{\theta}}(Y).$$

Let

$$F_{\boldsymbol{\theta}} = \alpha F_{\boldsymbol{\theta}}^{(1)} + (1 - \alpha) F_{\boldsymbol{\theta}}^{(2)}, \qquad 0 < \alpha < 1,$$

and U_i denote a random variable whose distribution function is $F_{\boldsymbol{\theta}}^{(i)}$, $i = 1, 2$. Here, $F_{\boldsymbol{\theta}}$ is the distribution function of Y. We then wish to show that

$$S(Y; \boldsymbol{\theta}) \leq \alpha S(U_1; \boldsymbol{\theta}) + (1 - \alpha) S(U_2; \boldsymbol{\theta}), \qquad \forall \boldsymbol{\theta}.$$

Let $\mu^{(i)}$ denote the mean of $U^{(i)}$, $i = 1, 2$, and D_i denote the row vector of partial derivatives of $\mu^{(i)}$ with respect to $\theta_1, \ldots, \theta_k$, $i = 1, 2$. Finally, let σ^2, σ_1^2 and σ_2^2 denote the variances of Y, U_1 and U_2, respectively. Then,

$$D = \alpha D_1 + (1 - \alpha) D_2$$

and

$$\sigma^2 = \alpha \sigma_1^2 + (1 - \alpha) \sigma_2^2 + \alpha (1 - \alpha) (\mu^{(1)} - \mu^{(2)})^2.$$

It can be verified that

$$S(Y; \boldsymbol{\theta}) \leq \alpha S(U_1; \boldsymbol{\theta}) + (1 - \alpha) S(U_2; \boldsymbol{\theta}) \tag{3.3}$$

holds if and only if

$$(\mu^{(1)} - \mu^{(2)})^2 \{\alpha D_1' D_1 \sigma_2^2 + (1 - \alpha) D_2' D_2 \sigma_1^2\} + (D_1 \sigma_2^2 - D_2 \sigma_1^2)'(D_1 \sigma_2^2 - D_2 \sigma_1^2) \geq 0 \tag{3.4}$$

holds. Since (3.4) is true, (3.3) holds and hence the theorem.

Remark 3.3. The convexity of S when $r > 1$ seems to be an open problem.

Remark 3.4. Ferentinos and Papaioannou (1981) have given an extensive list of desirable properties that are expected to be satisfied by any measure of information. Some of these desirable properties such as conditional inequality, strong additivity and subadditivity have been shown by Nagaraja (1994) to be not satisfied by the linear sensitivity measure in the scalar parameter case. Naturally, these properties will therefore be not satisfied by the linear sensitivity measure proposed here for the multiparameter case. Furthermore, as the examples presented in Section 7 reveal, the linear sensitivity measure proposed in this paper also suffers from the fact that it is invariant under sufficient transformations only when the sufficient statistic T for $\boldsymbol{\theta}$ is a linear function of $\boldsymbol{Y}$, as has been pointed out earlier by Nagaraja (1994) for the scalar parameter case.

4. Sensitivity measure and Fisher information matrix

Let $J(\boldsymbol{\theta})$ denote the Fisher information contained in the distribution of an unbiased estimator $\boldsymbol{\delta}$ of $\boldsymbol{\mu}(\boldsymbol{\theta})$. Under certain regularity conditions (Rao (1973)), the matrix $\Sigma_{\boldsymbol{\delta}} - DJ^{-1}(\boldsymbol{\theta})D'$ is non-negative definite, where $\Sigma_{\boldsymbol{\delta}}$ is the dispersion matrix of $\boldsymbol{\delta}$ and the (i,j)-th element of D is $\partial\mu_i/\partial\theta_j$ $\forall i,j$. Equivalently,

$$D'\{DJ^{-1}(\boldsymbol{\theta})D'\}^{-1}D - D'\Sigma_{\boldsymbol{\delta}}^{-1}D$$

is non-negative definite. In other words, $D'\{DJ^{-1}(\boldsymbol{\theta})D'\}^{-1}D$ may be viewed as an upper bound for the sensitivity measure $S(\boldsymbol{\delta},\boldsymbol{\theta})$.

In particular, when $r = k$ and D is non-singular, the linear sensitivity measure is a lower bound of Fisher's measure of information.

When $\boldsymbol{\theta}$ is scalar (that is, in the one-parameter case), Nagaraja (1994) has in fact shown this result. In addition, he has shown that the Fisher information contained in a statistic T about the parameter θ equals the linear sensitivity measure in T if and only if T has a one-parameter exponential family density; see also Kullback (1985) and Zografos and Ferentinos (1994) for some additional insight into this property. Using the same arguments here, it can be shown that $J(\boldsymbol{\delta},\boldsymbol{\theta})$ and $S(\boldsymbol{\delta},\boldsymbol{\theta})$ are equal if and only if the distribution of $\boldsymbol{\delta}$ is of exponential type.

5. Sensitivity measure and BLUE

In this section, a relation between sensitivity measure and the BLUE for the simultaneous estimation problem is established in the multiparameter situation.

THEOREM 5.1. *If there exists a non-singular matrix $C(\boldsymbol{\theta})$ such that $A_0 = CD'\Sigma^{-1}$ does not depend on $\boldsymbol{\theta}$, then $A_0\boldsymbol{Y}$ is BLUE for estimating $A_0\boldsymbol{\mu}$.*

PROOF. By Theorem 2.1,

$$D'\Sigma^{-1}D - D'A'(A\Sigma A')^{-1}AD \geq 0, \qquad \forall A. \tag{5.1}$$

Let AY be an arbitrary linear unbiased estimator of $A_0\boldsymbol{\mu}$ with finite dispersion matrix. Then

$$A\boldsymbol{\mu} = A_0\boldsymbol{\mu}, \qquad \forall\boldsymbol{\theta}$$

and therefore

(5.2) $$AD = A_0D, \qquad \forall\boldsymbol{\theta}.$$

By the definition of A_0, invoking Theorem 2.1, we have

$$\begin{aligned} D'\Sigma^{-1}D &= D'(A_0)'(A_0\Sigma A_0')^{-1}A_0D \\ &= D'A'(A_0\Sigma A_0')^{-1}AD, \end{aligned}$$

in view of (5.2).

Now (5.1) gives

$$(AD)'\{(A_0\Sigma A_0')^{-1} - (A\Sigma A')^{-1}\}AD \geq 0, \qquad \forall A.$$

In other words,

$$(A_0\Sigma A_0')^{-1} - (A\Sigma A')^{-1} \geq 0, \qquad \forall A.$$

This implies the non-negative definiteness of

$$A\Sigma A' - A_0\Sigma A_0', \qquad \forall A.$$

Hence, $A_0\boldsymbol{Y}$ is the BLUE of $A_0\boldsymbol{\mu}$.

6. Sensitivity measure and the location-scale model

Let the population from which the sample is drawn be a location-scale model with probability density function (pdf)

$$f(x;\xi,\tau) = \frac{1}{\tau}g((x-\xi)/\tau), \qquad x \in \boldsymbol{R}; \quad \xi \in \boldsymbol{R}, \ \tau > 0.$$

Let $Y_{i:n}$, $i = 1,2,\ldots,n$, denote the order statistics based on a random sample of size n. Let $Z_{i:n} = (Y_{i:n} - \xi)/\tau$, $i = 1,2,\ldots,n$, denote the standard order statistics, whose distributions do not depend on $\boldsymbol{\theta} = (\xi,\tau)'$. If $\boldsymbol{\alpha}$ and B denote the mean vector and the dispersion matrix of $\boldsymbol{Z} = (Z_{1:n},\ldots,Z_{n:n})'$, it is known that the mean vector and the dispersion matrix of $\boldsymbol{Y}$ are

$$\boldsymbol{\mu} = \xi\mathbf{1} + \tau\boldsymbol{\alpha} \qquad \text{and} \qquad \Sigma = \tau^2 B.$$

Thus, $D = (\mathbf{1}\ \boldsymbol{\alpha})$ and

$$D'\Sigma^{-1} = \frac{1}{\tau^2}D'B^{-1}.$$

Note that the matrices D and B do not depend on $\boldsymbol{\theta}$.

Since we are interested in finding the BLUE of $\boldsymbol{\theta}$, it is sufficient to find a matrix C such that $CD'\Sigma^{-1}Y$ is a statistic and is unbiased for $\boldsymbol{\theta}$.

Consider

$$E_{\boldsymbol{\theta}}(CD'\Sigma^{-1}\boldsymbol{Y}) = \boldsymbol{\theta}, \qquad \forall\boldsymbol{\theta}.$$

This implies that

$$C = (D'\Sigma^{-1}D)^{-1} = \tau^2(D'B^{-1}D)^{-1}.$$

Clearly, $CD'\Sigma^{-1} = (D'B^{-1}D)^{-1}D'B^{-1}$ and does not depend on $\boldsymbol{\theta}$. Thus, by Theorem 5.1, it follows that the BLUE of $\boldsymbol{\theta}$ is

$$\hat{\boldsymbol{\theta}} = (D'B^{-1}D)^{-1}D'B^{-1}\boldsymbol{Y}.$$

The dispersion matrix of $\hat{\boldsymbol{\theta}}$ is $\tau^2(D'B^{-1}D)^{-1}$ or $(D'\Sigma^{-1}D)^{-1}$. Thus, the dispersion matrix of the BLUE of $\boldsymbol{\theta}$ is the inverse of the sensitivity measure.

Remark 6.1. It may be noted that a similar observation was made by Nagaraja (1994) for the location model and the scale model. Further, the above discussion extends to the conventional Type-II censored, progressive Type-II censored, and generalized progressive Type-II censored samples. The above discussion also suggests that one can propose efficiency measures for linear unbiased estimators based on the matrix sensitivity measure or a real-valued function of the same.

7. Examples

In this section, we illustrate the results of the paper through two examples.

Example 7.1. Let $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ denote a set of order statistics based on a random sample of size n from $Uniform(\xi, \xi+\tau)$ distribution. Following the notations of Section 6, we have (Arnold *et al.* (1992), pp. 14–20)

$$\alpha_i = i/(n+1), \qquad i = 1, 2, \ldots, n,$$
$$b_{ij} = i(n-j+1)/\{(n+1)^2(n+2)\}, \qquad i \le j.$$

We wish to find the sensitivity measure associated with $\boldsymbol{Y} = (Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n})'$. Noting that

$$b^{ij} = \begin{cases} 2(n+1)(n+2), & |i-j| = 0 \\ -(n+1)(n+2), & |i-j| = 1 \\ 0, & |i-j| \ge 2 \end{cases}$$

where $B^{-1} = ((b^{ij}))$, we have

$$\begin{aligned} S(\boldsymbol{Y};\boldsymbol{\theta}) &= \frac{1}{\tau^2} D'B^{-1}D \\ &= \frac{(n+1)(n+2)}{\tau^2} \begin{bmatrix} 2 & 1 \\ 1 & n/(n+1) \end{bmatrix}. \end{aligned}$$

By definition,

$$\begin{aligned} S(Y_{1:n}, Y_{n:n};\boldsymbol{\theta}) &= \frac{(n+1)(n+2)}{(n-1)\tau^2} \begin{bmatrix} 1 & 1 \\ \dfrac{1}{n+1} & \dfrac{n}{n+1} \end{bmatrix} \begin{bmatrix} n & -1 \\ -1 & n \end{bmatrix} \begin{bmatrix} 1 & \dfrac{1}{n+1} \\ 1 & \dfrac{n}{n+1} \end{bmatrix} \\ &= \frac{(n+1)(n+2)}{\tau^2} \begin{bmatrix} 2 & 1 \\ 1 & \dfrac{n}{n+1} \end{bmatrix} \\ &= S(\boldsymbol{Y};\boldsymbol{\theta}). \end{aligned}$$

This $S(Y_{1:n}, Y_{n:n};\boldsymbol{\theta}) = S(\boldsymbol{Y};\boldsymbol{\theta}) \forall \boldsymbol{\theta}$ means that the linear sensitivity measure associated with the whole observed sample $\boldsymbol{Y}$ is exactly equal to the linear sensitivity measure associated with the two order statistics $(Y_{1:n}, Y_{n:n})$, which form a complete sufficient statistic.

Remark 7.1. In this case, Fisher information does not exist, while we have shown that the linear sensitivity measure does exist, and that it is the same as that of $(Y_{1:n}, Y_{n:n})$ which is a complete sufficient statistic. In addition, the results of David (1981) may be invoked to find similarly the linear sensitivity measure in a doubly Type-II censored sample as well.

Example 7.2. Consider a progressively censored sample from a two-parameter exponential distribution with the location-scale parameter $\boldsymbol{\theta} = (\xi, \tau)'$. Let $Y_{i:n:N}$, $i = 1, 2, \ldots, n$, denote the progressively censored order statistics based on a sample of size N, where n is the number of observed failures and $r_1, r_2, \ldots, r_{n-1}, r_n$ denote the number of items censored at each failure. Note that $r_n = N - \sum_{j=1}^{n-1} r_j - n$. Define

$$Z_i = (Y_{i:n:N} - \xi)/\tau, \qquad i = 1, 2, \ldots, n.$$

From Viveros and Balakrishnan (1994), we have

$$E(Z_1) = N^{-1}$$

and

$$E(Z_i) = N^{-1} + \sum_{k=2}^{i} \left\{ N - \sum_{j=1}^{k-1} (r_j + 1) \right\}^{-1}, \qquad i = 2, 3, \ldots, n.$$

For $i \leq j$,

$$\begin{aligned} b_{ij} &= \operatorname{cov}_{\boldsymbol{\theta}}(Z_i, Z_j) \\ &= N^{-2} + \sum_{k=2}^{i} \left\{ N - \sum_{\ell=1}^{k-1} (r_\ell + 1) \right\}^{-2}, \qquad j = i, i+1, \ldots, n; \quad i = 1, 2, \ldots, n; \end{aligned}$$

see also Balakrishnan and Aggarwala ((2000), p. 19). By Lemma 7.5.1 of Arnold *et al.* ((1992), pp. 174–175), we have $B^{-1} = ((b^{ij}))$ given by

$$b^{ij} = \begin{cases} N^2 + (N - r_1 - 1)^2, & i = j = 1 \\ \{N - \sum_{\ell=1}^{i-1} (r_\ell + 1)\}^2 + \{N - \sum_{\ell=1}^{i} (r_\ell + 1)\}^2, & i = j;\ i = 2, 3, \ldots, n \\ -\{N - \sum_{\ell=1}^{i} (r_\ell + 1)\}^2, & j = i + 1;\ i = 1, 2, \ldots, n - 1 \\ -\{N - \sum_{\ell=1}^{j} (r_\ell + 1)\}^2, & j = i - 1;\ i = 2, 3, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

In this case,

$$\alpha_i = E_{\boldsymbol{\theta}}(Z_i), \qquad i = 1, 2, \ldots, n.$$

Thus, we obtain

$$\begin{aligned} S(\boldsymbol{Y}; \boldsymbol{\theta}) &= \frac{1}{\tau^2} D' B^{-1} D \\ &= \frac{1}{\tau^2} \begin{bmatrix} N^2 & N \\ N & n \end{bmatrix}. \end{aligned}$$

We will now show that the sensitivity measure associated with $(Y_{1:n:N}, T_0)$, where $T_0 = \sum_{k=2}^{n} (r_k + 1)(Y_{k:n:N} - Y_{1:n:N})$, is same as that associated with the sample $\boldsymbol{Y}$.

Recall that (Viveros and Balakrishnan (1994))

$$E_{\boldsymbol{\theta}}(Y_{1:n:N}) = \xi + \tau/N, \quad V_{\boldsymbol{\theta}}(Y_{1:n:N}) = \tau^2/N^2,$$
$$E_{\boldsymbol{\theta}}(T_0) = (n-1)\tau, \qquad V_{\boldsymbol{\theta}}(T_0) = (n-1)\tau^2,$$

and $\text{cov}(Y_{1:n:N}, T_0) = 0$. Thus, it can be seen that

$$S(Y_{1:n:N}, T_0; \boldsymbol{\theta}) = S(\boldsymbol{Y}; \boldsymbol{\theta}), \qquad \forall \boldsymbol{\theta},$$

which means that the linear sensitivity measure associated with the observed progressively censored sample $\boldsymbol{Y}$ is exactly equal to the linear sensitivity measure associated with $(Y_{1:n:N}, T_0)$, which form a complete sufficient statistic for $\boldsymbol{\theta}$.

Remark 7.2. Following the arguments of Lehmann and Casella ((1998), p. 43), it can be observed that $(Y_{1:n:N}, T_0)$ is a complete sufficient statistic. Further taking $r_1 = r_2 = \cdots = r_{n-1} = 0$, we obtain the corresponding results for Type-II right censored sampling scheme. If, in addition, $N = n$ then the results for complete sample case may be obtained. In each of the above two examples, we observe a justification for Tukey's linear sensitivity measure while considering complete sufficient statistic in inference problems regarding $\boldsymbol{\theta}$.

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