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ON WAITING TIME FOR REVERSED PATTERNS IN RANDOM SEQUENCES*

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Abstract. By using a combinatorial method it is shown that for every finite pattern, the distribution of the waiting time for the reversed pattern coincides with that of the waiting time for the original pattern in a multi-state dependent sequence with a certain type of exchangeability. The number of the typical sequences until the occurrence of a given pattern and that of the typical sequences until the occurrence of the reversed pattern are shown to be equal. Further, the corresponding results for the waiting time for the r-th occurrence of the pattern, and for the number of occurrences of a specified pattern in n trials are also studied. Illustrative examples based on urn models are also given.

Key words and phrases: Discrete distribution, exchangeability, multi-state dependent sequence, reversed pattern, typical sequence, urn model, waiting time problem.

1. Introduction

Recently, exact distribution theory of patterns has been developed very much stimulated by some new techniques such as the finite Markov chain imbedding technique (Fu and Koutras (1994), Koutras and Alexandrou (1995), Fu (1996) and Koutras (1997)) and the method of conditional probability generating functions (p.g.f.'s) (Ebneshahrashoob and Sobel (1990) and Uchida (1998)). Though the distribution of the waiting time for the first occurrence of a pattern in a sequence of independent identically distributed (i.i.d.) discrete random variables were known (see Feller (1968), Johnson and Kotz (1977), Blom and Thorburn (1982)), the above new techniques enabled us to consider the problem in dependent sequences (Aki and Hirano (1993), Koutras (1997), Uchida (1998), Robin and Daudin (1999) and the references therein). In that case, needless to say, we can not evade that the results become complex and tedious. However, we can enjoy the theoretical results comfortably by using computer algebra systems.

We will begin by considering a simple example. Let $\xi_1, \xi_2, \dots$ be a sequence of $\{0,1\}$-valued i.i.d. random variables with $P(\xi_i = 1) = p = 1 - q$, $i = 1, 2, \dots$. We denote by $\phi((x_1, \dots, x_k); t)$ the p.g.f. of distribution of the waiting time for the first occurrence of the pattern $(x_1, \dots, x_k)$, $x_i = 0, 1$; $i = 1, \dots, k$, in the sequence ξ_1, $\xi_2, \dots$. Then we have 2^3 p.g.f.'s of the waiting time for each $\{0,1\}$-pattern of length three by using the method of probability generating function technique (e.g., Ebneshahrashoob and Sobel (1990) and Aki and Hirano (1993)). They do not necessarily coincide with each other

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even if $p = q = \frac{1}{2}$. However the p.g.f.'s of the waiting time for each {0,1}-pattern $(1,0,0), (0,0,1), (1,1,0)$ and $(0,1,1)$ become respectively

$$\phi((1,0,0);t) = \phi((0,0,1);t) = \frac{pq^2t^3}{(1-qt)(1-pt-pqt^2)},$$
$$\phi((1,1,0);t) = \phi((0,1,1);t) = \frac{p^2qt^3}{(1-pt)(1-qt-pqt^2)}$$

and we observe that $\phi((1,0,0);t)$ and $\phi((1,1,0);t)$ equal to $\phi((0,0,1);t)$ and $\phi((0,1,1);t)$, respectively. We also find that the patterns (1,0,0) and (1,1,0) are the reversed patterns of (0,0,1) and (0,1,1), respectively.

We deal with a more general case. We consider a finite string (succession of outcomes) composed of elements of a set $\{1,2,\dots,\mu\}$, and call it briefly a finite $\{1,2,\dots,\mu\}$-pattern. The purpose of this paper is to show that for every finite $\{1,2,\dots,\mu\}$-pattern, the distribution of the waiting time for the reversed pattern coincides with that for the original pattern even in multi-state dependent sequences with a certain type of exchangeability.

2. Waiting time for reversed patterns

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ be any $\{1,2,\dots,\mu\}$-patterns of length m and n, respectively. We denote by $\bar{\boldsymbol{a}} = (a_m, a_{m-1}, \dots, a_1)$ the reversed pattern of $\boldsymbol{a}$. For patterns $\boldsymbol{a}$ and $\boldsymbol{b}$, we let $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = (a_1, \dots, a_m, b_1, \dots, b_n)$ be the concatenated pattern. For $1 \le i \le j \le n$, we denote by $[\boldsymbol{b}]_i^j$ the subpattern $(b_i, b_{i+1}, \dots, b_j)$ of $\boldsymbol{b}$. We define $h(i, \boldsymbol{b})$ = number of i's in $\boldsymbol{b}$, $i = 1,2,\dots,\mu$. Let Z_ℓ be the set of all $\{1,2,\dots,\mu\}$-patterns of length ℓ. For $\{1,2,\dots,\mu\}$-pattern $\boldsymbol{x}$ of length m, we let

$$A_\ell(\boldsymbol{x}) = \{\boldsymbol{c} = (c_1, \dots, c_\ell) \in Z_\ell \mid [\langle \boldsymbol{x}, \boldsymbol{c} \rangle]_i^{i+m-1} \ne \boldsymbol{x}, i = 2, 3, \dots, \ell+1\}.$$

Let $h_1, h_2, \dots, h_\mu$ be nonnegative integers satisfying $h_1 + \cdots + h_\mu = \ell$. For such nonnegative integers $h_1, h_2, \dots, h_\mu$, we let

$$A_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{x}) = \{\boldsymbol{c} = (c_1, \dots, c_\ell) \in A_\ell(\boldsymbol{x}) \mid h(i, \boldsymbol{c}) = h_i, i = 1,2,\dots,\mu\}.$$

Then, we have the following theorem.

THEOREM 1. *Let $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ be sequences of $\{1,2,\dots,\mu\}$-valued random variables. For each $\ell = 1,2,\dots$ and nonnegative integers $h_1, \dots, h_\mu$ with $h_1 + \cdots + h_\mu = \ell$, assume that*

$$P((X_1, \dots, X_{\ell+m}) = \langle \bar{\boldsymbol{c}}, \bar{\boldsymbol{a}} \rangle) = P((Y_1, \dots, Y_{\ell+m}) = \langle \bar{\boldsymbol{d}}, \boldsymbol{a} \rangle)$$

for each $\boldsymbol{c} \in A_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{a})$ and $\boldsymbol{d} \in A_\ell^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}})$. Then, the distribution of the waiting time for the pattern $\bar{\boldsymbol{a}}$ in $X_1, X_2, \dots$ and that of the waiting time for the pattern $\boldsymbol{a}$ in $Y_1, Y_2, \dots$ are equal.

COROLLARY 1. *Let $X_1, X_2, \dots$ be a sequence of $\{1,2,\dots,\mu\}$-valued exchangeable random variables. Then, the distribution of the waiting time for the pattern $\boldsymbol{a}$ and that of the waiting time for the pattern $\bar{\boldsymbol{a}}$ in $X_1, X_2, \dots$ are equal.*

PROOF OF COROLLARY 1. Let $Y_1, Y_2, \ldots$ be the same as $X_1, X_2, \ldots$. Since the numbers of each letter in $\langle\bar{c},\bar{a}\rangle$ and in $\langle\bar{d},a\rangle$ are the same,

$$P((X_1, \ldots, X_{\ell+m}) = \langle\bar{c},\bar{a}\rangle) = P((Y_1, \ldots, Y_{\ell+m}) = \langle\bar{d},a\rangle)$$

holds for every $c \in A_\ell^{(h_1,\ldots,h_\mu)}(a)$ and $d \in A_\ell^{(h_1,\ldots,h_\mu)}(\bar{a})$ from the exchangeability. This completes the proof.

In order to prove Theorem 1 we need the following lemma.

LEMMA 1. *For every nonnegative integer ℓ and nonnegative integers $h_1, h_2, \ldots, h_\mu$ satisfying $h_1 + \cdots + h_\mu = \ell$,*

$$\#A_\ell^{(h_1,\ldots,h_\mu)}(a) = \#A_\ell^{(h_1,\ldots,h_\mu)}(\bar{a}) \tag{2.1}$$

holds. Here $\#A$ is the number of elements in a set A.

PROOF. We prove this by induction on ℓ. First, we consider the case $\ell = 1$. Suppose that $c = (c_1) \notin A_1(a)$ for a $c_1 \in M \equiv \{1, 2, \ldots, \mu\}$. Then, $(a_2, \ldots, a_m, c_1) = (a_1, \ldots, a_m)$ holds and hence $a_1 = a_2 = \cdots = a_m = c_1$ hold. Thus, we find that the pattern a is a run of c_1. Since the reversed pattern of a run is the same as the original pattern, $d = (c_1) \notin A_1(\bar{a})$. Consequently, we see that $\#A_1^{(h_1,\ldots,h_\mu)}(a) = \#A_1^{(h_1,\ldots,h_\mu)}(\bar{a})$.

Next, suppose that (2.1) holds for $\ell = n - 1$. Define

$$E_n(x) = \{c \in A_{n-1}(x) \times M \mid [\langle x,c\rangle]_{n+1}^{n+m} = x\}.$$

If we define $d = [\overline{\langle a, c\rangle}]_{m+1}^{n+m}$ for every $c \in E_n(a)$, the mapping $E_n(a) \ni c \mapsto d \in E_n(\bar{a})$ is a bijection and $h(i, c) = h(i, d)$ for every $i = 1, 2, \ldots, \mu$. This can be checked easily if $n \geq m$. Suppose that $n < m$. Then $(a_{n+1}, \ldots, a_m, c_1, \ldots, c_n) = (a_1, \ldots, a_m)$ and hence $(c_1, \ldots, c_n) = (a_{m-n+1}, \ldots, a_m)$ and $(d_1, \ldots, d_n) = \overline{(a_1, \ldots, a_n)}$. Then, we can see that $c_k = d_{n+1-f(k)}$ for $k = 1, 2, \ldots, n$, where

$$f(k) = \begin{cases} (m+k) \bmod n, & \text{if } ((m+k) \bmod n) \neq 0 \\ n, & \text{if } ((m+k) \bmod n) = 0. \end{cases}$$

Noting that $A_n(a) = (A_{n-1}(a) \times M) \setminus E_n(a)$ and $A_n(\bar{a}) = (A_{n-1}(\bar{a}) \times M) \setminus E_n(\bar{a})$ we have (2.1) for $\ell = n$ from the induction hypothesis. This completes the proof.

PROOF OF THEOREM 1. From the assumption of Theorem 1,

$$P((X_1, \ldots, X_{\ell+m}) = \langle\bar{c},\bar{a}\rangle) = P((Y_1, \ldots, Y_{\ell+m}) = \langle\bar{d},a\rangle)$$

holds for each $c \in A_\ell^{(h_1,\ldots,h_\mu)}(a)$ and $d \in A_\ell^{(h_1,\ldots,h_\mu)}(\bar{a})$. Hence, we write the probability value as $p_{h_1,\ldots,h_\mu}$. Let $W(a)$ ($W(\bar{a})$) denote the waiting time for the pattern a (resp. $\bar{a}$). Then, from Lemma 1, we have

$$\begin{aligned} &P(W(\bar{a}) = m + \ell) \\ &= \sum_{h_1+\cdots+h_\mu=\ell} \sum_{c \in A_\ell^{(h_1,\ldots,h_\mu)}(a)} P((X_1, \ldots, X_{\ell+m}) = \langle\bar{c},\bar{a}\rangle) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{h_1+\cdots+h_\mu=\ell} \#A_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{a})p_{h_1,\dots,h_\mu} \\
&= \sum_{h_1+\cdots+h_\mu=\ell} \#A_\ell^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}})p_{h_1,\dots,h_\mu} \\
&= \sum_{h_1+\cdots+h_\mu=\ell} \sum_{\boldsymbol{d}\in A_\ell^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}})} P((Y_1,\dots,Y_{\ell+m}) = \langle \bar{\boldsymbol{d}},\boldsymbol{a}\rangle) \\
&= P(W(\boldsymbol{a}) = m+\ell).
\end{aligned}
$$

This completes the proof.

Example 1 (The Pólya sampling). Let us consider an urn containing a white and b black balls. Assume that balls are randomly drawn one at a time and that each ball is returned to the urn along with c additional balls of the same color before the next drawing. Suppose that a finite {(W)hite, (B)lack}-pattern $\boldsymbol{a}$ is given. For example, let $\boldsymbol{a} = (W,B,W,W)$. Then, the distribution of the waiting time for the pattern $\boldsymbol{a}$ and that of the waiting time for the reversed pattern $\bar{\boldsymbol{a}} = (W,W,B,W)$ are equal, since the random sequence generated by the repeated drawing is exchangeable.

3. Waiting time for the r-th occurrence

Next, we compare the waiting times until the r-th overlapping occurrence. First, we will show the next lemma regarding the numbers of typical sequences to the next occurrence of the pattern $\boldsymbol{a}$ just after the occurrence of the pattern $\boldsymbol{a}$. Let $h_1,h_2,\dots,h_\mu$ be nonnegative integers with $h_1+\cdots+h_\mu=\ell$. We define

$$
\begin{aligned}
C_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{x}) = \{\boldsymbol{c} = (c_1,\dots,c_\ell) \in Z_\ell \mid [\boldsymbol{c}]_1^{\ell-1} \in A_{\ell-1}(\boldsymbol{x}), \\
\boldsymbol{c} \notin A_\ell(\boldsymbol{x}), h(i,\boldsymbol{c}) = h_i, i = 1,2,\dots,\mu\}.
\end{aligned}
$$

LEMMA 2. *Let $h_1,h_2,\dots,h_\mu$ be nonnegative integers with $h_1+\cdots+h_\mu=\ell$. Then,*

$$\#C_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{a}) = \#C_\ell^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}})$$

holds.

PROOF. For every $\boldsymbol{c} \in C_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{a})$, $[\langle\boldsymbol{a},\boldsymbol{c}\rangle]_1^m = (a_1,\dots,a_m)$ and $[\langle\boldsymbol{a},\boldsymbol{c}\rangle]_{\ell+1}^{m+\ell} = (a_1,\dots,a_m)$ hold. Then, we see that $[\overline{\langle\boldsymbol{a},\boldsymbol{c}\rangle}]_1^m = (a_m,\dots,a_1)$ and $[\overline{\langle\boldsymbol{a},\boldsymbol{c}\rangle}]_{\ell+1}^{m+\ell} = (a_m,\dots,a_1)$. Noting that $\overline{\langle\boldsymbol{a},\boldsymbol{c}\rangle} = \langle\bar{\boldsymbol{c}},\bar{\boldsymbol{a}}\rangle$ we have

$$[\langle\bar{\boldsymbol{c}},\bar{\boldsymbol{a}}\rangle]_{m+1}^{m+\ell} \in C_\ell^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}}),$$

even if ℓ is less than m. Moreover, it is easy to see that the mapping $\boldsymbol{c} \mapsto [\langle\bar{\boldsymbol{c}},\bar{\boldsymbol{a}}\rangle]_{m+1}^{m+\ell}$ is one-to-one. Similarly, we see that the mapping $C_\ell^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}}) \ni \boldsymbol{d} \mapsto [\overline{\langle\bar{\boldsymbol{a}},\boldsymbol{d}\rangle}]_{m+1}^{m+\ell} \in C_\ell^{(h_1,\dots,h_\mu)}(\boldsymbol{a})$ is also one-to-one. Therefore, the desired result holds.

THEOREM 2. *Let $X_1,X_2,\dots$ be a sequence of $\{1,2,\dots,\mu\}$-valued exchangeable random variables. Let r be a positive integer. Then the distribution of the waiting time for* *the r-th overlapping occurrence of a pattern and that for the r-th overlapping occurrence of the reversed pattern in the sequence are equal.*

PROOF. For simplicity we prove only the case $r = 2$. Let $W_2(\boldsymbol{a})$ $(W_2(\bar{\boldsymbol{a}}))$ be the waiting times for the second overlapping occurrence of the pattern $\boldsymbol{a}$ (resp. $\bar{\boldsymbol{a}}$). Let ℓ_1 and ℓ_2 be nonnegative integers with $\ell_1+\ell_2=\ell$. Let $h_1,\dots,h_\mu,k_1,\dots,k_\mu$ be nonnegative integers satisfying $h_1+\cdots+h_\mu=\ell_1$ and $k_1+\cdots+k_\mu=\ell_2$. Then, note that from the exchangeability of $X_1,X_2,\dots$, for every $\boldsymbol{c_1}\in A_{\ell_1}^{(h_1,\dots,h_\mu)}(\boldsymbol{a})$ and $\boldsymbol{d_1}\in C_{\ell_2}^{(k_1,\dots,k_\mu)}(\bar{\boldsymbol{a}})$, the probability

$$P((X_1,\dots,X_{\ell_1+m},X_{\ell_1+m+1},\dots,X_{\ell_1+m+\ell_2})=\langle\overline{\boldsymbol{c_1}},\bar{\boldsymbol{a}},\boldsymbol{d_1}\rangle)$$

is a constant and hence we write it as $p_{h_1+k_1+h(1,\bar{\boldsymbol{a}}),\dots,h_\mu+k_\mu+h(\mu,\bar{\boldsymbol{a}})}$.

Then, from Lemmas 1 and 2, we obtain the following.

$$\begin{aligned}
&P(W_2(\bar{\boldsymbol{a}})=\ell+m)\\
&=\sum_{\ell_1+\ell_2=\ell}\sum_{h_1+\cdots+h_\mu=\ell_1}\sum_{k_1+\cdots+k_\mu=\ell_2}\sum_{\boldsymbol{c_1}\in A_{\ell_1}^{(h_1,\dots,h_\mu)}(\boldsymbol{a})}\sum_{\boldsymbol{d_1}\in C_{\ell_2}^{(k_1,\dots,k_\mu)}(\bar{\boldsymbol{a}})}\\
&\qquad P((X_1,\dots,X_{\ell_1+m},X_{\ell_1+m+1},\dots,X_{\ell_1+m+\ell_2})=\langle\overline{\boldsymbol{c_1}},\bar{\boldsymbol{a}},\boldsymbol{d_1}\rangle)\\
&=\sum_{\ell_1+\ell_2=\ell}\sum_{h_1+\cdots+h_\mu=\ell_1}\sum_{k_1+\cdots+k_\mu=\ell_2}\\
&\qquad \#A_{\ell_1}^{(h_1,\dots,h_\mu)}(\boldsymbol{a})\#C_{\ell_2}^{(k_1,\dots,k_\mu)}(\bar{\boldsymbol{a}})p_{h_1+k_1+h(1,\bar{\boldsymbol{a}}),\dots,h_\mu+k_\mu+h(\mu,\bar{\boldsymbol{a}})}\\
&=\sum_{\ell_1+\ell_2=\ell}\sum_{h_1+\cdots+h_\mu=\ell_1}\sum_{k_1+\cdots+k_\mu=\ell_2}\\
&\qquad \#A_{\ell_1}^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}})\#C_{\ell_2}^{(k_1,\dots,k_\mu)}(\boldsymbol{a})p_{h_1+k_1+h(1,\boldsymbol{a}),\dots,h_\mu+k_\mu+h(\mu,\boldsymbol{a})}\\
&=\sum_{\ell_1+\ell_2=\ell}\sum_{h_1+\cdots+h_\mu=\ell_1}\sum_{k_1+\cdots+k_\mu=\ell_2}\sum_{\boldsymbol{d_2}\in A_{\ell_1}^{(h_1,\dots,h_\mu)}(\bar{\boldsymbol{a}})}\sum_{\boldsymbol{c_2}\in C_{\ell_2}^{(k_1,\dots,k_\mu)}(\boldsymbol{a})}\\
&\qquad P((X_1,\dots,X_{\ell_1+m},X_{\ell_1+m+1},\dots,X_{\ell_1+m+\ell_2})=\langle\overline{\boldsymbol{d_2}},\boldsymbol{a},\boldsymbol{c_2}\rangle)\\
&=P(W_2(\boldsymbol{a})=\ell+m).
\end{aligned}$$

This completes the proof.

Since a sequence of i.i.d. variables has the exchangeability property, it is clear that the outcomes of Corollary 1 and Theorem 2 hold true for sequences of multivalued i.i.d. trials as well.

Last in this section, we discuss the distributions of the number of occurrences of a specified pattern and the reversed pattern in n trials. Under a very general condition, the relationship between the distribution of the waiting times of a given event and that of number of occurrences of the event in the first n trials is investigated by Koutras (1997). By using the results and Theorem 2, we obtain the following theorem.

THEOREM 3. *Let $X_1,X_2,\dots$ be a sequence of $\{1,2,\dots,\mu\}$-valued exchangeable random variables. Let n be a positive integer. Then, the distribution of the number of overlapping occurrences of a pattern in $X_1,\dots,X_n$ and that of overlapping occurrences of the reversed pattern in $X_1,\dots,X_n$ are equal.*

PROOF. From Theorem 2, for every positive integer r, the distributions of the waiting times for the r-th overlapping occurrences of the pattern and the reversed pattern are equal. Let Z_n and W_n be the number of overlapping occurrences of the pattern and the reversed pattern in $X_1, \dots, X_n$, respectively. From Theorem 21.3.1 in Koutras (1997), $P(Z_n = r)$ and $P(W_n = r)$ are completely determined by the p.g.f.'s of the corresponding waiting times of the r-th and $(r+1)$-th occurrences. Therefore, we obtain the desired result.

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