ON THE DISTRIBUTION OF THE SUM OF n NON-IDENTICALLY DISTRIBUTED UNIFORM RANDOM VARIABLES

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Abstract. The distribution of the sum of independent identically distributed uniform random variables is well-known. However, it is sometimes necessary to analyze data which have been drawn from different uniform distributions. By inverting the characteristic function, we derive explicit formulae for the distribution of the sum of n non-identically distributed uniform random variables in both the continuous and the discrete case. The results, though involved, have a certain elegance. As examples, we derive from our general formulae some special cases which have appeared in the literature.

Key words and phrases: Uniform distribution, probability density, convolution, Fourier transform, sine integrals.

1. Introduction

The classical uniform distribution is perhaps the most versatile statistical model: applications abound in nonparametric statistics and Bayes procedures. Chu (1957) and Leon (1961) utilized uniform distributions in connection with sample quasi-ranges. Naus (1966) applied uniform distribution in a power comparison of tests of random clustering. For additional applications and examples, see Johnson *et al.* (1995).

Here, we are concerned with the distribution of the sum of n independent non-identically distributed uniform random variables. It is well-known that the probability density function of such a sum, in which the summands are uniformly distributed in a common interval $[-a, a]$, can be obtained via standard convolution formulae: see Feller ((1966), p. 27) or Rényi ((1970), pp. 196–197), for example. However, it is sometimes necessary to analyze data which have been drawn from non-identical uniform distributions. For example, measurements accurate to the nearest foot may be combined with measurements accurate to the nearest inch. In such cases, the distribution of the sum is more complicated. Tach (1958) gives tables to five decimal places of the cumulative distribution of the sum for $n = 2, 3$ and 4 for some special cases.

The first general result in this direction seems to have been made by Olds (1952), who derived the distribution of the sum $\sum_{j=1}^{n} X_j$, in which each X_j is uniformly distributed in an interval of the form $[0, a_j)$ with $a_j > 0$. The proof is by induction, and in that respect is somewhat unsatisfactory, since in general inductive proofs require knowing beforehand the formula to be proved. Subsequently, Roach (1963) deduced what is essentially Olds' formula using n-dimensional geometry. Later Mitra (1971), apparently unaware of these previous results, derived the distribution of the sum in which each ran-

dom variable is uniformly distributed in an interval of the form $[-\omega_j, \omega_j]$ using Nörlund's (1924) difference calculus.

Here, we derive an explicit formula for the slightly more general situation of the distribution of the sum $\sum_{j=1}^{n} X_j$, in which each X_j is uniformly distributed in an interval of the form $[c_j - a_j, c_j + a_j]$ with $a_j > 0$ and c_j an arbitrary real number. Of course, each of the aforementioned results can be obtained from ours by specializing the parameters c_j and a_j accordingly.

Our approach is via Fourier theory and is quite straightforward; specifically we invert the characteristic function. As a result, our formula differs somewhat in form from the special cases alluded to previously. However, the inversion technique is quite flexible, and readily lends itself to the study of other types of distributions, such as the discrete case, which seems not to have been discussed in the literature. Thus, in a similar fashion, we derive the distribution of the sum of n random variables with point mass at the integers in intervals of the form $[-m_j, m_j]$, in which each m_j is a positive integer. The formula in the discrete case is somewhat more complicated than the corresponding formula in the continuous case; nevertheless, they are clearly closely related, and there is a certain charm and elegance to both. Of course, the same results may be obtained using the standard transformation methods.

2. The continuous case

Fix a positive integer n, and let $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{c} = (c_1, c_2, \ldots, c_n)$ be vectors of real numbers with each $a_j > 0$. For each $j = 1, 2, \ldots, n$, we consider a random variable X_j uniformly distributed on the closed interval $[c_j - a_j, c_j + a_j]$. The step function $\chi_j : \mathbf{R} \to \mathbf{R}$ defined for real x by

(2.1)
$$2a_j \, \chi_j(x) = \begin{cases} 1, & \text{if } |x - c_j| < a_j, \\ \dfrac{1}{2}, & \text{if } |x - c_j| = a_j, \\ 0, & \text{if } |x - c_j| > a_j \end{cases}$$

represents the density of the random variable X_j for each $j = 1, 2, \ldots, n$. (When employing techniques from Fourier theory, it is convenient to define the densities at jump discontinuities so that the equation $\chi_j(x+) + \chi_j(x-) = 2\chi_j(x)$ is satisfied for all real x.) The corresponding characteristic function (Fourier transform) is given by

$$\hat{\chi}_j(t) := \frac{1}{2a_j} \int_{c_j - a_j}^{c_j + a_j} e^{itx} dx = e^{ic_j t} \operatorname{sinc}(a_j t), \quad t \in \mathbf{R},$$

where $\operatorname{sinc}(x) := x^{-1} \sin x$ if $x \neq 0$, and $\operatorname{sinc}(0) := 1$. For real x, the density of the sum $\sum_{j=1}^{n} X_j$ is given by the n-fold convolution $f_n(x) := (\chi_1 * \chi_2 * \cdots * \chi_n)(x)$. Thus,

$$f_n(x) = \int_{-\infty}^{\infty} \chi_1(x - y_2) \int_{-\infty}^{\infty} \chi_2(y_2 - y_3) \cdots \int_{-\infty}^{\infty} \chi_{n-1}(y_{n-1} - y_n) \chi_n(y_n) dy_2 \cdots dy_n.$$

In particular, if each of the n intervals is centered at 0, and we write $x_1 = x - y_2$, $x_n = y_n$ and $x_j = y_j - y_{j+1}$ for $1 < j < n$, then the conditions on the variables $x_1, x_2, \ldots, x_n$ are that each $|x_j| < a_j$ and $\sum_{j=1}^{n} x_j = x$. Thus, $f_n(x)$ is simply the volume (in the sense of

Lebesgue measure) of the $(n-1)$-dimensional region

$$\left\{ (x_1, x_2, \ldots, x_{n-1}) \in R^{n-1} : \left| x - \sum_{j=1}^{n-1} x_j \right| < a_n \text{ and } |x_j| < a_j \text{ for } 1 \le j < n \right\}$$

divided by the volume $\prod_{j=1}^{n} 2a_j$ of the n-dimensional hyperbox

(2.2) $\{ (x_1, x_2, \ldots, x_n) \in R^n : |x_j| < a_j \text{ for } j = 1, 2, \ldots, n \}.$

Despite the utility of these representations, it is desirable to have an explicit formula for f_n. In this vein, we have the following.

THEOREM 1.   *The density of the sum of n independent random variables, uniformly distributed in the intervals $[c_j - a_j, c_j + a_j]$ for $j = 1, 2, \ldots, n$, is given by*

$$(2.3) f_n(x) = \left[\sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(x + \sum_{j=1}^{n} (\varepsilon_j a_j - c_j) \right)^{n-1} \times \text{sign}\left(x + \sum_{j=1}^{n} (\varepsilon_j a_j - c_j) \right) \prod_{j=1}^{n} \varepsilon_j \right]$$

$$\Big/ \left[(n-1)! 2^{n+1} \prod_{j=1}^{n} a_j \right],$$

in which the sum is over all 2^n vectors of signs

$$\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{-1, 1\}^n \quad \text{i.e. each } \varepsilon_j = \pm 1$$

and

$$\text{sign}(y) := \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ -1, & \text{if } y < 0. \end{cases}$$

PROOF.   Since the random variables are assumed to be independent, the characteristic function of the distribution of the sum is the product of the characteristic functions of their distributions:

$$(2.4) \qquad \hat{f}_n(t) = \prod_{j=1}^{n} \hat{\chi}_j(t) = \prod_{j=1}^{n} e^{itc_j} \text{sinc}(a_j t).$$

Mitra ((1971), p. 195) remarks that "It is difficult to obtain an inverse Fourier transform of this product in a neat form." Indeed, his approach is to expand the product of sincs into a power series using generalized Bernoulli polynomials. However, we shall see that the inverse Fourier transform of (2.4) has an elegant representation involving a sum over the vertices of the hyperbox (2.2).

Since for all real x, $f_n(x) = \frac{1}{2} f_n(x+) + \frac{1}{2} f_n(x-)$ by continuity of f_n for $n > 1$, and by definition of χ_1 when $n = 1$, Fourier inversion gives

$$f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{f}_n(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \prod_{j=1}^{n} \text{sinc}(a_j t) dt,$$

where $y := x - \sum_{j=1}^{n} c_j$. Since sinc is an even function, making the change of variable $t \mapsto -t$ yields

$$(2.5) \qquad f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ity} \prod_{j=1}^{n} \mathrm{sinc}(a_j t) dt.$$

It remains only to evaluate the integral (2.5). Related integrals are studied in Borwein and Borwein (2001) using a version of the Parseval/Plancherel formula, and by expanding the product of sincs into a sum of cosines. Our approach is somewhat more direct. We first express the sinc functions using complex exponentials, so that

$$(2.6) \qquad f_n(x) = \frac{1}{2\pi} \left(\frac{1}{2i}\right)^n \left(\prod_{j=1}^{n} a_j^{-1}\right) \int_{-\infty}^{\infty} t^{-n} e^{ity} \prod_{j=1}^{n} (e^{ita_j} - e^{-ita_j}) dt.$$

For each of the 2^n vectors of signs $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, let

$$\rho_\varepsilon := \prod_{j=1}^{n} \varepsilon_j, \quad \text{and} \quad \vec{\varepsilon} \cdot \vec{a} = \sum_{j=1}^{n} \varepsilon_j a_j.$$

By carefully expanding the product of exponentials in (2.6), we find that

$$(2.7) \qquad \prod_{j=1}^{n} (e^{ita_j} - e^{-ita_j}) = \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \rho_\varepsilon \exp(it\vec{\varepsilon} \cdot \vec{a}).$$

It follows that

$$(2.8)\; f_n(x) = \frac{1}{2\pi} \left(\frac{1}{2i}\right)^n \left(\prod_{j=1}^{n} a_j^{-1}\right)^n \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \rho_\varepsilon \, \mathrm{P.V.} \int_{-\infty}^{\infty} t^{-n} \exp(it(y + \vec{\varepsilon} \cdot \vec{a})) dt.$$

Although each of the individual integrals in (2.8) is divergent, the singularities must cancel because (2.6) converges. Therefore, the required finite integral (2.6) is equal to its principal value, and hence by linearity is given by (2.8).

In view of the fact that (2.7) is entire with a zero of order n at $t = 0$, we may integrate (2.8) by parts $n - 1$ times and thereby obtain

$$(2.9) \qquad f_n(x) = \frac{1}{2\pi} \left(\frac{1}{2i}\right)^n \left(\prod_{j=1}^{n} a_j^{-1}\right) \frac{i^{n-1}}{(n-1)!} \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \rho_\varepsilon (y + \vec{\varepsilon} \cdot \vec{a})^{n-1}$$

$$\times \mathrm{P.V.} \int_{-\infty}^{\infty} t^{-1} \exp(it(y + \vec{\varepsilon} \cdot \vec{a})) dt.$$

But for any real number b, we have

$$\mathrm{P.V.} \int_{-\infty}^{\infty} t^{-1} \exp(itb) dt = \lim_{\varepsilon \to 0+} \left\{ \int_{\varepsilon}^{\infty} t^{-1} \exp(itb) dt + \int_{-\infty}^{-\varepsilon} t^{-1} \exp(itb) dt \right\}$$

$$= \lim_{\varepsilon \to 0+} \left\{ \int_{\varepsilon}^{\infty} t^{-1} \exp(itb) dt - \int_{\varepsilon}^{\infty} t^{-1} \exp(-itb) dt \right\}$$

$$= 2i \int_{0}^{\infty} t^{-1} \sin(tb) dt$$

$$= i\pi \, \mathrm{sign}(b).$$

Applying this latter result to (2.9) yields (2.3) and completes the proof of Theorem 1. $\square$

In some applications, it may be easier to work with powers of expressions involving the maximum function $y_+ := \max(y, 0)$ as opposed to the sign functions in (2.3). To this end, we make the following.

DEFINITION 1.   Let $\tau : R \to R$ be given by

(2.10)
$$\tau(x) = \begin{cases} 1, & \text{if } x > 0, \\ \dfrac{1}{2}, & \text{if } x = 0, \\ 0, & \text{if } x < 0, \end{cases}$$

and for y real and n a positive integer, let $y_+^{n-1} := y^{n-1}\tau(y)$. Note that $y_+^0 = \tau(y)$ and $y_+^n = (\max(y, 0))^n$ for $n > 0$.

Then we have the following corollary to Theorem 1.

COROLLARY 1.   *Let f_n be as in Theorem 1. Then*

(2.11) $$f_n(x) = \left[\sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(x + \sum_{j=1}^n (\varepsilon_j a_j - c_j) \right)_+^{n-1} \prod_{j=1}^n \varepsilon_j \right] \Bigg/ \left[(n-1)!\, 2^n \prod_{j=1}^n a_j \right].$$

PROOF.   Note that $\operatorname{sign}(x) = 2\tau(x) - 1$ holds for all real x. Hence, substituting the τ function for the sign function in (2.3), we see that it suffices to prove the identity

(2.12) $$\sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(x + \sum_{j=1}^n (\varepsilon_j a_j - c_j) \right)^{n-1} \prod_{j=1}^n \varepsilon_j = 0.$$

Since $e^{a_j t} - e^{-a_j t} = 2a_j t + O(t^2)$ as $t \to 0$, (2.12) follows easily on comparing coefficients of t^{n-1} in

$$\sum_{\vec{\varepsilon} \in \{-1,1\}^n} \exp\left\{ \left(x + \sum_{j=1}^n \varepsilon_j a_j - c_j \right) t \right\} \prod_{j=1}^n \varepsilon_j = e^{xt} \prod_{j=1}^n e^{-c_j t}(e^{a_j t} - e^{-a_j t}).$$

Alternatively, note that if $x \geq \sum_{j=1}^n (c_j + a_j)$, then $f_n(x) = 0$ by definition: since each X_j is $u[c_j - a_j, c_j + a_j]$, the sum $\sum_{j=1}^n X_j$ must fall within the interval $[\sum_{j=1}^n (c_j - a_j), \sum_{j=1}^n (c_j + a_j)]$. But, if $x \geq \sum_{j=1}^n (c_j + a_j)$, then we can drop the subscripted "+" from (2.11) since $x + \sum_{j=1}^n (\varepsilon_j a_j - c_j) \geq 0$ for each $\vec{\varepsilon} \in \{-1,1\}^n$. It follows that (2.12) holds for all $x \geq \sum_{j=1}^n (c_j + a_j)$. Since the left hand side of (2.12) is a polynomial in x which vanishes for all sufficiently large values of x, it must in fact vanish for all real x by the identity theorem. $\square$

Example 1.   When $n = 1$, formulae (2.3) and (2.11) give the central difference representations

$$\chi_1(x) = \frac{\operatorname{sign}(x - c_1 + a_1) - \operatorname{sign}(x - c_1 - a_1)}{4a_1} = \frac{\tau(x - c_1 + a_1) - \tau(x - c_1 - a_1)}{2a_1}$$

respectively. Both are equivalent to the definition (2.1) with $j = 1$.

Example 2. When $n = 2$, we know that

$$f_2(x) = (\chi_1 * \chi_2)(x) = \int_{-\infty}^{\infty} \chi_1(x - y)\chi_2(y)dy.$$

If $c_1 = c_2 = 0$, the convolution reduces to

$$f_2(x) = \frac{1}{4a_1a_2} \int_{\max(x-a_1,-a_2)}^{\min(x+a_1,a_2)} dy = \frac{\min(x + a_1, a_2) - \max(x - a_1, -a_2)}{4a_1a_2},$$

so that, in particular,

$$(2.13) \qquad f_2(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(a_1 t)}{a_1 t} \cdot \frac{\sin(a_2 t)}{a_2 t} dt = \frac{\min(a_1, a_2)}{2a_1a_2}.$$

On the other hand, in light of the fact that $y\,\text{sign}(y) = |y|$ for y real, formula (2.3) gives

$$\begin{aligned} f_2(0) &= \frac{|a_1 + a_2| - |a_1 - a_2| - |-a_1 + a_2| + |-a_1 - a_2|}{8a_1a_2} \\ &= \frac{2(a_1 + a_2) - 2|a_1 - a_2|}{8a_1a_2} \\ &= \frac{\min(a_1, a_2)}{2a_1a_2}, \end{aligned}$$

in agreement with (2.13). Alternatively, since $y_T(y) = y_+ = \max(y, 0)$, formula (2.3) gives

$$\begin{aligned} f_2(0) &= \frac{\max(a_1 + a_2, 0) - \max(a_1 - a_2, 0) - \max(-a_1 + a_2, 0) + \max(-a_1 - a_2, 0)}{4a_1a_2} \\ &= \frac{a_1 + a_2 - \max(a_1 - a_2, 0) - \max(a_2 - a_1, 0)}{4a_1a_2} \\ &= \frac{\min(a_1, a_2)}{2a_1a_2}. \end{aligned}$$

Example 3. If the random variables comprising the sum are uniformly distributed in a common interval centered at 0—say $[-a, a]$ with $a > 0$—then formula (2.11) gives

$$f_n(x) = \frac{1}{(n-1)!(2a)^n} \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(x + a \sum_{j=1}^{n} \varepsilon_j \right)_+^{n-1} \prod_{j=1}^{n} \varepsilon_j.$$

If k of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are negative and the remaining $n - k$ are positive, then summing over k yields

$$f_n(x) = \frac{1}{(n-1)!(2a)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + (n - 2k)a)_+^{n-1},$$

in agreement with Feller (1966).

Example 4. If for each $j = 1, 2, \ldots, n$ we set $c_j = a_j$ and then replace a_j by $a_j/2$, then X_j will be uniformly distributed in $[0, a_j]$ and will have density χ_j now given by

$$(2.14) \qquad a_j \chi_j(x) := \begin{cases} 1, & \text{if } \ 0 < x < a_j, \\ \dfrac{1}{2}, & \text{if } \ x = a_j \text{ or if } x = 0, \\ 0, & \text{if } \ x > a_j \text{ or if } x < 0. \end{cases}$$

Formula (2.11) of Corollary 1 now gives

$$f_n(x) = \left[\sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(x - \sum_{j=1}^{n} \left(\frac{1-\varepsilon_j}{2} \right) a_j \right)_+^{n-1} \prod_{j=1}^{n} \varepsilon_j \right] \Big/ \left[(n-1)! \prod_{j=1}^{n} a_j \right]$$

$$= \left[\sum_{\vec{s} \in \{0,1\}^n} (-1)^{\Sigma \vec{s}} (x - \vec{s} \cdot \vec{a})_+^{n-1} \right] \Big/ \left[(n-1)! \prod_{j=1}^{n} a_j \right],$$

where the sum is now over all 2^n vectors $\vec{s} = (s_1, s_2, \ldots, s_n)$ in which each component takes the value 0 or 1, $\sum \vec{s}$ denotes the sum of the components $s_1 + s_2 + \cdots + s_n$ and $\vec{s} \cdot \vec{a}$ denotes the dot product $s_1 a_1 + s_2 a_2 + \cdots + s_n a_n$. If we now break up the sum according to the number of non-zero components in the vector $\vec{s}$, we find that

$$f_n(x) = \left[x_+^{n-1} - \sum_{1 \le j_1 \le n} (x - a_{j_1})_+^{n-1} + \sum_{1 \le j_1 < j_2 \le n} (x - a_{j_1} - a_{j_2})_+^{n-1} - + \cdots \right.$$

$$\left. + (-1)^n \sum_{1 \le j_1 < j_2 < \cdots < j_n \le n} \left(x - \sum_{k=1}^{n} a_{j_k} \right)_+^{n-1} \right]$$

$$\Big/ \left[(n-1)! \prod_{j=1}^{n} a_j \right],$$

which is equivalent to the formula of Olds ((1952), p. 282 (1)) when $n > 1$. When $n = 1$, the two formulae differ at $x = 0$ and at $x = a_1$ because Olds uses the density which is $1/a_1$ for $0 \le x < a_1$ and 0 otherwise, in contrast to our more symmetrical density (2.14).

3. The discrete case

Here, we fix n positive integers $m_1, m_2, \ldots, m_n$. For each $j = 1, 2, \ldots, n$, we now consider a random variable X_j uniformly distributed on the set of $(2m_j + 1)$ integers contained in the closed interval $[-m_j, m_j]$, i.e. the set $\{-m_j, 1 - m_j, \ldots, m_j - 1, m_j\}$. The mass function of X_j is the rational-valued function of an *integer* variable given by

$$(3.1) \qquad (2m_j + 1)\chi_j(p) := \begin{cases} 1, & \text{if } \ |p| \le m_j, \\ 0, & \text{if } \ |p| > m_j. \end{cases}$$

We seek a formula analogous to (2.3) for the probability mass function of the sum $\sum_{j=1}^{n} X_j$, namely the n-fold convolution

$$(3.2) \qquad g_n(p) := (\chi_1 * \chi_2 * \cdots * \chi_n)(p) = \sum_{k_1+k_2+\cdots+k_n=p} \prod_{j=1}^{n} \chi_j(k_j),$$

where the sum is over all integers $k_1, k_2, \ldots, k_n$ such that $k_1 + k_2 + \cdots + k_n = p$. Although (3.2) is a formula of sorts, the various conditions on the summation indices make it inconvenient to apply. For example, when $n = 3$, let $u_k = u_k(p) := \min(m_2, p - k + m_3)$ and $v_k = v_k(p) := \max(-m_2, p - k - m_3)$. Then

$$g_3(p) = \left(\prod_{j=1}^{3} (2m_j + 1)^{-1} \right) \sum_{\substack{|k| \leq m_1 \\ u_k < v_k}} (u_k - v_k + 1),$$

where the sum is over all integers k for which $u_k < v_k$ and $|k| \leq m_1$. In general, one needs to consider cases which depend on the size of p in relation to various signed sums of subsets of the parameters $m_1, m_2, \ldots, m_n$. The number of cases to be delineated increases exponentially with n. Thus, the situation becomes rapidly unwieldy. Fortunately, there is alternative approach provided by Fourier theory. With the convolution $g_n : \mathbf{Z} \to \mathbf{Z}$ defined as above, we have

THEOREM 2. *For all integers* p,

$$(3.3) \qquad g_n(p) = \frac{M}{2^n} \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \mathrm{sign}\left(2p + \sum_{j=1}^{n} (2m_j + 1)\varepsilon_j \right) \left(\prod_{j=1}^{n} \varepsilon_j \right)$$

$$\times \sum_{k=0}^{(n-1)/2} (-1)^k b_{2k}^{(n)} \frac{\left(2p + \sum_{j=1}^{n} (2m_j + 1)\varepsilon_j \right)^{n-2k-1}}{(n - 2k - 1)!},$$

where

$$M := \prod_{j=1}^{n} (2m_j + 1)^{-1},$$

and the rational numbers $b_{2k}^{(n)}$ *are the coefficients in the Laurent series expansion*

$$(3.4) \qquad \left(\frac{1}{\sin x} \right)^n = \sum_{k=0}^{\infty} x^{2k-n} b_{2k}^{(n)}.$$

Explicitly,

$$(3.5) \qquad b_{2k}^{(n)} = (-1)^k \binom{n+2k}{n} \sum_{m=0}^{2k} \frac{n}{n+m} \binom{2k}{m} \frac{1}{2^m (2k+m)!}$$

$$\cdot \sum_{r=0}^{m} (-1)^r \binom{m}{r} (2r - m)^{2k+m}.$$

PROOF. The corresponding characteristic functions for $j = 1, 2, \ldots, n$ are now given by

$$(2m_j + 1)\hat{\chi}_j(t) = \sum_{|k| \le m_j} e^{itk} = \begin{cases} \dfrac{\sin((m_j + 1/2)t)}{\sin(t/2)}, & \text{if } t \notin 2\pi Z, \\ 2m_j + 1, & \text{if } t \in 2\pi Z, \end{cases}$$

which we recognize as the familiar Dirichlet kernel—see e.g. Körner ((1988), p. 68). Since for $t \notin 2\pi Z$, we have

$$(\chi_1 * \chi_2 * \cdots * \chi_n)\hat{}(t) = \prod_{j=1}^n \hat{\chi}_j(t) = M \prod_{j=1}^n \frac{\sin((m_j + 1/2)t)}{\sin(t/2)},$$

it follows by orthogonality of the exponential that

$$g_n(p) = \frac{M}{2\pi} \int_0^{2\pi} e^{ipt} \prod_{j=1}^n \frac{\sin((m_j + 1/2)t)}{\sin(t/2)} dt = \frac{M}{\pi} \int_0^\pi e^{2ipx} \prod_{j=1}^n \frac{\sin((2m_j + 1)x)}{\sin x} dx.$$

We remark in passing that the factors $\sin((2m_j + 1)x)/\sin x$ in this latter representation are simply the Chebyshev polynomials $U_{2m_j}(\cos x)$ of the second kind—see e.g. Abramowitz and Stegun ((1972), p. 766). The partial-fraction expansion

$$(3.6) \qquad \left(\frac{1}{\sin x}\right)^n = \sum_{k=0}^{(n-1)/2} b_{2k}^{(n)} \sum_{r=-\infty}^\infty \left(\frac{(-1)^r}{x + r\pi}\right)^{n-2k},$$

is a modified version of the formula in Schwatt ((1924), pp. 209–210), and yields

$$(3.7) \quad g_n(p) = \sum_{k=0}^{(n-1)/2} b_{2k}^{(n)} \sum_{r=-\infty}^\infty \frac{M}{\pi} \int_0^\pi e^{2ipx} \left(\frac{(-1)^r}{x + r\pi}\right)^{n-2k} \prod_{j=1}^n \sin((2m_j + 1)x) dx.$$

When $n - 2k > 1$, the bilateral series

$$\sum_{r=-\infty}^\infty \left(\frac{(-1)^r}{x + r\pi}\right)^{n-2k}$$

converges absolutely, and the interchange of summation and integration is easily justified using either Lebesgue's dominated convergence theorem or Fubini's theorem with both Lebesgue and counting measure. If $n - 2k = 1$, absolute convergence can be recovered by recasting the bilateral series in the form

$$\sum_{r=-\infty}^\infty \frac{(-1)^r}{x + r\pi} = \frac{1}{x} + \sum_{r=1}^\infty (-1)^r \left(\frac{2x}{x^2 - r^2\pi}\right),$$

and the justification proceeds as in the previous case.

We shall see that the inner sum of integrals in (3.7) can be expressed as a single integral over the whole real line. To this end, we compute

$$\sum_{r=-\infty}^\infty \frac{1}{\pi} \int_0^\pi e^{2ipx} \left(\frac{(-1)^r}{x + r\pi}\right)^{n-2k} \prod_{j=1}^n \sin((2m_j + 1)x) dx$$

$$= \sum_{r=-\infty}^{\infty} \frac{(-1)^{(n-2k)r}}{\pi} \int_{r\pi}^{(r+1)\pi} t^{2k-n} e^{2ip(t-r\pi)} \prod_{j=1}^{n} \sin\left((2m_j + 1)(t - r\pi)\right) dt$$

$$= \sum_{r=-\infty}^{\infty} \frac{(-1)^{(n-2k)r}}{\pi} \int_{r\pi}^{(r+1)\pi} t^{2k-n} e^{2ipt} \prod_{j=1}^{n} (-1)^{(2m_j+1)r} \sin((2m_j + 1)t) dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} t^{2k-n} e^{2ipt} \prod_{j=1}^{n} \sin((2m_j + 1)t) dt.$$

This latter integral can be evaluated just as we evaluated the integral (2.5). After expanding the product of sines as a sum over the constituent exponentials and integrating by parts $n - 2k - 1$ times, one finds that

$$(3.8) \qquad \frac{1}{\pi} \int_{-\infty}^{\infty} t^{2k-n} e^{2ipt} \prod_{j=1}^{n} \sin((2m_j + 1)t) dt$$

$$= \frac{(-1)^k}{2^n (n - 2k - 1)!} \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(2p + \sum_{j=1}^{n} (2m_j + 1)\varepsilon_j \right)^{n-2k-1}$$

$$\times \operatorname{sign}\left(2p + \sum_{j=1}^{n} (2m_j + 1)\varepsilon_j \right) \prod_{j=1}^{n} \varepsilon_j.$$

Substituting (3.8) into (3.7) and interchanging the order of summation completes the proof of Theorem 2. $\square$

Remark. The equations (3.4), (3.5), (3.6) can also be obtained using Jordan's ((1979), §74, p. 216) general formula for the higher derivatives of a power of a reciprocal function and then applying Mittag-Leffler's theorem.

As in the continuous case, we can replace the sign function in Theorem 2 with the τ function of Definition 1. Thus we obtain

COROLLARY 2. *Let* g_n, M, *and* $b_{2k}^{(n)}$ *be as in Theorem 2. Then, for all positive integers* n *and integer* p,

$$(3.9) \qquad g_n(p) = \frac{M}{2^{n-1}} \sum_{k=0}^{(n-1)/2} \frac{(-1)^k b_{2k}^{(n)}}{(n - 2k - 1)!}$$

$$\cdot \sum_{\vec{\varepsilon} \in \{-1,1\}^n} \left(2p + \sum_{j=1}^{n} (2m_j + 1)\varepsilon_j \right)_{+}^{n-2k-1} \prod_{j=1}^{n} \varepsilon_j,$$

where $y_+^{n-2k-1} = y^{n-2k-1} \tau(y)$ *as in Definition 1.*

PROOF. Making the substitution $\operatorname{sign}(y) = 2\tau(y) - 1$ in Theorem 2 and noting that the coefficient of t^{n-2k-1} in

$$\sum_{\vec{\varepsilon} \in \{-1,1\}^n} \exp\left\{ \left(2p + \sum_{j=1}^{n} (2m_j + 1)\varepsilon_j \right) t \right\} = e^{2pt} \prod_{j=1}^{n} (e^{(2m_j+1)t} - e^{-(2m_j+1)t})$$

vanishes for $0 \leq k \leq (n-1)/2$, we obtain the desired result. $\square$

Example 5. When $n = 1$, formulae (3.3) and (3.9) give the representations

$$(2m_1 + 1)\chi_1(p) = \frac{1}{2}\operatorname{sign}\left(p + m_1 + \frac{1}{2}\right) - \frac{1}{2}\operatorname{sign}\left(p - m_1 - \frac{1}{2}\right)$$

$$= \tau\left(p + m_1 + \frac{1}{2}\right) - \tau\left(p - m_1 - \frac{1}{2}\right),$$

respectively. Both are equivalent to the definition (3.1) with $j = 1$.

Example 6. When $n = 2$, we have $M := (2m_1 + 1)^{-1}(2m_2 + 1)^{-1}$, and

$$g_2(p) = \chi_1 * \chi_2(p) = \sum_{k+j=p} \chi_1(k)\chi_2(j).$$

Since this is simply the coefficient of t^p in the product

$$M \sum_{|k| \leq m_1} t^k \sum_{|j| \leq m_2} t^j,$$

letting $u(p) := \min(m_1, p + m_2)$ and $v(p) := \max(-m_1, p - m_2)$, we have

$$g_2(p) = M \times \begin{cases} u(p) - v(p) + 1 & \text{if } u(p) \geq v(p), \\ 0 & \text{if } u(p) < v(p). \end{cases}$$

On the other hand formula (3.3) gives the elegant representation

$$g_2(p) = \frac{1}{2}M(|p + m_1 + m_2 + 1| - |p + m_1 - m_2| - |p - m_1 + m_2|$$

$$+ |p - m_1 - m_2 - 1|),$$

in which we have used the relation $y \operatorname{sign}(y) = |y|$ for y real.

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