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BETA APPROXIMATION TO THE DISTRIBUTION OF KOLMOGOROV-SMIRNOV STATISTIC

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Abstract. The distribution of Kolmogorov-Smirnov statistic can be globally approximated by a general beta distribution. The approximation is very simple and accurate. It can be easily implemented in any statistical software. Therefore, we can use a beta distribution to find the practical p-value of a goodness-of-fit test, which is much simpler than existing methods in the literature.

Key words and phrases: Kolmogorov-Smirnov statistic, beta distribution, approximation, skewness, kurtosis, p-value.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous population with distribution function $F(x)$, and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be its order statistics. We wish to test the null hypothesis

$$H_0: F(x) = F_0(x), \qquad \text{for all} \quad x$$

against the alternative

$$H_1: F(x) \neq F_0(x), \qquad \text{for some} \quad x$$

where $F_0(x)$ is a completely specified distribution function.

The most well-known goodness-of-fit test on H_0 v.s. H_1 is Kolmogorov-Smirnov statistic

$$D_n = \sup_{-\infty<x<\infty} |F_n(x) - F_0(x)|,$$

where $F_n(x)$ is the empirical distribution function defined by

$$F_n(x) = \#\{1 \le j \le n, X_j \le x\}/n, \qquad x \in (-\infty, \infty).$$

The statistic D_n is distribution-free. The asymptotic distribution of D_n under the null hypothesis was derived by Kolmogorov (1933), and Smirnov (1939) gave a simpler proof. However, the exact null distribution for finite-sample case is complicated to express. Kolmogorov (1933) and Massey (1950) established recursive formulas for

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calculating the null probability $P(D_n < k/n)$ for integer values of k. Then Birnbaum (1952) tabulated these values for $n = 1, 2, \ldots, 100$ and $k = 1, 2, \ldots, 15$.

Since the exact null distribution of D_n is only available at k/n for limited integer values of k, approximate methods have been explored. For example, some critical values of D_n based on interpolation were given by Massey (1951) and Birnbaum (1952), and the most common-used approximate critical values in statistical tables and literature were from Miller (1956). However, the approximation is only valid for the upper tail of the distribution, since the critical values (with level α) are approximated by the exact ones (with level $\alpha/2$) for one-sided test. See Conover (1980) and Gibbons (1992).

Research on the Kolmogorov-Smirnov statistics and their sampling distributions remains very active. See, for instance, Cabaña (1996), Cabaña and Cabaña (1994, 1997), Friedrich and Schellhaas (1998), Justel *et al.* (1997), Kim (1999), Kulinskaya (1995), Paramasamy (1992) and Rama (1993) among others.

In this paper we use a general beta distribution to approximate the sampling distribution of D_n, which will enable us to obtain the practical p-value of Kolmogorov-Smirnov test with sufficient closeness to the true one. It can be easily implemented in any statistical software. Note that traditional methods of approximating the p-value are more complicated and less accurate. For example, the current approximation method used in S-Plus is based on interpolation for small sample ($n \leq 50$) or the limiting distribution for $n > 50$, which may not be accurate enough (see Section 2). Of course, with now-days computers, it is likely for people to compute the p-value or critical point by Monte Carlo methods rather than interpolating, but our beta approximation is easier to perform and rather accurate.

The paper is arranged as follows: In Section 2 we will show that the distribution of D_n can be globally approximated by a general beta distribution, and the approximation is very simple and accurate. Therefore, by using a beta distribution, it is easy to get the practical p-value for the Kolmogorov-Smirnov test. Approximation for one-sided test is discussed in Section 3.

2. Beta approximation to the distribution of D_n

Let $B_{p,q}$ denote a random variable having standard beta distribution $Beta(p, q)$ with density

$$b_{p,q}(x) = x^{p-1}(1-x)^{q-1}/B(p,q), \qquad 0 < x < 1,$$

and distribution function

$$B_{p,q}(x) = \int_{-\infty}^{x} b_{p,q}(t)dt, \qquad -\infty < x < \infty,$$

where $B(p,q)$ is beta function with p, $q > 0$.

Our simulation study shows that the distribution of Kolmogrov-Smirnov statistic D_n approximately equals that of a general beta variable $aB_{p,q} + b$, where constants a, b, p, q are chosen such that D_n and $aB_{p,q} + b$ have the same first four moments, or equivalently have the same mean μ, standard deviation σ, skewness $r_1 = \bar{\mu}_3/\sigma^3$ and kurtosis $r_2 = \bar{\mu}_4/\sigma^4$, where $\bar{\mu}_k$ denotes the k-th central moment.

Let μ_n, σ_n, r_{n1} and r_{n2} be, respectively, the mean, standard deviation, skewness and kurtosis of D_n. It is easy to prove that D_n and $aB_{p,q} + b(a > 0)$ have the same mean, standard deviation, skewness and kurtosis (or equivalently have the same first four moments) if and only if

$$\begin{cases} \mu_n = \dfrac{ap}{p+q} + b, & r_{n1} = \dfrac{2(q-p)}{p+q+2}\sqrt{\dfrac{p+q+1}{pq}}, \\ \sigma_n = \dfrac{a}{p+q}\sqrt{\dfrac{pq}{p+q+1}}, & r_{n2} = \dfrac{3(p+q+1)}{p+q+3}\left[\dfrac{2(q-p)^2}{pq(p+q+2)} + 1\right]. \end{cases}$$

Note that

$$\begin{cases} p+q = P(r_{n1}, r_{n2}) & \text{with } P = P(x,y) = 6(y - x^2 - 1)/(3x^2 - 2y + 6), \\ pq = Q(r_{n1}, r_{n2}) & \text{with } Q(x,y) = 4P^2/[16 + x^2(P+2)^2/(P+1)]. \end{cases}$$

Hence, $a, b, p, q (a > 0)$ are uniquely decided by

$$\begin{cases} p, q = [P(r_{n1}, r_{n2}) \pm \sqrt{P(r_{n1}, r_{n2})^2 - 4Q(r_{n1}, r_{n2})}]/2, \\ a = \sigma_n(p+q)\sqrt{(p+q+1)/(pq)}, \qquad b = \mu_n - ap/(p+q), \end{cases} \tag{2.1}$$

with $q > p (q \leq p)$ if $r_{n1} > 0 (r_{n1} \leq 0)$.

Then D_n and $aB_{p,q} + b$ have the exactly same first four moments, as well as the approximately same moments of higher order based on our simulation (see below). Therefore, they have approximately the same moment generating function or characteristic function, and thus they have approximately the same distribution.

As a result, $F_{D_n}(x)$ and $f_{D_n}(x)$, the distribution and density functions of D_n, can be simply approximated by those of $aB_{p,q} + b$, i.e.,

$$F_{D_n}(x) \approx B_{p,q}\left(\frac{x-b}{a}\right) \quad \text{and} \quad f_{D_n}(x) \approx b_{p,q}\left(\frac{x-b}{a}\right) \Big/ a, \tag{2.2}$$

where a, b, p, q are given by (2.1) and will be approximated by (2.3).

Usually, having the same first four moments is not enough to guarantee a very good approximation, but (2.2) is a special case where the two distributions also have sufficiently close moments of higher order. For $n = 10$, 100 and 1000, for example, the first ten standard moments of the two variables are listed in Table 1, where the upper numbers in double entries are the moments of D_n based on simulation with size of one million, and the lower numbers correspond to $aB_{p,q} + b$. The specific values of a, b, p, q are given in Table 2.

Table 1. The first ten standard moments of D_n and $aB_{p,q} + b$.

n	μ	σ	$\bar{\mu}_3/\sigma^3$	$\bar{\mu}_4/\sigma^4$	$\bar{\mu}_5/\sigma^5$	$\bar{\mu}_6/\sigma^6$	$\bar{\mu}_7/\sigma^7$	$\bar{\mu}_8/\sigma^8$	$\bar{\mu}_9/\sigma^9$	$\bar{\mu}_{10}/\sigma^{10}$
10	0.25916	0.079832	0.8180	3.697	8.237	29.78	98.51	377.7	1500	6329
	0.25916	0.079832	0.8180	3.697	8.399	30.60	104.1	408.7	1672	7315
100	0.08519	0.025916	0.8561	3.869	9.209	34.70	124.2	516.0	2235	10533
	0.08519	0.025916	0.8561	3.869	9.311	35.10	127.1	529.6	2329	10999
1000	0.02730	0.008229	0.8616	3.884	9.285	34.93	125.2	517.0	2248	10403
	0.02730	0.008229	0.8616	3.884	9.399	35.47	129.0	539.2	2380	11286

Table 2. The first four standard moments of D_n and corresponding a, b, p, q.

n	μ_n	σ_n	r_{n1}	r_{n2}	a	b	p	q
5	0.35826	0.109496	0.7583	3.495	1.1571	0.14674	2.867	12.82
10	0.25916	0.079832	0.8180	3.697	1.0186	0.10590	2.980	16.83
20	0.18636	0.057362	0.8389	3.784	0.8215	0.07555	3.093	19.84
30	0.15331	0.047046	0.8457	3.818	0.7153	0.06190	3.165	21.60
50	0.11967	0.036602	0.8559	3.862	0.5974	0.04830	3.229	23.80
70	0.10150	0.031013	0.8562	3.863	0.5053	0.04106	3.224	23.73
100	0.08519	0.025916	0.8561	3.869	0.4327	0.03445	3.267	24.59
150	0.06985	0.021213	0.8599	3.887	0.3659	0.02823	3.298	25.70
200	0.06063	0.018404	0.8607	3.878	0.3051	0.02481	3.225	24.25
300	0.04960	0.015020	0.8594	3.864	0.2401	0.02053	3.171	23.02
500	0.03851	0.011636	0.8574	3.863	0.1878	0.01590	3.201	23.38
1000	0.02730	0.008229	0.8616	3.884	0.1385	0.01125	3.245	24.76

It can be seen from Table 1 that they do have the same first four moments and similar moments of higher order. Unfortunately, the exact first four moments of D_n are not available to determine a, b, p, q, so we use Monte Carlo approach. For some selected values of n, Table 2 lists the first four standard moments of D_n obtained by a one-million-size simulation together with the corresponding values of a, b, p, q calculated from (2.1).

Since D_n is distribution-free, its moments depend only on n, so do a, b, p, q in (2.1). For simplicity, linear functions of n^{-1} and n^d (d is fixed) are used to approximate them. Of course, better approximations may be made by using more complicated functions at the price of the simplicity, the most important feature of this paper.

Directly fitting the data of a, b, p, q in Table 2 does not work well and could destroy their structure in (2.1). which enable the approximate distribution to have correct first four moments. Instead we fit the moments first. A linear regression model $y = \beta_0 + \beta_1 n^{-1} + \beta_2 n^d$ is used to fit (by least squares approach) the data of μ_n, σ_n, r_{n1}, r_{n2} (against n) in Table 2 respectively. For different values of d, we have different models to fit the data. We choose a d which roughly corresponds to the best fit by the following approach. One can choose any initial value of d, and then fit the model to the data. If the fit is satisfactory, stop. Otherwise, increase or decrease the value of d and fit the model again. Then choose the d which corresponds to a better fit. Repeat this process until a satisfactory fit is obtained. In this way, the best d can be roughly reached within a few steps by our experience. The results are as follows:

$$\begin{cases} \hat{\mu}_n = -0.00008631 - 0.1348/n + 0.8587/n^{0.498}, & \hat{r}_{n1} = 0.861 - 0.3748/n - 0.6908/n^2, \\ \hat{\sigma}_n = 0.0004787 - 0.09059/n + 0.296/n^{0.525}, & \hat{r}_{n2} = 3.884 - 1.815/n - 0.6549/n^2. \end{cases}$$

Then, using them as the mean, standard deviation, skewness and kurtosis of D_n, we can get a new set of data for a, b, p, q (against n) via (2.1). The new data is well fitted by

$$(2.3)\quad \begin{cases} \hat{a} = 0.003326 - 6.012/n + 5.52/n^{0.53}, & \hat{p} = 3.258 - 3.727/n + 4.607/n^{1.6}, \\ \hat{b} = -0.0004245 - 0.003397/n + 0.3204/n^{0.48}, & \hat{q} = 25 - 161.2/n + 162.2/n^{1.3}, \end{cases}$$

which thus well keeps the original structure of (2.1).

With a, b, p and q approximated by (2.3), the distribution of Kolmogorov-Smirnov statistic D_n can be simply approximated by a completely known beta distribution in (2.2).

We now discuss the accuracy of the approximation. For $n = 10, 50, 100, 200$ and 500, Table 3 lists three sets of percentage points of D_n for comparison. The first line in multiple entries is obtained from (2.2) with a, b, p and q approximated by (2.3); the second line is based on a Monte Carlo simulation of size 100,000; the third line lists the most common-used approximate values given by Miller (1956), which are only available for upper tail (asymptotic values are used if $n > 100$). See also Conover (1980) and Gibbons (1992).

It can be seen from Table 3 that (a) compared with the simulation results, our approximate values are very accurate in the whole region (lower, central and upper parts) of the distribution, and the higher the percentage level, the more accurate the approximation; (b) at the upper tail they are consistent with Miller's approximate results for $n \leq 100$ but are better than asymptotic values, which are always a little bit larger than real ones, especially when $n \leq 200$.

The exact sampling distribution of D_n is complicated. Kolmogorov (1933) and Massey (1950) established recursive formulas for calculating the null probability $P(D_n <$

Table 3. Percentage points for D_n.

n	0.01	0.05	0.10	0.20	0.50	0.80	0.90	0.95	0.99
10	0.1300	0.1512	0.1667	0.1897	0.2479	0.3239	0.3698	0.4103	0.4910
	0.1273	0.1518	0.1673	0.1896	0.2468	0.3222	0.3691	0.4099	0.4885
						0.3226	0.3687	0.4093	0.4889
50	0.0606	0.0703	0.0773	0.0877	0.1139	0.1482	0.1691	0.1878	0.2256
	0.0596	0.0706	0.0778	0.0881	0.1140	0.1482	0.1693	0.1883	0.2265
						0.1484	0.1696	0.1884	0.2260
100	0.0433	0.0503	0.0553	0.0627	0.0813	0.1057	0.1206	0.1339	0.1608
	0.0426	0.0504	0.0556	0.0630	0.0812	0.1055	0.1207	0.1341	0.1608
						0.1056	0.1207	0.1340	0.1608
200	0.0310	0.0359	0.0394	0.0447	0.0579	0.0752	0.0858	0.0952	0.1144
	0.0305	0.0361	0.0396	0.0447	0.0577	0.0750	0.0856	0.0950	0.1145
						0.0759	0.0865	0.0960	0.1151
500	0.0197	0.0229	0.0251	0.0284	0.0368	0.0478	0.0545	0.0605	0.0726
	0.0194	0.0229	0.0252	0.0285	0.0366	0.0477	0.0543	0.0603	0.0723
						0.0480	0.0547	0.0607	0.0728

Table 4. Exact values and the beta approximations for $P(D_n < k/n)$ when $n = 40$.

k	3	4	5	6	7	8	9	10	11	12
exact	0.0345	0.2182	0.4808	0.7016	0.8471	0.9295	0.9708	0.9891	0.9964	0.9989
appr.	0.0344	0.2224	0.4812	0.7021	0.8488	0.9311	0.9716	0.9894	0.9964	0.9989

k/n) for integer values of k. Note that the recursive formulas only apply to integer k. Birnbaum (1952) tabulated these values for $n = 1, 2, \ldots, 100$ and $k = 1, 2, \ldots, 15$. We can use Birnbaum's tables to check the accuracy of our approximation. Table 4 are such exact values for n=40 compared with the values obtained by the beta approximation given by (2.2). It can be seen that the values corresponding to the same k are almost equal, especially for large k. The situation is similar for other sample sizes.

We conclude from above that (2.2) globally gives very simple and accurate approximations to the distribution and density functions of D_n, especially at the upper tail (the most important part). Hence, we can easily use a beta distribution to find the practical p-value of the Kolmogorov-Smirnov test, which is simpler and more accurate than existing methods in the literature. For example, the current approximation method used in S-Plus is based on interpolation on limited values of exact distribution for small sample ($n \leq 50$), or the limiting distribution for $n > 50$, which has been shown in Table 3 that it may not give a good approximation to the true value if $n < 200$.

3. Approximation for one-sided test

If we want to test one-sided hypothesis, say

$$H_0: \ F(x) \leq F_0(x), \qquad \text{for all} \quad x$$

against the alternative

$$H_1: \ F(x) > F_0(x), \qquad \text{for some} \quad x,$$

the corresponding Kolmogorov-Smirnov statistic is

$$D_n^+ = \sup_{-\infty < x < \infty} [F_n(x) - F_0(x)].$$

The exact distribution of D_n^+ has been found by Birnbaum and Tingey (1951), but the computation is not easy. Miller (1956) tabulated its critical values for $n = 1, 2, \ldots, 100$ and $\alpha = 0.10, 0.05, 0.025, 0.05, 0.005$.

As in Section 2, for $n = 10, 100$ and 1000 Table 5 lists the first ten standard moments of D_n^+ and $aB_{p,q}+b$ (the values of a, b, p, q are given in Table 6). We can see that their high order moments are not so close as the two-sided test. Therefore, the beta approximation here could be less accurate than before.

Table 6 gives the first four standard moments of D_n^+ obtained by a one-million-size simulation together with the corresponding a, b, p, and q. Similarly, the data in Table 6 are fitted by

Table 5. The first ten standard moments of D_n^+ and $aB_{p,q} + b$.

n	μ	σ	$\bar{\mu}_3/\sigma^3$	$\bar{\mu}_4/\sigma^4$	$\bar{\mu}_5/\sigma^5$	$\bar{\mu}_6/\sigma^6$	$\bar{\mu}_7/\sigma^7$	$\bar{\mu}_8/\sigma^8$	$\bar{\mu}_9/\sigma^9$	$\bar{\mu}_{10}/\sigma^{10}$
10	0.1830	0.1020	0.5953	3.116	5.362	19.10	52.60	183.0	624.4	2324
	0.1830	0.1020	0.5953	3.116	5.220	18.52	48.96	166.3	540.7	1927
100	0.06105	0.03267	0.6278	3.235	5.955	21.62	64.07	235.2	868.9	3491
	0.06105	0.03267	0.6278	3.235	5.770	20.75	58.38	206.3	713.7	2691
1000	0.01963	0.01036	0.6321	3.249	6.025	21.93	65.58	242.8	908	3701
	0.01963	0.01036	0.6321	3.249	5.840	21.03	59.59	211.5	737	2796

Table 6. The first four standard moments of D_n^+ and corresponding a, b, p, q.

n	μ_n	σ_n	r_{n1}	r_{n2}	a	b	p	q
5	0.25093	0.14236	0.5521	2.983	1.0897	−0.046026	2.893	7.723
10	0.18298	0.10196	0.5953	3.116	0.8774	−0.032409	3.122	9.595
20	0.13245	0.07273	0.6140	3.173	0.6594	−0.021709	3.208	10.516
30	0.10924	0.05951	0.6213	3.202	0.5580	−0.017693	3.288	11.164
50	0.08547	0.04621	0.6284	3.232	0.4493	−0.013792	3.373	11.896
70	0.07254	0.03905	0.6267	3.220	0.3733	−0.010934	3.324	11.541
100	0.06105	0.03267	0.6278	3.235	0.3204	−0.009412	3.409	12.094
150	0.05009	0.02671	0.6282	3.238	0.2630	−0.007585	3.421	12.180
200	0.04350	0.02317	0.6328	3.245	0.2283	−0.006293	3.394	12.168
300	0.03564	0.01891	0.6329	3.247	0.1870	−0.005049	3.406	12.244
500	0.02771	0.01464	0.6286	3.236	0.1437	−0.003852	3.406	12.102
1000	0.01963	0.01036	0.6321	3.249	0.1031	−0.002733	3.436	12.407

Table 7. Some critical values for D_n^+.

n	0.10	0.05	0.025	0.01	0.005
10	0.3238	0.3706	0.4114	0.4582	0.4893
	0.3226	0.3687	0.4093	0.4566	0.4889
50	0.1486	0.1701	0.1891	0.2111	0.2260
	0.1484	0.1696	0.1884	0.2107	0.2260
100	0.1059	0.1212	0.1347	0.1504	0.1611
	0.1056	0.1207	0.1340	0.1499	0.1608

$$\begin{cases} \hat{\mu}_n = -0.0001739 - 0.1251/n + 0.6134/n^{0.496}, & \hat{r}_{n1} = 0.6322 - 0.331/n - 0.351/n^2, \\ \hat{\sigma}_n = 0.0002374 - 0.04877/n + 0.3451/n^{0.51}, & \hat{r}_{n2} = 3.248 - 1.375/n + 0.2648/n^2 \end{cases}$$

and

$$\begin{cases} \hat{a}=0.002816 - 3.063/n + 3.99/n^{0.53}, & \hat{p}=3.426 - 4.28/n + 5.061/n^{1.7}, \\ \hat{b}=-0.0002485 + 0.02671/n - 0.1283/n^{0.57}, & \hat{q}=12.34 - 37.79/n + 45.63/n^{1.7}. \end{cases} \tag{3.1}$$

Then the distribution of D_n^+ can be approximated by that of beta like (2.2) with a, b, p and q approximated by (3.1). The approximation is poor at the lower tail (a less important part) but is still as accurate as before in other parts of the distribution. For example, for n=10, 50, 100 Table 7 compares the approximate critical values of D_n^+ (the upper numbers in double entries) with the exact ones (the lower numbers) given by Miller (1956).

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