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TESTS OF PARAMETERS OF SEVERAL GAMMA DISTRIBUTIONS WITH INEQUALITY RESTRICTIONS

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Abstract. Gamma distribution is one of the most used methods of modeling lifetime data. However, testing homogeneity of parameters of $m \geq 3$ gamma distributions against order restrictions is almost non-existent in the current literature. We propose two methods to this end: one uses quadratic forms involving ratios of cumulants as test statistic and the other is a stepwise procedure which uses Fisher's method of combining p-values when shape parameters are equal but unknown. Both procedures allow use of arbitrary sample sizes of m populations. Test of the inequality restrictions as a null hypothesis against unrestricted alternatives is also considered. A Monte Carlo study of power at various alternatives shows that both methods are competitive when they are applicable.

Key words and phrases: Approximate tests, cumulants, Fisher's combination method, Monte Carlo studies, order restricted tests, quadratic forms.

1. Introduction

Over last few decades, the gamma distribution has arisen as one of the most important vehicles to model life-testing situations. Because of the flexibility in choice of the shape and scale parameters, a wide variety of lifetime data fits quite adequately to it. Among situations that lead to the gamma distribution are waiting time problems as it is well-known that the time to k-th occurrence of a Poisson process follows a gamma distribution. In reliability studies and in life testing, the gamma distribution is used as a generalization of the exponential distribution which is also a popular choice for modeling purposes. The gamma distribution is suggested as the failure time model for a system under continuous maintenance, where the reliability may experience some initial growth or decay but then reaches a stable state as time goes on. The gamma distribution has also been used in weather analysis. In theoretical calculations, the gamma distribution arises as the sum of independent, identically distributed exponential random variables. Gamma distribution evolves in the testing of equality of variances of several independent normal distributions. Johnson and Kotz (1970) provides a good review of the gamma distribution including several applications in various fields.

The gamma distribution denoted by $G(x; \theta, \gamma)$, with θ and γ being the scale and the shape parameters respectively, has density as

$$(1.1)\qquad \frac{1}{\Gamma(\gamma)\theta^{\gamma}} x^{\gamma-1} e^{-x/\theta}, \qquad x > 0,\ \theta, \gamma > 0.$$

The maximum likelihood estimation of γ can be approximated by the empirically determined formulas of Greenwood and Durand (1960) as given below

$$\begin{aligned}\hat{\gamma} &= \frac{0.5000876 + 0.1648852S - 0.0544274S^2}{S}, \qquad 0 < S \leq 0.5772 \\ &= \frac{8.898919 + 9.059950S + 0.9775373S^2}{S(17.79728 + 11.968477S + S^2)}, \qquad 0.5772 < S \leq 17 \\ &= \frac{1}{S}, \qquad 17 < S\end{aligned}$$

where $S = \ln(A/G)$, and A and G denote the arithmetic and geometric means respectively.

Inferences concerning the parameters of the gamma distribution are rather difficult mainly due to the fact that they are not of the conventional location-scale type. Bain and Engelhardt (1975) derived the exact tests of γ with θ being an unknown nuisance parameter. Engelhardt and Bain (1977) developed a conditional test for testing the scale parameter of a gamma distribution with unknown shape parameter. Grice and Bain (1980) proposed an approximate test for the mean of a gamma distribution with both parameters unknown. For the two-sample situation, Shiue and Bain (1983) proposed an approximate test for testing the equality of the scale parameters of two gamma distributions with unknown but common shape parameter. Also, Shiue *et al.* (1988) proposed an approximate procedure for testing the equality of means of two gamma distributions with unknown and unequal shape parameters.

Inference for parameters of more than two gamma distributions is quite rare in the existing literature. Tripathi *et al.* (1993) proposed a test for the parameters of $m \geq 2$ gamma distributions based on a generalized minimum chi-square procedure. This m-sample test is applicable in versatile testing situations with general unrestricted alternatives, it is asymptotic in nature, and for $m = 2$, the authors found the test by Shiue and Bain (1983) performed better in the equal but unknown shape parameter case. Mudholkar *et al.* (1993) considered a test of equality of variances of m normal distributions against restricted alternatives which essentially reduces to testing the equality of the scale parameters of m gamma distributions with shape parameters being a function of the sample sizes and hence known. Robertson *et al.* (1988) considered likelihood ratio tests for trend in the scale parameters of $m \geq 2$ gamma distributions when the shape parameters are known. Large sample approximations for the significance levels are obtained in terms of chi-bar squared distributions. For the simple order alternative and equal sample size case, approximations are provided in terms of special functions. Recently, Bhattacharya (2001) considered the problem of testing equality of scale parameters of $m \geq 3$ gamma distributions against nonincreasing order restrictions with a common but unknown shape parameter using the Fisher's method of combination of p-values. This testing procedure is also applicable against various other order restrictions.

In this paper we consider hypotheses tests involving general linear combinations of parameters for $m \geq 3$ gamma distributions against inequality restrictions. The first test statistic we propose uses quadratic forms involving ratios of cumulants of the gamma distributions as in Tripathi *et al.* (1993). We show that the asymptotic distribution of this test statistic is of the form of chi-bar squared, a weighted combination of chi-square distributions mixed over their degrees of freedom. We also consider testing the inequality restrictions as a null hypothesis against unrestricted alternatives. The test statistic in this case, also an appropriately defined quadratic form, has an asymptotic chi-bar square distribution as well. This is the content of Section 2.

In Section 3, we consider several applications of the results of Section 2. In particular, we have discussed the testing procedure for testing the equality of the means, equality of the scale parameters, and equality of the shape parameters against nonincreasing alternatives. We have considered the calculations of the weights of the chi-bar square distributions, known as the *level probabilities*, which are different for different situations. We also point out that the choice of the matrix $\boldsymbol{w}^*$ given on page 778 of Tripathi *et al.* (1993) when testing for the coefficient of variations is incorrect, and show the correct matrix $\boldsymbol{w}^*$ to be used.

In Section 4, we derive a closed form expression of power of the testing problem discussed by Bhattacharya (2001). In Section 5, we compare the power of this procedure with the previous in a Monte Carlo study at selected alternatives, shape parameters and various sample sizes. In Section 6, we make some concluding remarks.

2. Using ratios of cumulants

We consider general linear hypotheses regarding means and scale parameters of m gamma populations. We use notations similar to Tripathi *et al.* (1993). The hypotheses of interests are

$$H_0 : \boldsymbol{C\theta} = \boldsymbol{\Phi}_0, \qquad H_1 : \boldsymbol{C\theta} \geq \boldsymbol{\Phi}_0, \qquad H_2 : \boldsymbol{\theta} \text{ unrestricted} \tag{2.1}$$

where $\boldsymbol{C}$ is a $r \times 2m$ given matrix of rank r, $\boldsymbol{\Phi}_0$ is a given r-vector. Also $\boldsymbol{\theta}' = [\boldsymbol{\theta}_1^{*\prime}, \boldsymbol{\theta}_2^{*\prime}, \dots, \boldsymbol{\theta}_m^{*\prime}]$ with $\boldsymbol{\theta}_i^{*\prime} = [\theta_{i1}^*, \theta_{i2}^*] = [\gamma_i\theta_i, \theta_i]$, $i = 1, \dots, m$. The inequalities in H_1 in (2.1) are coordinatewise.

Let $X_{i1}, X_{i2}, \dots, X_{in_i}$ denote a random sample of size n_i from the i-th gamma population, $i = 1, \dots, m$, and assume that these m random samples are independent. Let the j-th cumulant of the i-th population be denoted by κ_{ij}. Then it can be easily seen that

$$\kappa_{i1} = \gamma_i\theta_i, \qquad \kappa_{ij} = (j-1)!\gamma_i\theta_i^j, \quad j > 1, \ \kappa_{i,j+1}/\kappa_{ij} = j\theta_i.$$

Let $\eta_{i0} = \kappa_{i1}, \eta_{ij} = \kappa_{i,j+1}/\kappa_{ij}$, $j \geq 1$. As in Tripathi *et al.* (1993), we use first four cumulants. This uses upto eighth order sample moments to estimate the population moments occuring in the covariance matrix of the $\boldsymbol{h}$ vector (defined below). The higher moments we use, more fluctuations there are in the estimate of the asymptotic covariance matrix. Let $\boldsymbol{\eta}_i' = (\eta_{i0}, \eta_{i1}, \eta_{i2}, \eta_{i3})$, $i = 1, \dots, m, \boldsymbol{\eta}' = [\boldsymbol{\eta}_1', \boldsymbol{\eta}_2', \dots, \boldsymbol{\eta}_m']$. Clearly, $\boldsymbol{\eta}_i = \boldsymbol{w}^*\boldsymbol{\theta}_i^*, i = 1, \dots, m$ with

$$\boldsymbol{w}^{*\prime} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Letting $\boldsymbol{w} = \operatorname{diag}(\boldsymbol{w}^*, \boldsymbol{w}^*, \dots, \boldsymbol{w}^*)$, we get the linear relation $\boldsymbol{\eta} = \boldsymbol{w\theta}$ where $\boldsymbol{w}$ is a $4m \times 2m$ matrix of known constants.

Let the sample counterpart of κ_{ij} be denoted by k_{ij}. Also let $h_{i0} = k_{i1}, h_{ij} = k_{i,j+1}/k_{i,j}$ where $h_{i0}, h_{ij}, \boldsymbol{h}_i, \boldsymbol{h}$ are sample counterparts of $\eta_{i0}, \eta_{ij}, \boldsymbol{\eta}_i, \boldsymbol{\eta}$, respectively. By the multivariate delta method $\boldsymbol{h}$ has an asymptotic normal distribution with mean $\boldsymbol{\eta}$ and variance-covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{J}_2\boldsymbol{J}_1\boldsymbol{V}\boldsymbol{J}_1'\boldsymbol{J}_2'$ where $\boldsymbol{V} = \operatorname{diag}(\boldsymbol{V}_1, \boldsymbol{V}_2, \dots, \boldsymbol{V}_m)$ with

$$\boldsymbol{V}_i = \frac{1}{n_i}\begin{bmatrix} \mu_{i2} - \mu_{i1}^2 & & & \\ \mu_{i3} - \mu_{i2}\mu_{i1} & \mu_{i4} - \mu_{i2}^2 & & \\ \mu_{i4} - \mu_{i3}\mu_{i1} & \mu_{i5} - \mu_{i3}\mu_{i2} & \mu_{i6} - \mu_{i3}^2 & \\ \mu_{i5} - \mu_{i4}\mu_{i1} & \mu_{i6} - \mu_{i4}\mu_{i2} & \mu_{i7} - \mu_{i4}\mu_{i3} & \mu_{i8} - \mu_{i4}^2 \end{bmatrix}$$

is the asymptotic covariance matrix of $(m_{i1}, m_{i2}, m_{i3}, m_{i4})$, a vector of the sample moments from the i-th sample, μ_{ij} is the j-th raw moment from the i-th population. The matrices $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are the Jacobians given by

$$\boldsymbol{J}_1 = \operatorname{diag}(\boldsymbol{J}_{11}, \boldsymbol{J}_{12}, \ldots, \boldsymbol{J}_{1m}), \quad \boldsymbol{J}_2 = \operatorname{diag}(\boldsymbol{J}_{21}, \boldsymbol{J}_{22}, \ldots, \boldsymbol{J}_{2m})$$

where $\boldsymbol{J}_{1i}$ and $\boldsymbol{J}_{2i}$ correspond to the transformations

$$\boldsymbol{J}_{1i} : (\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}) \rightarrow (\kappa_{i1}, \kappa_{i2}, \kappa_{i3}, \kappa_{i4}),$$
$$\boldsymbol{J}_{2i} : (\kappa_{i1}, \kappa_{i2}, \kappa_{i3}, \kappa_{i4}) \rightarrow (\eta_{i0}, \eta_{i1}, \eta_{i2}, \eta_{i3}).$$

The elements of $\boldsymbol{J}_{1i}$ and $\boldsymbol{J}_{2i}$ are

$$\boldsymbol{J}_{1i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2\mu_{i1} & 1 & 0 & 0 \\ -3(\mu_{i2} - 2\mu_{i1}^2) & -3\mu_{i1} & 1 & 0 \\ -4(\mu_{i3} - 6\mu_{i1}^2\mu_{i2}^2 + 6\mu_{i1}^3) & -6(\mu_{i2} - 2\mu_{i1}^2) & -4\mu_{i1} & 1 \end{bmatrix}$$

$$\boldsymbol{J}_{2i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\kappa_{i2}/\kappa_{i1}^2 & 1/\kappa_{i1} & 0 & 0 \\ 0 & -\kappa_{i3}/\kappa_{i2}^2 & 1/\kappa_{i2} & 0 \\ 0 & 0 & -\kappa_{i4}/\kappa_{i3}^2 & 1/\kappa_{i3} \end{bmatrix}.$$

To test H_0 against $H_1 - H_0$, we propose the following test statistic

$$(2.2) \qquad T_{01} = \min_{C\theta = \Phi_0} (\boldsymbol{h} - \boldsymbol{w\theta})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{h} - \boldsymbol{w\theta}) - \min_{C\theta \geq \Phi_0} (\boldsymbol{h} - \boldsymbol{w\theta})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{h} - \boldsymbol{w\theta})$$

where $\hat{\boldsymbol{\Sigma}}$ is a consistent estimate of $\boldsymbol{\Sigma}$. We reject H_0 for large values of T_{01}. The test statistic T_{01} can also be expressed as

$$(2.3) \qquad T_{01} = \min_{C\theta = \Phi_0} (\boldsymbol{z} - \boldsymbol{\theta})'\hat{\boldsymbol{\Omega}}^{-1}(\boldsymbol{z} - \boldsymbol{\theta}) - \min_{C\theta \geq \Phi_0} (\boldsymbol{z} - \boldsymbol{\theta})'\hat{\boldsymbol{\Omega}}^{-1}(\boldsymbol{z} - \boldsymbol{\theta})$$

where $\hat{\boldsymbol{\Omega}} = (\boldsymbol{w}'\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{w})^{-1}$ and $\boldsymbol{z} = (\boldsymbol{w}'\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{w})^{-1}\boldsymbol{w}'\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{h}$. It can be seen that the matrix $\hat{\boldsymbol{\Omega}} = \operatorname{diag}(\hat{\boldsymbol{\Omega}}_1, \hat{\boldsymbol{\Omega}}_2, \ldots, \hat{\boldsymbol{\Omega}}_m)$ is a block diagonal matrix where each diagonal element $\hat{\boldsymbol{\Omega}}_i = (\boldsymbol{w}'\hat{\boldsymbol{\Sigma}}_i^{-1}\boldsymbol{w})^{-1}$ is (2×2). The alternative expression in (2.3) is, however, not needed for hypotheses testing purposes.

It is easy to show that the minimum in the first term in (2.2) is attained at $\hat{\boldsymbol{\theta}}_0 = \boldsymbol{z} - \hat{\boldsymbol{\Omega}}\boldsymbol{C}'(\boldsymbol{C}\hat{\boldsymbol{\Omega}}\boldsymbol{C}')^{-1}(\boldsymbol{Cz} - \boldsymbol{\Phi}_0)$. However the minimum in the second term in (2.2) must be found by numerical methods. Let the minimum in the second term in (2.2) be attained at $\hat{\boldsymbol{\theta}}_1$. The distribution of T_{01} is given by Theorem 1 below.

When testing H_1 against $H_2 - H_1$, we propose the following test statistic

$$(2.4) \qquad T_{12} = \min_{C\theta \geq \Phi_0} (\boldsymbol{h} - \boldsymbol{w\theta})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{h} - \boldsymbol{w\theta}) - \min_{\theta \text{ unrestricted}} (\boldsymbol{h} - \boldsymbol{w\theta})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{h} - \boldsymbol{w\theta}).$$

We reject H_1 for large values of T_{12}. It is easy to show that the minimum in the second term in (2.4) is attained at the value $\hat{\boldsymbol{\theta}} = \boldsymbol{z}$. As earlier the test statistic T_{12} can be expressed as

$$(2.5) \qquad T_{12} = \min_{C\theta \geq \Phi_0} (\boldsymbol{z} - \boldsymbol{\theta})'\hat{\boldsymbol{\Omega}}^{-1}(\boldsymbol{z} - \boldsymbol{\theta}) - \min_{\theta \text{ unrestricted}} (\boldsymbol{z} - \boldsymbol{\theta})'\hat{\boldsymbol{\Omega}}^{-1}(\boldsymbol{z} - \boldsymbol{\theta})$$

although this alternative expression in (2.5) is not needed for hypotheses testing purposes. The distribution of T_{12} is also given by Theorem 1 below.

Let $\boldsymbol{\Omega} = (\boldsymbol{w}'\boldsymbol{\Sigma}^{-1}\boldsymbol{w})^{-1}$. For $0 \le i \le r$, let $P(i, m-1, \boldsymbol{C\Omega C}')$, called the *level probabilities*, be the probability that $\boldsymbol{C}\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\Phi}_0$ has i distinct positive components under H_0. The proof of Theorem 1 follows from the work of Shapiro (1985, 1988) and Kudô (1963).

THEOREM 1. *For a constant c_1, the asymptotic distribution of T_{01} under H_0 is given by*

$$\lim_{n_i \to \infty, \forall i} P(T_{01} \ge c_1) = \sum_{i=0}^{r} P(i, m-1, \boldsymbol{C\Omega C}') P(\chi_i^2 \ge c_1)$$

where χ_i^2 is a chi-square random variable with i degrees of freedom with $\chi_0^2 \equiv 0$.

For T_{12}, H_0 is least favorable within H_1, and for a constant c_2, its asymptotic distribution under H_0 is given by

$$\lim_{n_i \to \infty, \forall i} P(T_{12} \ge c_2) = \sum_{i=0}^{r} P(r-i, m-1, \boldsymbol{C\Omega C}') P(\chi_i^2 \ge c_2).$$

For computations, the $\boldsymbol{\Omega}$ appearing in the asymptotic distributions above is estimated by $\hat{\boldsymbol{\Omega}}$ given earlier. In the next section, we discuss several applications of Theorem 1.

3. Applications of Theorem 1

The procedure described in the previous section may be used in a versatile ways for hypothesis testing purposes. We discuss testing homogeneity of the means, the scale parameters and the shape parameters of m gamma distributions against nonincreasing order in the next three examples. We also discuss the calculations of the level probabilities.

Example 1. When testing equality of the means of m gamma distributions against nonincreasing order, the hypotheses are

$$H_0 : \theta_{11}^* = \theta_{21}^* = \cdots = \theta_{m1}^* \quad \text{versus} \quad H_1 : \theta_{11}^* \ge \theta_{21}^* \ge \cdots \ge \theta_{m1}^*$$

which can be expressed as in (2.1) using

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 \end{bmatrix}$$

and $\boldsymbol{\Phi}_0' = (0, 0, \ldots, 0)$ with $r = m-1$. The asymptotic distributions of the test statistics T_{01} and T_{12} are obtained from Theorem 1.

Now we consider approximations for the level probabilities in this case. Let $\boldsymbol{W} = \boldsymbol{C}\hat{\boldsymbol{\Omega}}\boldsymbol{C}'$. When $m = 2$, we have $P(0, 1, \boldsymbol{W}) = P(1, 1, \boldsymbol{W}) = .5$. When $m = 3$, we have

$P(0,2,\boldsymbol{W}) = .5 - (\cos^{-1}\rho_{12})/2\pi, P(1,2,\boldsymbol{W}) = .5, P(2,2,\boldsymbol{W}) = (\cos^{-1}\rho_{12})/2\pi$, where ρ_{12} is the (1,2)-th element of the matrix $[\text{diag}(\boldsymbol{W})]^{-1/2}[\boldsymbol{W}][\text{diag}(\boldsymbol{W})]^{-1/2}$, and can be expressed as $\rho_{12} = -\hat{\Omega}_2^{11}/\sqrt{(\hat{\Omega}_1^{11}+\hat{\Omega}_2^{11})(\hat{\Omega}_2^{11}+\hat{\Omega}_3^{11})}$ where $\hat{\Omega}_k^{ij}$ is the (i,j)-th element of $\hat{\Omega}_k$. If $\rho_{12} < 0$, the approximate critical values of T_{01} and T_{12} can be taken from Table A1 of Robertson *et al.* (1988) by setting their ρ equal to ρ_{12}.

For $m = 4$, we have

$$P(0,3,\boldsymbol{W}) = \frac{1}{2} - (\cos^{-1}\rho_{12} + \cos^{-1}\rho_{13} + \cos^{-1}\rho_{23})/4\pi,$$
$$P(1,3,\boldsymbol{W}) = \frac{3}{4} - (\cos^{-1}\rho_{12\cdot 3} + \cos^{-1}\rho_{13\cdot 2} + \cos^{-1}\rho_{23\cdot 1})/4\pi,$$
$$P(2,3,\boldsymbol{W}) = \frac{1}{2} - P(0,3,\boldsymbol{W}), \quad \text{and} \quad P(3,3,\boldsymbol{W}) = \frac{1}{2} - P(1,3,\boldsymbol{W})$$

where $\rho_{ij\cdot k} = (\rho_{ij} - \rho_{ik}\rho_{jk})/\sqrt{(1-\rho_{ik}^2)(1-\rho_{jk}^2)}$ with

$$\rho_{12} = -\frac{\hat{\Omega}_2^{11}}{\sqrt{(\hat{\Omega}_1^{11}+\hat{\Omega}_2^{11})(\hat{\Omega}_2^{11}+\hat{\Omega}_3^{11})}}, \quad \rho_{13} = 0, \quad \rho_{23} = -\frac{\hat{\Omega}_3^{11}}{\sqrt{(\hat{\Omega}_2^{11}+\hat{\Omega}_3^{11})(\hat{\Omega}_3^{11}+\hat{\Omega}_4^{11})}}.$$

For $m \geq 5$, expressions for the level probabilities are available in terms of orthant probabilities for a multivariate normal distribution. However, numerical techniques are needed to compute these arbitrary orthant probabilities. For this purpose, the programs of Bohrer and Chow (1978) and Sun (1988) are useful.

Example 2. When testing equality of the scale parameters of m gamma distributions against nonincreasing order, the hypotheses are

$$H_0 : \theta_{12}^* = \theta_{22}^* = \cdots = \theta_{m2}^* \quad \text{versus} \quad H_1 : \theta_{12}^* \geq \theta_{22}^* \geq \cdots \geq \theta_{m2}^*$$

which can be expressed as in (2.1) using

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & -1 \end{bmatrix}$$

and $\boldsymbol{\Phi}_0' = (0,0,\ldots,0)$ with $r = m-1$. Again, the asymptotic distributions of the test statistics T_{01} and T_{12} are obtained from Theorem 1.

Note that we do not assume equality of the shape parameters as in Bhattacharya (2001) and Shiue and Bain (1983). The calculations for the level probabilities are similar as in Example 1, except replace $\hat{\Omega}_k^{11}$ by $\hat{\Omega}_k^{22}$ for $k = 1,2,3,4$.

Example 3. Tripathi *et al.* (1993) proposed a test of equality of the coefficients of variation of m gamma distributions which essentially reduces to testing equality of their shape parameters. This requires modifying the $\boldsymbol{\theta}$ parameters from Section 2. However the derivation given in page 778 of Tripathi *et al.* (1993) is incorrect. In the following we propose a correct procedure for testing the equality of the shape parameters of m gamma distributions against nonincreasing order.

We reparameterize $\boldsymbol{\theta}' = [\boldsymbol{\theta}_1^{*\prime}, \boldsymbol{\theta}_2^{*\prime}, \ldots, \boldsymbol{\theta}_m^{*\prime}]$ with $\boldsymbol{\theta}_i^{*\prime} = [\theta_{i1}^*, \theta_{i2}^*] = [\ln \gamma_i, \ln \theta_i], i = 1, \ldots, m$. To test hypotheses as in (2.1) with suitably chosen $\boldsymbol{C}$ and $\boldsymbol{\Phi}_0$, we let $\eta_{i0} = \ln \kappa_{i1}$, and $\eta_{ij} = \ln (\kappa_{i,j+1}/j!)$, $j \geq 1$, and $\boldsymbol{\eta}_i' = (\eta_{i0}, \eta_{i1}, \eta_{i2}, \eta_{i3})$. This produces the relationship $\boldsymbol{\eta}_i = \boldsymbol{w}^* \boldsymbol{\theta}_i, i = 1, \ldots, m$ where

$$\boldsymbol{w}^{*\prime} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

The overall linear relationship $\boldsymbol{\eta} = \boldsymbol{w}^* \boldsymbol{\theta}$ is obtained as before. If $\boldsymbol{h}$ is the sample counterpart of $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\Sigma}}$ is a consistent estimate of the covariance matrix $\boldsymbol{\Sigma}$ of $\boldsymbol{h}$, then the test procedures described in the previous section can be followed, and the asymptotic distributions can be obtained.

4. Using combination of p-values

Bhattacharya (2001) considered the problem of testing equality of scale parameters against nonincreasing order of m gamma distributions with equal but unknown shape parameters using Fisher's method of combination of p-values. For smaller sample sizes, this test is approximate in nature as it depends on the MLE's of the common shape parameter in each step. Bhattacharya (2001) studied the actual size of the test in various situations including when $\gamma \to \infty$ and $\gamma \to 0$. In this section we derive an expression for the power function of this test for $m = 3$.

Let independent random samples of sizes n_i are available from m gamma distributions $G(\cdot; \theta_i, \gamma)$ with corresponding sample means as $\bar{X}_i$, $i = 1, \ldots, m$. The problem of testing $H_0 : \theta_1 = \cdots = \theta_m$ against $H_a - H_0$ where $H_a : \theta_1 \geq \cdots \geq \theta_m$ may be considered as the conjunction of $m-1$ nested problems of testing $H_{0i} : \theta_1 = \cdots = \theta_i = \theta_{i+1}$ versus $H_{ai} : \theta_1 = \cdots = \theta_i > \theta_{i+1}$ for $i = 1, \ldots, m-1$. For each i, the test of H_{0i} versus H_{ai} is based on the two-sample case of Shiue and Bain (1983) as follows. Let

$$F_i = \frac{\sum_{j=1}^{i} n_j \bar{X}_j}{N_i \bar{X}_{i+1}}$$

where $N_i = \sum_{j=1}^{i} n_j$. We reject H_{0i} if $F_i \geq F_{2N_i\hat{\gamma}_i, 2n_{i+1}\hat{\gamma}_i}(\alpha)$ where $\hat{\gamma}_i$ is the MLE of γ obtained in the i-th step by combining all related $i+1$ samples. It can be shown that the statistics $F_1, \ldots, F_{m-1}$ are mutually independent under H_0.

Let $P_i(\hat{\gamma}_i) = \Pr(F_{2N_i\hat{\gamma}_i, 2n_{i+1}\hat{\gamma}_i} \geq F_i)$ denote the (approximate) p-value associated with the test statistic $F_i, i = 1, \ldots, m-1$. The Fisher's method of combining independent p-values is based on the test statistic

$$\psi_F(\hat{\gamma}_1, \ldots, \hat{\gamma}_{m-1}) = -2 \sum_{i=1}^{m-1} \ln P_i(\hat{\gamma}_i)$$

which has a χ^2_{2m-2} distribution (approximately) under H_0. Thus we reject H_0 if $\psi_F(\hat{\gamma}_1, \ldots, \hat{\gamma}_{m-1}) \geq \chi^2_{2m-2}(\alpha)$ for given α.

The actual size of this test is $P(\gamma, \alpha) = \Pr[\psi_F(\hat{\gamma}_1, \ldots, \hat{\gamma}_{m-1}) \geq \chi^2_{2m-2}(\alpha)]$. Under H_0, this probability does not depend on θ since the joint density of F_i and $\hat{\gamma}_i$ is free of

θ. For $m = 3$, Bhattacharya (2001) have estimated values of $P(\gamma, \alpha)$ for various values of γ, α, n_i using Monte Carlo simulation. The case of the limiting values $P(0, \alpha) = \lim_{\gamma \to 0} P(\gamma, \alpha)$, and, $P(\infty, \alpha) = \lim_{\gamma \to \infty} P(\gamma, \alpha)$ are also derived. It is seen that in each case considered the true level $P(\gamma, \alpha)$ is slightly above the prescribed level α. Also, these values are nearly constant for fixed values of α and n_i's and are close to the limiting value $P(\infty, \alpha)$. Thus one may like to modify the test for small sample sizes so that the actual level is closer to the prescribed nominal level. Since $P(\gamma, \beta) \approx P(\infty, \beta)$ for small n_i's, an approximate α level test of H_0 versus $H_a - H_0$ is to reject H_0 if $\psi_F(\hat{\gamma}_1, \ldots, \hat{\gamma}_{m-1}) \geq \chi^2_{2m-2}(\beta)$ where $P(\infty, \beta) = \alpha$. Bhattacharya (2001) gave a table of β values for some commonly used α values for various small sample sizes for $m = 3$.

Since the statistics F_1, F_2 are independent of $\hat{\gamma}_1$, $\hat{\gamma}_2$, the power function for $m = 3$ of the test is given by

$$
\begin{aligned}
(4.1) \qquad P_{\theta_1,\theta_2}(\gamma, \alpha) &= \Pr_{\theta_1,\theta_2}[\psi_F(\hat{\gamma}_1, \hat{\gamma}_2) \geq \chi^2_4(\alpha)] \\
&= \int_0^\infty \int_0^\infty \Pr_{\theta_1,\theta_2}[\psi_F(\hat{\gamma}_1, \hat{\gamma}_2) \geq \chi^2_4(\alpha) \mid \hat{\gamma}_1 = a_1, \hat{\gamma}_2 = a_2] \\
&\quad \cdot f_{\hat{\gamma}_1,\hat{\gamma}_2}(a_1, a_2) da_1\, da_2 \\
&= \int_0^\infty \int_0^\infty \Pr_{\theta_1,\theta_2}[\psi_F(a_1, a_2) \geq \chi^2_4(\alpha) \mid \hat{\gamma}_1 = a_1, \hat{\gamma}_2 = a_2] \\
&\quad \cdot f_{\hat{\gamma}_1,\hat{\gamma}_2}(a_1, a_2) da_1\, da_2 \\
&= \int_0^\infty \int_0^\infty \Pr_{\theta_1,\theta_2}[\psi_F(a_1, a_2) \geq \chi^2_4(\alpha)] f_{\hat{\gamma}_1,\hat{\gamma}_2}(a_1, a_2) da_1\, da_2
\end{aligned}
$$

where $f_{\hat{\gamma}_1,\hat{\gamma}_2}$ is the joint density of $(\hat{\gamma}_1, \hat{\gamma}_2)$.

Since

$$
F_1 = \frac{\theta_1}{\theta_2} F_1', \qquad \text{and,} \qquad F_2 = \left(\frac{\theta_2 + \theta_1 F_1'}{\theta_3 (1 + F_1')} \right) F_2'
$$

where

$$
F_1' = \frac{n_1 \bar{X}_1 / \theta_1}{n_2 \bar{X}_2 / \theta_2} \sim F_{2n_1\gamma, 2n_2\gamma}, \qquad \text{and,} \qquad F_2' = \frac{(n_1 \bar{X}_1/\theta_1) + (n_2 \bar{X}_2/\theta_2)}{N_2 \bar{X}_3/\theta_3} \sim F_{2N_2\gamma, 2n_3\gamma}
$$

so we can express

$$
F_1 = \frac{\theta_1}{\theta_2} \mathcal{F}^{-1}_{2n_1\gamma, 2n_2\gamma}(U_1), \qquad \text{and,} \qquad F_2 = \left(\frac{\theta_2 + \theta_1 \mathcal{F}^{-1}_{2N_1\gamma, 2n_2\gamma}(U_1)}{\theta_3 (1 + \mathcal{F}^{-1}_{2N_1\gamma, 2n_2\gamma}(U_1))} \right) \mathcal{F}^{-1}_{2N_2\gamma, 2n_3\gamma}(U_2)
$$

where $\mathcal{F}_{\nu_1,\nu_2}(\cdot)$ is the cumulative distribution function of the F distribution with ν_1, ν_2 degrees of freedom and U_1, U_2 are independent $U(0,1)$ random variables.

When $\hat{\gamma}_i = a_i$, the component p-values can be expressed as

$$
\begin{aligned}
P_1(a_1) &\stackrel{\mathcal{D}}{=} 1 - \mathcal{F}_{2N_1a_1, 2n_2a_1} \left(\frac{\theta_1}{\theta_2} \mathcal{F}^{-1}_{2N_1\gamma, 2n_2\gamma}(U_1) \right), \\
(4.2) \qquad P_2(a_2) &\stackrel{\mathcal{D}}{=} 1 - \mathcal{F}_{2N_2a_2, 2n_3a_2} \left(\left(\frac{\theta_2 + \theta_1 \mathcal{F}^{-1}_{2N_1\gamma, 2n_2\gamma}(U_1)}{\theta_3 (1 + \mathcal{F}^{-1}_{2N_1\gamma, 2n_2\gamma}(U_1))} \right) \mathcal{F}^{-1}_{2N_2\gamma, 2n_3\gamma}(U_2) \right),
\end{aligned}
$$

and then we obtain

$$
\begin{aligned}
(4.3) \quad \Pr_{\theta_1,\theta_2}[\psi_F(a_1, a_2) \geq \chi^2_4(\alpha)] &= \Pr_{\theta_1,\theta_2}[-2 \ln P_1(a_1) - 2 \ln P_2(a_2) \geq \chi^2_4(\alpha)] \\
&= \Pr_{\theta_1,\theta_2}[P_1(a_1) P_2(a_2) \leq e^{-\chi^2_4(\alpha)/2}].
\end{aligned}
$$

In derivation of the last expression in (4.3), the convolution formulas may not be used to find the distribution of $-2 \ln P_1(a_1) - 2 \ln P_2(a_2)$ as under the alternative $P_1(a_1)$ and $P_2(a_2)$ are not independent. However using the conditional argument as in Mudholkar *et al.* (1993), the last expression in (4.3) can be written as

$$(4.4) \quad 1 - \int_0^y \mathcal{F}_{2N_2\gamma, 2n_3\gamma}\left(\left(\frac{\theta_2 + \theta_1 F_1'}{\theta_3(1 + F_1')}\right) \mathcal{F}^{-1}_{2N_2a_2, 2n_3a_2}\left(1 - \frac{e^{-\chi_4^2(\alpha)/2}}{P_1(a_1)}\right)\right) du$$

where $y = \mathcal{F}_{2n_1\gamma, 2n_2\gamma}(\frac{\theta_1}{\theta_2}\mathcal{F}^{-1}_{2n_1a_1, 2n_2a_1}(1 - e^{-\chi_4^2(\alpha)/2}))$. Substituting (4.4) in (4.1) we obtain the power function of the test as

$$(4.5) \quad E_{\hat{\gamma}_1, \hat{\gamma}_2}\left[1 - \int_0^y \mathcal{F}_{2N_2\gamma, 2n_3\gamma}\left(\left(\frac{\theta_2 + \theta_1 F_1'}{\theta_3(1 + F_1')}\right) \mathcal{F}^{-1}_{2N_2a_2, 2n_3a_2}\left(1 - \frac{e^{-\chi_4^2(\alpha)/2}}{P_1(a_1)}\right)\right) du\right].$$

For general m, the expressions for the power function may be derived by tedious calculations and are not included here.

5. A Monte Carlo study

It is of interest to study the power of the two procedures described in the previous sections in the case of Example 2 of Section 2. For $m = 3$ the asymptotic power of the procedures are computed at various combinations of sample sizes and alternative scale parameter points using a common value of the shape parameter. Different shape parameter values (bell-shaped: $\gamma < 1$, and reverse J shaped $\gamma > 1$) are considered.

We have used the IMSL subroutines to generate gamma random variates. Each simulated value for the two procedures (G for the first and F for the second) in Table 1 is based on 3,000 replications using $\alpha = .05$. As both tests are asymptotic in nature, we have chosen sample sizes 30, 50, and 100. The null hypothesis values of the scale parameters are taken as 1. The alternative scale parameter values range from 2.5 to 1 in a decreasing order with various combinations. For the first procedure, a consistent estimate $\hat{\Sigma}$ of Σ is obtained by using MLE's of the parameters γ_i's and θ_i's as given in Section 1. For the second procedure, although we have derived a closed form expression for the power function in Section 4, to stay on the same footing, we have simulated its power as well (it is seen that the simulated values are either very close or equal to the actual values for $m = 3$). We randomly generate m samples of size n_i from $G(x; \theta, \gamma)$ populations, compute $P_i(\hat{\gamma}_i)$, $1 \leq i \leq m - 1$ and $\psi_F(\hat{\gamma}_1, \dots, \hat{\gamma}_{m-1})$, then determine whether $\psi_F(\hat{\gamma}_1, \dots, \hat{\gamma}_{m-1}) \geq \chi^2_{2m-2}(\alpha)$. The critical points used under the F Column uses the values given in Table 2 of Bhattacharya (2001).

It is seen that for both procedures the powers increase as the sample sizes increase. Also as γ increases the powers increase in both cases. We also observe that for the first procedure the power is low at or near H_0 especially for smaller γ which is not the case with the second procedure. However as the θ values move away from H_0, the power increases fairly quickly in both cases even for moderate sample sizes. Similar results were obtained at other combinations of γ, α, n_i's which are not reported here for brevity. Thus we recommend that the second procedure may be preferred over the first when it is reasonable to assume that the shape parameters are equal especially for smaller sample sizes. In all other cases the first procedure should be used.

Table 1. Power comparison of the two methods for $m = 3$ (G=first, F=second).

						$\gamma = .5$		$\gamma = 1.0$		$\gamma = 2.0$	
n_1	n_2	n_3	θ_1	θ_2	θ_3	G	F	G	F	G	F
30	30	30	1	1	1	.039	.051	.046	.041	.061	.048
			1.5	1	1	.209	.322	.455	.517	.744	.765
			1.5	1.5	1	.248	.241	.485	.394	.781	.664
			2	1	1	.461	.657	.850	.892	.992	.994
			2	1.5	1	.465	.517	.814	.797	.979	.971
			2	2	1	.559	.514	.889	.820	.994	.985
			2.5	1	1	.682	.849	.976	.986	1.0	1.0
			2.5	1.5	1	.660	.750	.951	.957	1.0	.999
			2.5	2	1	.734	.727	.973	.957	1.0	.999
			2.5	2.5	1	.802	.755	.986	.971	1.0	1.0
50	50	50	1	1	1	.034	.046	.042	.043	.054	.047
			1.5	1	1	.314	.450	.641	.699	.917	.933
			1.5	1.5	1	.356	.335	.668	.583	.925	.886
			2	1	1	.714	.844	.972	.982	1.0	1.0
			2	1.5	1	.672	.726	.946	.941	.998	.998
			2	2	1	.778	.744	.982	.969	1.0	.999
			2.5	1	1	.918	.973	1.0	1.0	1.0	1.0
			2.5	1.5	1	.868	.915	.997	.999	1.0	1.0
			2.5	2	1	.914	.909	.999	.998	1.0	1.0
			2.5	2.5	1	.953	.941	1.0	.999	1.0	1.0
100	100	100	1	1	1	.035	.046	.047	.044	.051	.053
			1.5	1	1	.572	.692	.903	.926	.998	.999
			1.5	1.5	1	.610	.587	.915	.871	.997	.994
			2	1	1	.956	.984	1.0	1.0	1.0	1.0
			2	1.5	1	.924	.939	.999	.998	1.0	1.0
			2	2	1	.973	.965	1.0	1.0	1.0	1.0
			2.5	1	1	.998	1.0	1.0	1.0	1.0	1.0
			2.5	1.5	1	.993	.996	1.0	1.0	1.0	1.0
			2.5	2	1	.997	.997	1.0	1.0	1.0	1.0
			2.5	2.5	1	.999	.999	1.0	1.0	1.0	1.0

6. Concluding remarks

Tests for gamma distribution for $m \geq 3$ have been treated in literature very rarely. We have proposed two simple testing procedures of equality against inequality restrictions for parameters of $m \geq 3$ gamma distributions with any sample sizes. The first procedure is quite general and may be used to test for equality of the means, scale or shape parameters. It does not assume any condition on the nuisance parameter. The second procedure which tests equality of the scale parameters against inequality restrictions under assumption of equality of the shape parameters performs better in the vicinity of H_0 and is also applicable with a variety of order restrictions. The critical value of the first

Table 1. (continued)

						$\gamma = .5$		$\gamma = 1.0$		$\gamma = 2.0$	
n_1	n_2	n_3	θ_1	θ_2	θ_3	G	F	G	F	G	F
30	50	100	1	1	1	.015	.046	.030	.043	.042	.051
			1.5	1	1	.159	.403	.449	.605	.793	.860
			1.5	1.5	1	.355	.472	.729	.730	.963	.955
			2	1	1	.442	.766	.877	.949	.996	.998
			2	1.5	1	.626	.778	.947	.963	1.0	1.0
			2	2	1	.805	.884	.994	.994	1.0	1.0
			2.5	1	1	.696	.924	.985	.996	1.0	1.0
			2.5	1.5	1	.798	.920	.994	.997	1.0	1.0
			2.5	2	1	.921	.958	.999	.999	1.0	1.0
			2.5	2.5	1	.966	.987	1.0	1.0	1.0	1.0
100	50	30	1	1	1	.077	.055	.077	.046	.068	.043
			1.5	1	1	.545	.548	.838	.824	.981	.978
			1.5	1.5	1	.382	.259	.650	.468	.883	.760
			2	1	1	.909	.915	.998	.998	1.0	1.0
			2	1.5	1	.765	.707	.965	.941	.999	.999
			2	2	1	.765	.595	.968	.907	1.0	.997
			2.5	1	1	.992	.993	1.0	1.0	1.0	1.0
			2.5	1.5	1	.944	.933	.998	.998	1.0	1.0
			2.5	2	1	.929	.864	.998	.993	1.0	1.0
			2.5	2.5	1	.938	.841	.999	.992	1.0	1.0
100	30	50	1	1	1	.064	.056	.057	.042	.058	.045
			1.5	1	1	.536	.556	.841	.828	.980	.979
			1.5	1.5	1	.442	.371	.756	.634	.950	.904
			2	1	1	.909	.921	.998	.997	1.0	1.0
			2	1.5	1	.845	.815	.990	.984	1.0	1.0
			2	2	1	.857	.781	.993	.979	1.0	1.0
			2.5	1	1	.992	.994	1.0	1.0	1.0	1.0
			2.5	1.5	1	.977	.976	1.0	1.0	1.0	1.0
			2.5	2	1	.970	.958	1.0	1.0	1.0	1.0
			2.5	2.5	1	.975	.957	1.0	1.0	1.0	1.0

procedure depends on a chi-bar square distribution whereas that of the second depends on a chi-square distribution. Thus in practice both tests would be useful in appropriate situations.

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