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COMPOUND POISSON APPROXIMATION FOR MULTIPLE RUNS IN A MARKOV CHAIN

OURANIA CHRYSSAPHINOU AND EUTICHIA VAGGELATOU

Department of Mathematics, University of Athens, Panepistimiopolis, 15784 Athens, Greece

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Abstract. We consider a sequence $X_1, \dots, X_n$ of r.v.'s generated by a stationary Markov chain with state space $\mathcal{A} = \{0, 1, \dots, r\}$, $r \geq 1$. We study the *overlapping* appearances of runs of k_i consecutive i's, for all $i = 1, \dots, r$, in the sequence $X_1, \dots, X_n$. We prove that the number of overlapping appearances of the above multiple runs can be approximated by a Compound Poisson r.v. with compounding distribution a *mixture* of *geometric* distributions. As an application of the previous result, we introduce a specific Multiple-failure mode reliability system with Markov dependent components, and provide lower and upper bounds for the reliability of the system.

Key words and phrases: Multiple runs, Stein-Chen method, Kolmogorov distance, Compound Poisson approximation, consecutive-$k_1, \dots, k_r$-out-of-n: MFM system.

1. Introduction

During the last decade, there has been an increasing interest in studying several ways of counting the number of runs which appear in sequences of trials. This is due to the wide applicability of runs in many areas of science, such as quality control, reliability, non-parametric statistical analysis etc.

Most of the papers dealing with runs concern the appearances of success runs in sequences of {0,1} trials. Also, there has been an interest to obtain results for the appearances of more than one kind of runs. On that we refer the works of Aki (1992), Chryssaphinou *et al.* (1994), Uschida and Aki (1995) and Han and Aki (1999), who have studied waiting time problems for the appearances of several kinds of runs, using probability generating functions arguments and imbedding Markov techniques. For more literature, one may consult Godbole and Papastavridis (1994), Koutras (2000) and the references therein.

On the other hand, many authors have looked into the matter of runs from a different point of view; they approximate the distribution of the number of runs by well known distributions such as Poisson and Compound Poisson. The Stein-Chen method has proved to be a powerful tool in computing the error bound of the above approximations. The work done on this field concerns appearances of success runs and words. On that we refer Arratia *et al.* (1990), Barbour *et al.* (1992*b*), Roos (1993), Roos and Stark (1996), Erhardsson (1997), Eichelsbacher and Roos (1998), Reinert and Schbath (1998), Barbour and Xia (2000), Barbour *et al.* (2000) and the references therein. For general literature on the Stein-Chen method see Stein (1972), Chen (1975), Arratia *et al.* (1989), (1990), Barbour *et al.* (1992*a*), (1992*b*), Roos (1993), (1994), Erhardsson (1997), (1999), Barbour and Utev (1998), (1999), Barbour and Xia (1999), (2000).

In this paper we are going to study the following model. We consider a stationary irreducible and aperiodic Markov chain $\{X_a\}_{a\in Z}$ on a finite state space $\mathcal{A} = \{0, 1, \dots, r\}$, $r \geq 1$, with transition probability matrix $P = (p_{zw})_{(z,w)\in\mathcal{A}\times\mathcal{A}}$ and stationary distribution $\pi = (\pi_z)_{z\in\mathcal{A}}$. For $i \in \mathcal{A}$, let $I_{i,a} := I[(X_{a-k_i+1} \cdots X_a) = (i \cdots i)]$ be the indicator of the event that a run of k_i consecutive i's ends at index $a \in Z$.

We define the r.v. $W := \sum_{a=1}^{n} \sum_{i=1}^{r} I_{i,a}$, which enumerates the overlapping runs of k_i consecutive i's, $i = 1, \dots, r$, that end in the finite sequence $X_1, \dots, X_n$. In order to avoid edge effects and to facilitate calculations, we assume that the doubly infinite sequence $\{X_a\}_{a\in Z}$ is observed.

We are interested in approximating the distribution $\mathcal{L}(W)$ of W. Since these runs tend to occur in clumps, a Compound Poisson distribution seems to be the appropriate approximating distribution for W. Furthermore, we will see that the compounding distribution is a mixture of geometric distributions, as it is expected by intuition. The interesting fact of this model is its potentiality of applications in a variety of fields such as reliability, randomness tests, waiting time problems etc. Since the motivation of the present work arises from a specific reliability system, we do not count runs of 0's (see Section 3). However, the case of taking into account runs of 0's can be treated in exactly the same way, using the r.v. $\sum_{a=1}^{n} \sum_{i=0}^{r} I_{i,a}$ instead of W (Remark 2.2).

We will exploit the results obtained by Barbour and Xia (2000), in order to get bounds for the distance with respect to the Kolmogorov metric between $\mathcal{L}(W)$ and an appropriate Compound Poisson distribution. For that purpose, we describe in brief their results.

The Compound Poisson distribution $CP(\lambda, \mu)$, where μ is a probability measure and $\lambda > 0$, is defined by

$$CP(\lambda, \mu) = \mathcal{L}\left(\sum_{i=1}^{N} Y_i\right) = \mathcal{L}\left(\sum_{i\geq 1} iZ_i\right),$$

where Y_i, $i \geq 1$, are independent r.v.'s with distribution μ which are also independent of N. The latter follows the Poisson distribution with parameter λ. Moreover, the Z_i, $i \geq 1$, are independent Poisson r.v.'s with parameter $\lambda\mu_i$.

Let W be a non-negative integer valued r.v. and let $\mu_i \geq 0$, $i \in N$, be such that $\sum_{i\geq 1} \mu_i = 1$ and $m_1 = \sum i\mu_i < +\infty$. Suppose that, for some $\lambda > 0$, for all bounded $g : N \to R$ and for small ε_0 and ε_1, it holds true that

$$\left| \mathbb{E}\left\{ \sum_{i\geq 1} i\lambda\mu_i g(W+i) - Wg(W) \right\} \right| \leq \varepsilon_0 M_0(g) + \varepsilon_1 M_1(g),$$

where $M_0(g) := \sup_{j\geq 0} |g(j)|$ and $M_1(g) := \sup_{j\geq 0} |g(j+1) - g(j)|$. Then

$$\begin{aligned} d_K(\mathcal{L}(W), CP(\lambda, \mu)) &:= \sup_{f\in\mathcal{F}_K} |\mathbb{E}f(W) - CP(\lambda, \mu)\{f\}| \\ &\leq \varepsilon_0 \sup_{f\in\mathcal{F}_K} M_0(g_f) + \varepsilon_1 \sup_{f\in\mathcal{F}_K} M_1(g_f), \end{aligned} \tag{1.1}$$

where $\mathcal{F}_K = \{f_k, k \in N : f_k(x) = 1_{[k,+\infty)}(x)\}$. The function g_f is the solution of the

Compound Poisson Stein equation

$$\sum_{i\geq 1} i\lambda\mu_i g(j+i) - jg(j) = f(j) - CP(\lambda,\mu)\{f\}, \qquad j \geq 0.$$

Barbour and Xia (2000) proved that, under the condition

$$k\mu_k \geq (k+1)\mu_{k+1}, \qquad k \geq 1,$$

it holds

$$\sup_{f\in\mathcal{F}_K} M_0(g_f) \leq \min\left\{1, \sqrt{\frac{2}{e\lambda\mu_1}}\right\} \qquad \text{and} \qquad \sup_{f\in\mathcal{F}_K} M_1(g_f) \leq \min\left\{\frac{1}{2}, \frac{1}{\lambda\mu_1+1}\right\}.$$

Using the above inequalities and relation (1.1), it follows that

$$d_K(\mathcal{L}(W), CP(\lambda,\mu)) \leq \varepsilon_0 \min\left\{1, \sqrt{\frac{2}{e\lambda\mu_1}}\right\} + \varepsilon_1 \min\left\{\frac{1}{2}, \frac{1}{\lambda\mu_1+1}\right\}. \tag{1.2}$$

For suitably chosen ε_0 and ε_1, (1.2) can lead to very satisfactory results by giving sharp bounds on the error of approximation, even for large $\mathbb{E}W$. Finally, for more discussion about ε_0 and ε_1 and the above approximations, we refer Barbour and Chryssaphinou (2001).

This paper is organized as follows. In Section 2, we present bounds for the Kolmogorov distance between the distribution of W and a suitable Compound Poisson distribution for the Markov case (Theorem 2.1). The i.i.d. analogue is an immediate consequence (Theorem 2.2). Under some restrictions on k_i and p_{ii}, $i = 1,\dots,r$, the obtained error bounds are of order at most $O(\psi \max_{1\leq i\leq r} k_i)$, where $\psi := \sum_{i=1}^r \mathbb{E}I_{i,a}$, and tend to zero as $n \to +\infty$ giving us the limit result of Theorem 2.3. In Section 3, employing the previous results, we derive upper and lower bounds for the reliability of a specific Multiple-failure mode system: the Consecutive-$k_1,\dots,k_r$-out-of-n: MFM system with Markov dependent components. This system is a generalization of the consecutive-k, r -out-of-n: DFM system (dual-failure mode) with independent components, which was introduced by Koutras (1997). Finally, numerical examples are presented to illustrate the satisfactory behaviour of the above obtained bounds.

2. Main results

As we have mentioned in the previous section, our main aim is to bound the distance $d_K(\mathcal{L}(W), CP(\lambda,\mu))$. Instead of W, it is easier to approximate the *declumped* r.v.

$$\widetilde{W} = \sum_{a=1}^{n}\sum_{k\geq 1} k\Psi_{a,k},$$

where $\Psi_{a,k}$ denotes the indicator of the event that a k-clump of runs appears at a (the first run ends at index a). In particular, if C_i^k denotes a run of consecutive i's with length $k_i + k - 1$ (in other words k-overlapping appearances of runs of i's, each one with length k_i) and $xC_i^k y = x\, \underbrace{ii\dots i}_{k_i+k-1}\, y$, with x, $y \in \mathcal{A}$ and x, $y \neq i$, then

$$\Psi_{a,k} = \sum_{i=1}^{r}\sum_{x,y\neq i} I_{a+k}(xC_i^k y). \tag{2.1}$$

Here, in general, $I_a(Y)$ denotes the indicator of the event that a pattern Y ends at position a, i.e. if $Y = y_1, \ldots, y_v$, with $y_1, \ldots, y_v \in \mathcal{A}$, then $I_a(Y) = I[(X_{a-v+1}, \ldots, X_a) = (y_1, \ldots, y_v)]$. The r.v. $\Psi_{a,k}$ has mean

$$\mathbb{E}\Psi_{a,k} = \sum_{i=1}^{r}(1-p_{ii})^2 p_{ii}^{k-1}\psi_i, \tag{2.2}$$

where we set $\psi_i := \mathbb{E}I_{i,a} = \pi_i p_{ii}^{k_i-1}$, for $i = 1, \ldots, r$.

Now, applying the triangular inequality

$$d_K\left(\mathcal{L}(W), CP(\lambda,\mu)\right) \le d_K(\mathcal{L}(W), \mathcal{L}(\widetilde{W})) + d_K(\mathcal{L}(\widetilde{W}), CP(\lambda,\mu)), \tag{2.3}$$

we notice that the main problem concentrates on the bounding of the distances at the right hand side of this inequality.

In Section 1, we assumed that the doubly infinite sequence $\{X_a\}_{a\in Z}$ is observed so that edge effects can be ignored. This introduces an error which results from the difference between W and $\widetilde{W}$. We observe that

$$\{W \ne \widetilde{W}\} \subseteq \left\{(I_{i,0}=1)\bigcap(I_{i,1}=1)\right\} \cup \left\{(I_{i,n}=1)\bigcap(I_{i,n+1}=1)\right\}.$$

In other words W differs from $\widetilde{W}$ when, in the infinite sequence $\{X_a\}_{a\in Z}$, a run of k_i+1 consecutive i's, for some $i = 1, \ldots, r$, ends at position 1 or at position $n+1$. Thus, due to symmetry, we have

$$\begin{aligned}\mathbb{P}(W \ne \widetilde{W}) &\le 2\sum_{i=1}^{r}\mathbb{P}(\text{a run of } i \text{ 's with length } k_i+1 \text{ ends at position 1})\\ &\le 2\psi,\end{aligned}$$

and since $d_K(\mathcal{L}(W), \mathcal{L}(\widetilde{W})) \le \mathbb{P}(W \ne \widetilde{W})$, it follows

$$d_K(\mathcal{L}(W), \mathcal{L}(\widetilde{W})) \le 2\psi. \tag{2.4}$$

So it remains to bound the distance $d_K(\mathcal{L}(\widetilde{W}), CP(\lambda,\mu))$. Here, $CP(\lambda,\mu)$ denotes the Compound Poisson distribution of $\sum_{k\ge1} kZ_k$, where Z_k are independent Poisson r.v.'s with parameter $\lambda\mu_k$, which is the mean of the r.v. $\sum_{a=1}^{n}\Psi_{a,k}$. Using (2.2), it is easy to see that

$$\lambda = n\sum_{i=1}^{r}(1-p_{ii})\psi_i; \qquad \mu_k = \sum_{i=1}^{r} w_i(1-p_{ii})p_{ii}^{k-1}, \quad k \ge 1,$$

where $0 < w_i = (1-p_{ii})\psi_i/\sum_{j=1}^{r}(1-p_{jj})\psi_j < 1$. Thus the compounding distribution μ is a mixture of r Geometric distributions $Ge(1-p_{ii})$, $i = 1, \ldots, r$.

Let $\mathcal{B}(a,k) \subset \{1, \ldots, n\} \times N$ denote the set which contains the indices (b,l) for which $\Psi_{b,l}$ is "strongly" dependent on $\Psi_{a,k}$. Set

$$X_{a,k} := \sum_{(a,k)\ne(b,l)\in\mathcal{B}(a,k)} l\Psi_{b,l}; \qquad Z_{a,k} := \sum_{(b,l)\notin\mathcal{B}(a,k)} l\Psi_{b,l}.$$

Then $\widetilde{W} = k\Psi_{a,k} + X_{a,k} + Z_{a,k}$ and observing that $\mathbb{E}(\Psi_{a,k}g(\widetilde{W})) = \mathbb{E}(\Psi_{a,k}g(k + X_{a,k} + Z_{a,k}))$, it can be verified that

$$\left|\mathbb{E}\left\{\sum_{k\geq 1} k\lambda\mu_k g(\widetilde{W}+k) - \widetilde{W}g(\widetilde{W})\right\}\right| \leq M_0(g)b_3^* + M_1(g)(b_1^* + b_2^*),$$

where

$$b_1^* := \sum_{a=1}^{n}\sum_{k\geq 1}\sum_{(b,l)\in\mathcal{B}(a,k)} kl\mathbb{E}\Psi_{a,k}\mathbb{E}\Psi_{b,l};$$

$$b_2^* := \sum_{a=1}^{n}\sum_{k\geq 1}\sum_{(a,k)\neq(b,l)\in\mathcal{B}(a,k)} kl\mathbb{E}(\Psi_{a,k}\Psi_{b,l});$$

$$b_3^* := \sum_{a=1}^{n}\sum_{k\geq 1} k\mathbb{E}|\mathbb{E}\{\Psi_{a,k} - \mathbb{E}\Psi_{a,k}\|\sigma(\Psi_{b,l} : (b,l)\in\mathcal{B}(a,k)^c)\}|.$$

The set $\mathcal{B}(a,k)$ is given by:

$$\mathcal{B}(a,k) = \{(b,l) : a - M - l - s + 1 \leq b \leq a + M + k + s - 1, l \geq 1\}, \tag{2.5}$$

where $M := \max_{1\leq i\leq r} k_i$ and $s \geq 1$. This choice of $\mathcal{B}(a,k)$ arises from the fact that $\Psi_{a,k}$ is measurable with respect to $\sigma(X_{a-M},\ldots,X_{a+k})$, so that $\Psi_{a,k}$ and $\Psi_{b,l}$ are "weakly" dependent whenever $b - M \geq a + k + s$ or $a - M \geq b + l + s$ for s sufficiently large. We note that s results from the Markovian structure of the problem, which necessitates a larger neighborhood than in the i.i.d. case. Obviously in the latter, one should take $s = 1$.

Considering the above notations, relation (1.2) takes the form

$$d_K(\mathcal{L}(\widetilde{W}), CP(\lambda,\mu)) \leq \min\left\{\frac{1}{2}, \frac{1}{\lambda\mu_1 + 1}\right\}(b_1^* + b_2^*) + \min\left\{1, \sqrt{\frac{2}{e\lambda\mu_1}}\right\}b_3^*. \tag{2.6}$$

Inequality (2.6) is valid under the restriction $k\mu_k \geq (k+1)\mu_{k+1}, k \geq 1$, which is statisfied when $\max_{1\leq i\leq r} p_{ii} \leq 1/2$. Making use of the neighborhood (2.5) and bounding the quantities b_1^*, b_2^* and b_3^* in a nice way, we will get at the desired bound for the distance $d_K(\mathcal{L}(W), CP(\lambda,\mu))$ in the case of monotone μ_k's. But before presenting this result, we need some additional notation.

Let $p_{xy}^{(t)}$ denote the t-order transition probability of the irreducible and aperiodic Markov chain $\{X_a\}_{a\in Z}$ and $p_{(R)xy}^{(t)} = \frac{\pi_y p_{yx}^{(t)}}{\pi_x}$ the t-order transition probability of the reversed Markov chain $\{X_a^R\}_{a\in Z}$, for $t \geq 1$. Following a coupling argument (Lindvall (1992), p. 96), we deduce

$$\max_{x\in\mathcal{A}}\max\left\{\sum_{y\in\mathcal{A}}|p_{xy}^{(t)} - \pi_y|, \sum_{y\in\mathcal{A}}|p_{(R)xy}^{(t)} - \pi_y|\right\} \leq 2\varrho^t, \qquad \forall t \geq 1, \tag{2.7}$$

where $\varrho = 1 - \min\{\sum_{y\in\mathcal{A}}\min_{x\in\mathcal{A}} p_{xy}, \sum_{y\in\mathcal{A}}\min_{x\in\mathcal{A}} p_{(R)xy}\}$. In the case where $\varrho = 1$ (this happens when there is at least one zero in each of the columns of P), one may do the following. Since the state space $\mathcal{A}$ is finite, there exists an $l > 0$ such that $p_{xy}^{(l)} > 0$, for all $x, y \in \mathcal{A}$. So we take $\varrho = 1 - \min\{\sum_{y\in\mathcal{A}} \min_{x\in\mathcal{A}} p_{xy}^{(l)}, \sum_{y\in\mathcal{A}} \min_{x\in\mathcal{A}} p_{(R)xy}^{(l)}\}$ and then replace ϱ^t by $\varrho^{[\frac{t}{l}]}$ in relation (2.7).

Furthermore, we define the following quantities:

$$m := \min_{1\le i\le r} k_i, \qquad \Delta := \sup_{t\ge 1} \max_{z,w\in\{1,\dots,r\}} \frac{p_{zw}^{(t)}}{\pi_w}. \tag{2.8}$$

Now we are ready to state the first main result of this work.

THEOREM 2.1. *If* $\max_{1\le i\le r} p_{ii} \le 1/2$, *then*

$$d_K(\mathcal{L}(W), CP(\lambda,\mu)) \le B_1(s) := \min\left\{\frac{1}{2}, \frac{1}{\lambda\mu_1+1}\right\} D_1(s) + \min\left\{1, \sqrt{\frac{2}{e\lambda\mu_1}}\right\} D_0(s) + 2\psi, \qquad s \ge 1$$

where

$$D_1(s) := \left\{2\mathbb{E}W(M+s+1)\psi + (M-m+s+1)\frac{\Delta}{\psi}\left(\sum_{i=1}^{r}\frac{\psi_i}{1-p_{ii}}\right)^2 + 2\sum_{i=1}^{r}\frac{\psi_i p_{ii}}{1-p_{ii}}\right\}$$

$$D_0(s) := 2\mathbb{E}W\varrho^s\{2 + 2\varrho^s + \varrho^{M+s+1}\}$$

and $CP(\lambda,\mu)$ *denotes the Compound Poisson distribution with* $\lambda = n\sum_{i=1}^{r}(1-p_{ii})\psi_i$ *and compounding distribution* $\mu_k = \sum_{i=1}^{r} w_i(1-p_{ii})p_{ii}^{k-1}$, $w_i = (1-p_{ii})\psi_i/\sum_{j=1}^{r}(1-p_{jj})\psi_j$, *for* $k \ge 1$ *and* $i = 1,\dots,r$.

PROOF. We will use the triangular inequality (2.3). A bound for the distance $d_K(\mathcal{L}(W), \mathcal{L}(\widetilde{W}))$ has been already found in (2.4). Thus it remains to bound the distance $d_K(\mathcal{L}(\widetilde{W}), CP(\lambda,\mu))$ through the quantities b_1^*, b_2^* and b_3^*. First we treat the term b_1^*. Making use of the neighborhood (2.5), we have

$$\begin{aligned} b_1^* &= \sum_{a=1}^{n}\sum_{k\ge1}\sum_{l\ge1}\sum_{b=a-M-l-s+1}^{a+M+k+s-1} kl\mathbb{E}\Psi_{a,k}\mathbb{E}\Psi_{b,l} \\ &\le 2\sum_{a=1}^{n}\sum_{k\ge1}\sum_{l\ge1}\sum_{b=a}^{a+M+k+s-1} kl\mathbb{E}\Psi_{a,k}\mathbb{E}\Psi_{b,l} \\ &\le 2\sum_{a=1}^{n}\sum_{k\ge1} k\mathbb{E}\Psi_{a,k}\sum_{b=a}^{a+M+k+s-1}\sum_{l\ge1} l\mathbb{E}\Psi_{b,l} \\ &= 2\psi\sum_{a=1}^{n}\sum_{k\ge1}(M+k+s)k\mathbb{E}\Psi_{a,k} \\ &= 2\mathbb{E}W\left\{(M+s+1)\psi + 2\sum_{i=1}^{r}\psi_i p_{ii}/(1-p_{ii})\right\}, \end{aligned} \tag{2.9}$$

where it can be easily verified that $\sum_{l\ge1} l\mathbb{E}\Psi_{b,l} = \psi$ and $\sum_{l\ge1} l^2\mathbb{E}\Psi_{b,l} = \psi + 2\sum_{i=1}^{r}\psi_i p_{ii}/(1-p_{ii})$ in virtue of (2.2). Obviously, $\mathbb{E}W = n\psi$.

For the quantity b_2^*, we do the following

$$
\begin{aligned}
b_2^* &= 2\sum_{a=1}^{n}\sum_{k\geq 1}\sum_{l\geq 1}\sum_{b=a+1}^{a+M+k+s-1} kl\mathbb{E}(\Psi_{a,k}\Psi_{b,l}) \\
&= 2\sum_{a=1}^{n}\sum_{k,l\geq 1}\sum_{i,j=1}^{r}\sum_{x,y\neq i}\sum_{\tilde{x},\tilde{y}\neq j}\sum_{b=a+k+k_j-1}^{a+M+k+s-1} kl\mathbb{E}\{I_{a+k}(xC_i^k y)I_{b+l}(\tilde{x}C_j^l\tilde{y})\},
\end{aligned}
$$

because for $b = a+1,\dots,a+k+k_j-2$ it holds that $\mathbb{E}\{I_{a+k}(xC_i^k y)I_{b+l}(\tilde{x}C_j^l\tilde{y})\} = 0$. If $b = a+k+k_j-1,\dots,a+M+k+s-1$, then xC_i^k and $C_j^l\tilde{y}$ do not overlap. Besides, it holds $\sum_{y\neq i} I_{a+k}(xC_i^k y) \leq I_{a+k-1}(xC_i^k)$ and $\sum_{\tilde{x}\neq j} I_{b+l}(\tilde{x}C_j^l\tilde{y}) \leq I_{b+l}(C_j^l\tilde{y})$, so that the term $\sum_{y\neq i}\sum_{\tilde{x}\neq j}\mathbb{E}\{I_{a+k}(xC_i^k y)I_{b+l}(\tilde{x}C_j^l\tilde{y})\}$ is bounded by

$$
\begin{aligned}
\sum_{y\neq i}\sum_{\tilde{x}\neq j}\mathbb{E}\{I_{a+k}(xC_i^k y)I_{b+l}(\tilde{x}C_j^l\tilde{y})\} &\leq \mathbb{E}I_{a+k-1}(xC_i^k)\mathbb{E}I_{b+l}(C_j^l\tilde{y})p_{ij}^{(b-a-k-k_j+2)}/\pi_j \\
&\leq \Delta\mathbb{E}I_{a+k-1}(xC_i^k)\mathbb{E}I_{b+l}(C_j^l\tilde{y}),
\end{aligned}
$$

where Δ is given by (2.8). Since $\mathbb{E}I_{a+k-1}(xC_i^k) = \pi_x p_{xi} p_{ii}^{k_i+k-2}$, it follows

$$
\begin{aligned}
(2.10)\qquad b_2^* &\leq 2\Delta\sum_{a=1}^{n}\sum_{k,l\geq 1}\sum_{i,j=1}^{r}\sum_{x\neq i}\sum_{\tilde{y}\neq j}\sum_{b=a+k+k_j-1}^{a+M+k+s-1} kl\mathbb{E}I_{a+k-1}(xC_i^k)\mathbb{E}I_{b+l}(C_j^l\tilde{y}) \\
&\leq 2(M-m+s+1)\Delta\sum_{a=1}^{n}\left(\sum_{i=1}^{r}(1-p_{ii})\psi_i\sum_{k\geq 1}kp_{ii}^{k-1}\right)^2 \\
&= 2\mathbb{E}W\Delta(M-m+s+1)\left(\sum_{i=1}^{r}\frac{\psi_i}{1-p_{ii}}\right)^2/\psi.
\end{aligned}
$$

Next, we turn our attention to the term b_3^*. It holds true that $\sigma(\Psi_{b,l} : (b,l) \in \mathcal{B}(a,k)^c) \subseteq \sigma(X_1,\dots,X_{a-M-s},X_{a+k+s},\dots,X_n)$, and using the Markov property and relation (2.1), we obtain

$$
\begin{aligned}
b_3^* &\leq \sum_{a=1}^{n}\sum_{k\geq 1} k\mathbb{E}|\mathbb{E}\{\Psi_{a,k}-\mathbb{E}\Psi_{a,k}\|\sigma(X_1,\dots,X_{a-M-s},X_{a+k+s},\dots,X_n)\}| \\
&\leq \sum_{a=1}^{n}\sum_{k\geq 1} k\sum_{i=1}^{r}\sum_{x,y\neq i}\mathbb{E}\left|\mathbb{E}\{I_{a+k}(xC_i^k y)\|\sigma(X_{a-M-s},X_{a+k+s})\} - \mathbb{E}I_{a+k}(xC_i^k y)\right| \\
(2.11)\qquad &= \sum_{a=1}^{n}\sum_{k\geq 1} k\sum_{z,w\in\mathcal{A}}\sum_{i=1}^{r}\sum_{x,y\neq i}|\mathbb{P}(I_{a+k}(xC_i^k y)=1, X_{a-M-s}=z, X_{a+k+s}=w) \\
&\qquad -\mathbb{E}I_{a+k}(xC_i^k y)\mathbb{P}(X_{a-M-s}=z, X_{a+k+s}=w)|.
\end{aligned}
$$

We have

$$
\mathbb{P}(X_{a-M-s}=z, X_{a+k+s}=w) = \pi_z p_{zw}^{(M+k+2s)}
$$

and

$$\mathbb{P}(I_{a+k}(xC_i^k y) = 1, X_{a-M-s} = z, X_{a+k+s} = w) = p_{(R)xz}^{(M+s-k_i)} \mathbb{E} I_{a+k}(xC_i^k y) p_{yw}^{(s)},$$

where $p_{(R)xz}^{(t)} = \pi_z p_{zx}^{(t)}/\pi_x$ is the transition probability of order t of the reversed Markov chain. Furthermore, if we set

$$\varepsilon_{xy}^{(t)} := |p_{xy}^{(t)} - \pi_y| \quad \text{and} \quad \varepsilon_{(R)xy}^{(t)} := |p_{(R)xy}^{(t)} - \pi_y|,$$

then

$$\begin{aligned} &|\mathbb{P}(I_{a+k}(xC_i^k y) = 1, X_{a-M-s} = z, X_{a+k+s} = w) \\ &\quad - \mathbb{E} I_{a+k}(xC_i^k y) \mathbb{P}(X_{a-M-s} = z, X_{a+k+s} = w)| \\ &\quad \le \mathbb{E} I_{a+k}(xC_i^k y) \{\varepsilon_{(R)xz}^{(M+s-k_i)} \varepsilon_{yw}^{(s)} + \pi_w \varepsilon_{(R)xz}^{(M+s-k_i)} + \pi_z \varepsilon_{yw}^{(s)} + \pi_z \varepsilon_{zw}^{(M+k+2s)}\} \end{aligned}$$

and substituting this to relation (2.11), we get

$$\begin{aligned} b_3^* &\le \sum_{a=1}^{n} \sum_{k \ge 1} k \sum_{i=1}^{r} \sum_{x,y \ne i} \mathbb{E} I_{a+k}(xC_i^k y) \\ &\quad \sum_{z,w \in \mathcal{A}} \{\varepsilon_{(R)xz}^{(M+s-k_i)} \varepsilon_{yw}^{(s)} + \pi_w \varepsilon_{(R)xz}^{(M+s-k_i)} + \pi_z \varepsilon_{yw}^{(s)} + \pi_z \varepsilon_{zw}^{(M+k+2s)}\} \\ &= \sum_{a=1}^{n} \sum_{k \ge 1} k \sum_{i=1}^{r} \sum_{x,y \ne i} \mathbb{E} I_{a+k}(xC_i^k y) \\ &\quad \left\{ \sum_{z,w \in \mathcal{A}} \varepsilon_{(R)xz}^{(M+s-k_i)} \varepsilon_{yw}^{(s)} + \sum_{z \in \mathcal{A}} \varepsilon_{(R)xz}^{(M+s-k_i)} + \sum_{w \in \mathcal{A}} \varepsilon_{yw}^{(s)} + \sum_{w \in \mathcal{A}} \varepsilon_{zw}^{(M+k+2s)} \right\}. \end{aligned}$$

Relation (2.7) implies

$$\sum_{y \in \mathcal{A}} \varepsilon_{xy}^{(t)} \le 2\varrho^t \quad \text{and} \quad \sum_{y \in \mathcal{A}} \varepsilon_{(R)xy}^{(t)} \le 2\varrho^t, \qquad x \in \mathcal{A},$$

and taking into account that $M + s - k_i \ge s$ and $M + k + 2s \ge M + 1 + 2s$, we obtain

$$b_3^* \le 2\varrho^s \sum_{a=1}^{n} \sum_{k \ge 1} k \sum_{i=1}^{r} \sum_{x,y \ne i} \mathbb{E} I_{a+k}(xC_i^k y) \{2 + 2\varrho^s + \varrho^{M+s+1}\}.$$

Hence, in virtue of $\sum_{k \ge 1} k \mathbb{E} \Psi_{a,k} = \psi$, it follows

$$b_3^* \le 2\mathbb{E} W \varrho^s \{2 + 2\varrho^s + \varrho^{M+s+1}\}. \tag{2.12}$$

We remind that inequality (2.6) is valid under the assumption $k\mu_k \ge (k+1)\mu_{k+1}$, $k \ge 1$, which is statisfied when $\max_{1 \le i \le r} p_{ii} \le 1/2$. Finally, combining (2.12) with (2.9), (2.10), (2.6) and (2.4), we derive the required result. □

Remarks 2.1. In Theorem 2.1 the quantity s is chosen so as to minimize the error estimate $B_1(s)$. Given the values of r, ϱ, Δ, ψ_i and k_i, $i = 1, \ldots, r$, it only requires simple calculus to find the optimum value of s which minimizes the bound. In practice, this is an easy task, especially when using a computer program (for example Mathematica).

Remarks 2.2. Theorem 2.1 can be slightly modified to hold for the case of counting all kinds of runs (i.e. runs of k_i consecutive i's for all $i = 0, \ldots, r$). Then we take $m := \min_{0 \le i \le r} k_i$, $\Delta := \sup_{t \ge 1} \max_{z,w \in \mathcal{A}} \frac{p^{(t)}_{zw}}{\pi_w}$ and we consider all summations over $i = 0, \ldots, r$.

Erhardsson (1997) studied the overlapping appearances of words following a different and more complicated approach than ours. Applying Theorem 2.2. of Erhardsson's thesis, one could derive a bound concerning our model. However, as it has been noticed by the author (Erhardsson (1997), p. 90), the evaluation of the bound is rather complicated, involving the computation of eigenvalues. For the specific case of runs, our approach is simpler and leads to an easily evaluated bound useful for relative applications.

A bound for the i.i.d. case results from Theorem 2.1 immediately, if we take $s = 1$, $\Delta = 1$ and $\varrho = 0$. Here, the probabilities ψ_i and ψ are equal to $p_i^{k_i}$ and $\sum_{i=1}^r p_i^{k_i}$ respectively, where p_i denotes the probability $\mathbb{P}(X_j = i)$, for $i = 0, 1, \ldots, r$. Thus we obtain the following:

THEOREM 2.2. *If* $\max_{1 \le i \le r} p_i \le 1/2$, *then*

$$d_K(\mathcal{L}(W), CP(\lambda^*, \mu^*)) \le B_2 := \min\left\{\frac{1}{2}, \frac{1}{\lambda^* \mu_1^* + 1}\right\} D_1^* + 2\sum_{i=1}^r p_i^{k_i},$$

where

$$D_1^* := 2\mathbb{E}W \left\{ (M+2)p_i^{k_i} + (M - m + 2)\frac{1}{\sum_{i=1}^r p_i^{k_i}} \left(\sum_{i=1}^r \frac{p_i^{k_i}}{1 - p_i}\right)^2 + 2\sum_{i=1}^r \frac{p_i^{k_i+1}}{1 - p_i} \right\}$$

and $CP(\lambda^*, \mu^*)$ *is the Compound Poisson distribution with* $\lambda^* = n\sum_{i=1}^r (1 - p_i)p_i^{k_i}$ *and compounding distribution* $\mu_k^* = \sum_{i=1}^r w_i^*(1 - p_i)p_i^{k-1}$, $w_i^* = (1 - p_i)p_i^{k_i} / \sum_{i=1}^r (1 - p_i)p_i^{k_i}$, $k \ge 1$, $i = 1, \ldots, r$.

A natural consequence of the above approximations is the following limit result.

THEOREM 2.3. *Let* n, $k_i \to +\infty$ *so that* $n(1 - p_{ii})\psi_i \to \lambda_i < +\infty$, $\forall i = 1, \ldots, r$. *If* $\frac{M}{n} \to 0$, *then*

$$W \xrightarrow{D} CP(\lambda, \mu),$$

where $\lambda = \sum_{i=1}^r \lambda_i$ *and* $\mu_k = \sum_{i=1}^r \frac{\lambda_i}{\lambda}(1 - p_{ii})p_{ii}^{k-1}$, $k \ge 1$.

PROOF. In order to show that W converges in distribution to $CP(\lambda, \mu)$, it suffices to prove that $d_K(\mathcal{L}(W),\ CP(\lambda, \mu)) \to 0$, as $n \to +\infty$. The assumptions that $\max_{1 \le i \le r} p_{ii} \le 1/2$ and $n\sum_{i=1}^r (1 - p_{ii})\psi_i \to \lambda$ imply that $\mathbb{E}W = n\psi$ is bounded. Using the latter, it is easy to verify that, for $s = M$, the bound $B_1(M)$ of Theorem 2.1 is of order $O(M\psi) = O(\frac{M}{n})$ and the proof is complete. □

The distribution of W can lead to results which are associated with waiting time problems. In particular, if $\mathcal{E}_i$ denotes a run of k_i consecutive i's, with $i \in \mathcal{A}$, and T_k the waiting time (number of trials) until the k-th appearance of a run among $\mathcal{E}_1, \ldots, \mathcal{E}_r$, then

$$\mathbb{P}(T_k > n) = \mathbb{P}(W < k).$$

(note that all k appearances could be of the same kind of runs). Hence, for large n, the distribution function $F_k(n) := \mathbb{P}(T_k \leq n)$ of the waiting time T_k can be approximated by

$$F_k(n) \simeq e^{-\lambda} \sum_{w \geq k} \sum_{i_1+2i_2+3i_3+\cdots=w} \prod_{t\geq 1} \frac{(\lambda\mu_t)^{i_t}}{i_t!}, \qquad k \geq 1,$$

and the corresponding error is bounded by

$$\left| F_k(n) - e^{-\lambda} \sum_{w \geq k} \sum_{i_1+2i_2+3i_3+\cdots=w} \prod_{t\geq 1} \frac{(\lambda\mu_t)^{i_t}}{i_t!} \right| \leq \inf_{s\geq 1} B_1(s),$$

where $B_1(s)$ is the bound of Theorem 2.1. For $k = 1$, we get

$$F_1(n) \simeq 1 - e^{-\lambda}.$$

3. An application to reliability theory: Consecutive $k_1, \ldots, k_r$-out-of-n: MFM system

Starting from the late 50's, there has been an increasing interest in studying reliability models subject to more than one failures (see Satoh *et al.* (1993), Koutras (1997) and the references therein). Specifically, Koutras (1997) introduced a new model with components subject to two different kinds of failures; the consecutive-k, r-out-of-n: DFM system, which is an extension of the much studied consecutive-k-out-of-n: F system. He provides recursive formula for the evaluation of the reliability of such a system to the case where the components are independent. Recently, Boutsikas and Koutras (2002) study a class of Multiple-Failure mode (MFM) systems with independent components, using techniques involving the corresponding structure functions.

Here, we introduce a new system called "consecutive-$k_1, \ldots, k_r$-out-of-n: MFM system", which generalizes the consecutive-k, r-out-of-n: DFM system, from the following points of view: first, we consider more than two kinds of failure and second, the components are related with Markovian dependence. Analytically, the system is described as follows. We consider n linearly arranged components associated with a sequence $X_1, \ldots, X_n$ of r.v.'s, which are produced by a stationary Markov chain on $\mathcal{A} = \{0, \ldots, r\}$. The state $\{0\}$ denotes a correctly functioning component, whereas the state $\{i\}$ corresponds to a defect component of type i, for $i = 1, \ldots, r$. We assume that the system fails if at least k_i consecutive components have the defect of type i, for any $i = 1, \ldots, r$.

Let R denote the reliability of the above defined system. Then it holds that $R = \mathbb{P}(\widehat{W} = 0)$, where $\widehat{W}$ is the number of k_i consecutive components with the defect of type i, for all $i = 1, \ldots, r$, in the sequence of n components. In other words, $\widehat{W}$ enumerates the runs of k_i consecutive i's that begin and end in the sequence $X_1, \ldots, X_n$, i.e. $\widehat{W} = \sum_{i=1}^{r} \sum_{a=k_i}^{n} I_{a,i}$. Therefore, upper and lower bounds for R can be derived immediately from Theorems 2.1 and 2.2. As these theorems provide bounds referring to the r.v. $W = \sum_{i=1}^{r} \sum_{a=1}^{n} I_{a,i}$, we need an additional error which results from the difference between W and $\widehat{W}$. This error is given by

$$\mathbb{P}(W \neq \widehat{W}) \leq \sum_{i=1}^{r} (k_i - 1)\psi_i.$$

The related results are the following.

COROLLARY 3.1. *If* $\max_{1\le i\le r} p_{ii} \le 1/2$, *then*

$$e^{-\lambda} - \inf_{s\ge 1} B_1(s) - \sum_{i=1}^{r}(k_i-1)\psi_i \le R \le e^{-\lambda} + \inf_{s\ge 1} B_1(s) + \sum_{i=1}^{r}(k_i-1)\psi_i,$$

where $\lambda = n\sum_{i=1}^{r}(1-p_{ii})\psi_i$ *and* $B_1(s)$ *is the bound of Theorem* 2.1.

The result that follows concerns the i.i.d. case.

COROLLARY 3.2. *If* $\max_{1\le i\le r} p_i \le 1/2$, *then*

$$e^{-\lambda^*} - B_2 - \sum_{i=1}^{r}(k_i-1)p_i^{k_i} \le R \le e^{-\lambda^*} + B_2 + \sum_{i=1}^{r}(k_i-1)p_i^{k_i},$$

where $\lambda^* = n\sum_{i=1}^{r}(1-p_i)p_i^{k_i}$ *and* B_2 *is the bound of Theorem* 2.2.

Next, we present three numerical examples, to illustrate the behaviour of the bounds established in Theorems 2.1 and 2.2. All examples concern the evaluation of upper and lower bounds for the reliability of the consecutive-k_1, k_2, k_3-out-of-n: MFM system, for specific values of k_1, k_2, k_3 and n. For simplicity, the upper and lower bounds of Corollary 3.1 will be denoted by U_1 and L_1 respectively, and those of Corollary 3.2 by U_2 and L_2 respectively.

In Tables 1 and 2 we considered the consecutive-k_1, k_2, k_3-out-of-n: MFM system with components related to trials generated by a Markov chain with transition probability matrix:

$$P_1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{3}{4} & \frac{1}{20} & \frac{1}{10} & \frac{1}{10} \\ \frac{3}{4} & \frac{1}{10} & \frac{1}{20} & \frac{1}{10} \\ \frac{3}{4} & \frac{1}{10} & \frac{1}{10} & \frac{1}{20} \end{pmatrix}.$$

The quantities ϱ and Δ are 0.1 and 1.2 respectively. For the evaluation of the lower and upper bounds L_1 and U_1, we applied Corollary 3.1. In both tables we used the same values for n. On the other hand, in Table 1 we considered slightly smaller values of k_i's than in Table 2. One can observe that, for larger k_i's, the difference $U_1 - L_1$ becomes tighter, in that way providing more accurate interval estimation for the reliability.

In Tables 3 and 4 the computations were conducted again according to Corollary 3.1, using the same values for n as in the previous example. However, we considered a different transition probability matrix with *larger* probabilities p_{11}, p_{22} and p_{33}:

Table 1. Numerical values for L_1, U_1 with $k_1 = 4$, $k_2 = 5$, $k_3 = 4$.

n	L_1	U_1
20	0.9995	0.9997
50	0.9989	0.9991
100	0.9979	0.9981
1000	0.9798	0.9800
5000	0.9034	0.9037
10000	0.8161	0.8166
20000	0.6661	0.6669
50000	0.3618	0.3635

Table 2. Numerical values for L_1, U_1 with $k_1 = 4$, $k_2 = 6$, $k_3 = 5$.

n	L_1	U_1
20	0.9997	0.9998
50	0.9994	0.9995
100	0.9989	0.9990
1000	0.9896	0.9897
5000	0.9492	0.9493
10000	0.9010	0.9012
20000	0.8118	0.8121
50000	0.5939	0.5943

Table 3. Numerical values for L_1, U_1 with $k_1 = 7$, $k_2 = 6$, $k_3 = 7$.

n	L_1	U_1
20	0.9992	0.9997
50	0.9984	0.9989
100	0.9971	0.9976
1000	0.9733	0.9739
5000	0.8743	0.8752
10000	0.7645	0.7659
20000	0.5844	0.5867
50000	0.2599	0.2648

Table 4. Numerical values for L_1, U_1 with $k_1 = 8$, $k_2 = 6$, $k_3 = 7$

n	L_1	U_1
20	0.9994	0.9998
50	0.9987	0.9991
100	0.9976	0.9980
1000	0.9781	0.9786
5000	0.8960	0.8967
10000	0.8030	0.8040
20000	0.6447	0.6465
50000	0.3331	0.3367

$$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{10} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix}.$$

Here, ϱ and Δ are equal to 0.2667 and 1.6776 respectively. In this case, we observe that, in order to obtain tight approximating intervals, it is essential to use larger k_i's than in the previous example.

Finally, upper and lower bounds for the reliability of the consecutive-k_1, k_2, k_3-out-of-n: MFM system with independent and identical components are presented in Table 5. These bounds were computed according to Corollary 3.2. For the calculations, we used $n = 2000$, $k_1 = 4$, $k_2 = 3$, $k_3 = 4$ and several values for the probabilities p_1, p_2 and p_3.

The bounds were computed in Mathematica 2.2 (Wolfram (1998)).

Table 5. Numerical values for L_2, U_2 with $n = 2000$, $k_1 = 4$, $k_2 = 3$, $k_3 = 4$.

p_0	p_1	p_2	p_3	L_2	U_2
0.76	0.10	0.08	0.06	0.3077	0.3278
0.79	0.09	0.07	0.05	0.4583	0.4684
0.82	0.08	0.06	0.04	0.6125	0.6172
0.85	0.07	0.05	0.03	0.7519	0.7540
0.88	0.06	0.04	0.02	0.8624	0.8632
0.91	0.05	0.03	0.01	0.9376	0.9379
0.931	0.04	0.02	0.009	0.9796	0.9796
0.952	0.03	0.01	0.008	0.9964	0.9965
0.964	0.02	0.009	0.007	0.9982	0.9982

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