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ON BIVARIATE LACK OF MEMORY PROPERTY AND A NEW DEFINITION

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Abstract. The earlier concept of bivariate lack of memory property has been examined and an alternative concept of bivariate lack of memory property has been presented along with a set of equivalent conditions. In this process, the link relations known in the literature on univariate lack of memory property have been extended to the bivariate set-up. A multivariate extension of the lack of memory property has also been proposed.

Key words and phrases: Lack of memory property, failure rate, mean residual life, coefficient of variation of residual life, bivariate extension.

1. Introduction

It is fortunate and convenient that in the case of univariate normal distribution, one of the most widely used stochastic model, almost all ways of generating a bivariate version lead to the same distributional form. Such is not the case for the exponential distribution, the other most commonly used stochastic model. There are quite a few bivariate exponential distributions developed in the literature from different considerations, viz., modelling, characterization, functional equation, regression equation and geometric considerations. To mention a few, Freund (1961), Downton (1970), Hawks (1972), Sarkar (1987), Cowan (1987) and Mukherjee and Roy (1996) have put forward different bivariate exponential distribution based different approaches and each of these works is having its own merits and limitations.

Amongst all these approaches the characterization approach has been highly appreciated by the researchers (see Galambos and Kotz (1978)). And amongst all the characterizing properties the most popular one is the lack of memory property. A corresponding Bivariate Lack of Memory Property (BLMP) was put forward by Marshall and Olkin (1967) in connection with the development of a multivariate exponential distribution. The purpose of the present work is to make a critical examination of the BLMP proposed in Marshall and Olkin (1967), suggest an alternative definition for BLMP and establish a few interlinked results.

2. Present definition of BLMP

In terms of the survival function $S(x_1, x_2)$ of a non-negative continuous vector variable $\boldsymbol{X} = (X_1, X_2)$, the BLMP due to Marshall and Olkin (to be referred by BLMP_1) can be written as

$$(2.1) \qquad S(x_1 + t, x_2 + t) = S(x_1, x_2)S(t, t) \qquad \text{for all} \quad x_1 \geq 0,\ x_2 \geq 0,\ t \geq 0.$$

BLMP_1 has been later used by many authors in different contexts and multivariate version of the same is also available in the literature (see Galambos and Kotz (1978)). Writing $U = \text{Min}(X_1, X_2)$, $V = X_1 - X_2$, Block (1977) presented a characterization of BLMP_1 in terms of exponential law for U and independence of U and V. Earlier, Block and Basu (1974) examined the distribution of V under BLMP_1. Samanta (1975) considered n random observations (X_{1i}, X_{2i}), $i = 1, 2, \ldots, n$ and proved that, for $\boldsymbol{Z} = \text{Min}\{X_{1i}, X_{2i}, 1 \le i \le n\}$, independence of $\boldsymbol{Z}$ and $\boldsymbol{U} = (X_{13} - Z, X_{23} - Z, \ldots, X_{1n} - Z, X_{2n} - Z)$ is necessary and sufficient for BLMP_1. Roy and Mukherjee (1989) considered some equivalent versions of BLMP_1 which can be viewed as random versions of the condition (2.1).

This definition due to Marshall and Olkin (1967) is based on joint survival function. An alternative approach could have been to consider the conditional survival functions. Before doing that, let us examine the present definition first. For this purpose, we start with the observation that any bivariate extension of a univariate property becomes meaningful only when it reduces to the univariate one under a reduction of the underlying dimension. Here, the condition (2.1) does not satisfy this basic requirement because it cannot be reduced to univariate lack of memory property for the marginal distributions. In fact a close look at the BLMP_1 reveals that the same is an extension of LMP for $\text{Min}\{X_1, X_2\}$ from independent set-up to dependent set-up. As a result, attention of the bivariate analysis got prodirected towards minimum-exponential property.

Further, from the celebrated discussion on univariate LMP in Galambos and Kotz (1978) we may insist upon the following requirements for BLMP:

(BP1) BLMP $\Leftrightarrow$ constancy of bivariate failure rates, in some sense,

(BP2) BLMP $\Leftrightarrow$ constancy of bivariate mean residual lives, in some sense, and

(BP3) BLMP $\Leftrightarrow$ unity of the coefficient of variations of the residual lives,

where requirement (BP1) is comparable with (P1) of Galambos and Kotz ((1978), p. 12) and requirement (BP2) is comparable with (P2) of the same reference. Requirement (BP3) arises out of the works of Mukherjee and Roy (1986) on higher moments. By global constancy we mean independence of the bivariate function with respect to both the variables and by local constancy we mean the independence of the bivariate function with respect to one variable only.

It may be noted that for BLMP_1, even with additional conditions of marginal exponentiality, the above conditions do not hold true. Following Johnson and Kotz (1975) if we define the bivariate failure rates as $r_i(x_1, x_2)$, $i = 1, 2$, where $r_i(x_1, x_2) = \partial/\partial x_i[-\log S(x_1, x_2)]$, we may observe that Marshall and Olkin (1967)'s Bivariate Exponential Distribution (BED), uniquely satisfying BLMP_1 under exponential marginals, has

$$r_i(x_1, x_2) = \begin{cases} \lambda_1 & \text{if} \quad x_i \prec x_{3-i} \\ \lambda_i + \lambda_3 & \text{if} \quad x_i \succ x_{3-i} \end{cases}$$

$i = 1, 2$, where the survival function is given by

$$S(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)].$$

Thus, $r_i(x_1, x_2)$, $i = 1, 2$ are neither globally nor locally constants. Further, mean residual lives $M_i(x_1, x_2)$, $i = 1, 2$, work out as

$$\begin{aligned} M_i(x_1, x_2) &= E(X_i - x_i \mid X_1 \ge x_1, X_2 \ge x_2) \\ &= \begin{cases} 1/(\lambda_i + \lambda_3) & \text{for} \quad x_i \succ x_{3-i} \\ 1/\lambda_i + [1/(\lambda_i + \lambda_3) - 1/\lambda_i] \exp[-\lambda_i(x_{3-j} - x_i)] & \text{for} \quad x_i \prec x_{3-i}, \end{cases} \end{aligned}$$

$i = 1, 2$. Here, again mean residual lives are neither globally nor locally constants. Thus, BLMP_1 has some limitations as a bivariate generalisation of the univariate LM property. This does not mean that the bivariate exponential distribution of Marshall and Olkin is not a suitable bivariate model. It has its own merits when viewed from the point of shock process and would continue to remain one of the most useful bivariate exponential law for real life applications. We are only concerned with the BLMP aspect in this paper.

3. A new definition

In view of the above discussions we make an attempt to introduce a new concept for BLMP which will be abbreviated as BLMP_2. This we do by imposing the condition that each of the conditional distributions of $\{X_1 \mid X_2 \succeq x_2\}$ and $\{X_2 \mid X_1 \succeq x_1\}$ should follow univariate LMP. Rewriting those conditions in terms of survival function we get the following definition:

DEFINITION 3.1. A survival function $S(x_1, x_2)$ is said to possess BLMP_2 if and only if for all choices of non-negative x_1, x_2, y_1, y_2

$$S(x_1 + y_1, x_2)S(0, x_2) = S(x_1, x_2)S(y_1, x_2) \tag{3.1}$$

and

$$S(x_1, x_2 + y_2)S(x_1, 0) = S(x_1, x_2)S(x_1, y_2). \tag{3.2}$$

That this definition reduces to nivariate LMP can be easily verified with a choice of $x_2 = 0$ in (3.1) or $x_1 = 0$ in (3.2). The following theorem will ensure that BLMP_2 satisfies the properties (BP1), (BP2) and (BP3).

THEOREM 3.1. *The folowing statements are equivalent*:

(i) $\boldsymbol{X}$ *follows* BLMP_2,

(ii) *failure rates*, $r_1(x_1, x_2)$ *and* $r_2(x_1, x_2)$, *of* $\boldsymbol{X}$ *are locally constants*,

(iii) *mean residual lives*, $M_1(x_1, x_2)$ *and* $M_2(x_1, x_2)$, *of* $\boldsymbol{X}$ *are locally constants*,

(iv) *coefficients of variation of the residual lives are unity, and*

(v) $\boldsymbol{X}$ *follows Bivariate Exponential Distribution due to Gumbel* (*BED-G*) (1960) *with survival function*

$$S(x_1, x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_1 x_2) \tag{3.3}$$

where $\lambda_1 \succ 0$, $\lambda_2 \succ 0$, $0 \preceq \lambda_3 \preceq \lambda_1\lambda_2$.

PROOF. We shall prove that under the existence of failure rates (i)<=>(ii). Further, (i)=>(iii)=>(iv)=>(i). In view of the second implication existence of failure rates will follow automatically.

To prove that (i)=>(ii) we take logarithm of (3.1) and differentiate w.r.t. y_1 so as to obtain the relation $r_1(x_1 + y_1, x_2) = r_1(y_1, x_2)$ for all choices of non-negative x_1, y_1 and x_2. Hence, $r_1(x_1, x_2) = r_1(0, x_2)$ i.e. locally constant w.r.t. x_1. A similar treatement of (3.2) will ensure local constancy of $r_2(x_1, x_2)$. To prove the converse let $r_1(x_1, x_2) = k_i(x_{3-i})$, $i = 1, 2$. Then noting that $r_i(x_1, x_2)$ is the failure rate of $\{X_i \mid X_{3-i} \succeq x_{3-i}\}$, $i = 1, 2$ we conclude that the corresponding survival functions must satisfy univariate LMP for each choice of x_{3-i}. But the survival function of $\{X_1 \mid X_2 \succeq x_2\}$ is

$\{S(x_1,x_2)/S(0,x_2)\}$ and of $\{X_2 \mid X_1 \succeq x_1\}$ is $\{S(x_1,x_2)/S(x_1,0)\}$. Hence follows (3.1) and (3.2). Thus, (ii)$\prec\!=\!\succ$(i).

To prove that (i)$=\succ$ (iii) we note that

$$\begin{aligned} M_1(x_1,x_2) &= \int_0^\infty \{S(x_1+t,x_2)/S(x_1,x_2)\}dt \\ &= \int_0^\infty \{S(t,x_2)/S(0,x_2)\}dt, \qquad \text{under (3.1)}, \end{aligned}$$

which is locally constant in x_1. Similarly, $M_2(x_1,x_2)$ is locally constant in x_2.

To prove that (iii)$\Rightarrow$(iv) let $M_i(x_1,x_2)=k_i(x_{3-i})$, $i=1,2$. Then integration of the following w.r.t. x_1

$$\{1/M_1(x_1,x_2)\} = \left\{S(x_1,x_2)\Big/\int_{x_1}^\infty S(t,x_2)dt\right\}$$

results in

$$x_1/k_1(x_2) = -\log\left\{\int_{x_1}^\infty S(t,x_2)dt\right\} + c$$

where c is the constant of integration. Differentiating the above with respect to x_1 we get after usual simplification

$$S(x_1,x_2) = [\exp\{-x_1/k_1(x_2)\}]/k_1(x_2). \tag{3.4}$$

Now,

$$\operatorname{Var}(X_1-x_1 \mid X_1 \succeq x_1, X_2 \succeq x_2) = 2\int_0^\infty t\{S(x_1+t,x_2)/S(x_1,x_2)\}dt - M_1^2(x_1,x_2). \tag{3.5}$$

Under (3.4) it simplifies to

$$\begin{aligned} \operatorname{Var}(X_1-x_1 \mid X_1 \succeq x_1, X_2 \succeq x_2) &= 2\int_0^\infty t\exp[-t/k_1(x_2)]dt - k_1^2(x_2) \\ &= k_1^2(x_2). \end{aligned}$$

Thus $C_1(x_1,x_2)$, the coefficient of variation of $\{X_1-x_1 \mid X_1 \succeq x_1, X_2 \succeq x_2\}$ is unity. Similarly $C_2(x_1,x_2)$, the coefficient of variation of $\{X_2-x_2 \mid X_1 \succeq x_1, X_2 \succeq x_2\}$ also reduces to unity.

To prove that (iv)$\Rightarrow$(v) we note that the condition $C_1(x_1,x_2)=1$ implies

$$\int_0^\infty t\{S(x_1+t,x_2)/S(x_1,x_2)\}dt = \left[\int_0^\infty \{S(x_1+t,x_2)/S(x_1,x_2)\}dt\right]^2$$

$$\frac{S(x_1,x_2)}{\int_{x_1}^\infty S(u,x_2)du} = \frac{\int_{x_1}^\infty S(u,x_2)du}{\int_{x_1}^\infty (u-x_1)S(u,x_2)du}. \tag{3.6}$$

Integrating both sides of (3.6) w.r.t. x_1, we get

$$\log\int_{x_1}^\infty S(u,x_2)du = \log A_1(x_2) + \log\int_{x_1}^\infty (u-x_1)S(u,x_2)du$$

$$\int_{x_1}^{\infty} S(u, x_2)du = A_1(x_2)\int_{x_1}^{\infty}(u - x_1)S(u, x_2)du \tag{3.7}$$

where $A_1(x_2)$ is an integration constant. Differentiating (3.7) w.r.t. x_1 we get

$$S(x_1, x_2) = A_1(x_2)\int_{x_1}^{\infty} S(u, x_2)du. \tag{3.8}$$

Thus,

$$\frac{S(x_1, x_2)}{\int_{x_1}^{\infty} S(u, x_2)du} = A_1(x_2)$$

which results in, for some $B_1(x_2)$,

$$S(x_1, x_2) = B_1(x_2)\exp[-x_1 A_1(x_2)]. \tag{3.9}$$

A similar treatment of $C_2(x_1, x_2) = 1$ will give rise to

$$S(x_1, x_2) = B_2(x_1)\exp[-x_2 A_2(x_1)]. \tag{3.10}$$

Now comparing (3.9) with (3.10) we get as an identity

$$B_2(0)\exp[-x_1 A_1(x_2) - x_2 A_2(0)] = B_1(0)\exp[-x_1 A_1(0) - x_2 A_2(x_1)].$$

Taking logarithm of both sides of the above and observing that the LHS is linear in x_1 we conclude that the RHS is necessarily linear in x_1. Thus $A_2(x_1) = \alpha + \beta x_1$ for some α and β. Thus from (3.10) we observe, after simplification, $S(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_1 x_2]$. It is easy to note that $\lambda_1 \succ 0$, $\lambda_2 \succ 0$, $0 \preceq \lambda_3 \preceq \lambda_1\lambda_2$.

Lastly, we need to show that (v)⇒(i). This is an easy consequence of (3.3). □

Remark 1. From Theorem 3.1 one may obtain as corollary three characterization results presented in Johnson and Kotz (1975), Zahedi (1985) and Roy and Gupta (1996). Further, in our process of unification we have not only done away with restrictive assumptions used therein but also made a general networking similar to that of Galambos and Kotz (1978)'s univariate result.

Remark 2. A multivariate extension of BLMP_2 (to be abbreviated as MLMP_2) in terms of multivariate survival function $S(x_1, x_2, \ldots, x_p)$ is given by the condition

$$\begin{aligned} &S(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_p)S(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p) \\ &\quad = S(x_1, \ldots, x_i, \ldots, x_p)S(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p) \end{aligned}$$

for all non-negative choices of the arguments and for all $i = 1, 2, \ldots, p$. A result similar to Theorem 3.1 can also be easily ensured using mathematical induction principle. In fact, it can be shown that the unique determination of the corresponding mulivariate exponential distribution is given by the survival function

$$S(x_1, \ldots, x_1, \ldots, x_p) = \exp\left[-\sum \lambda_i x_i - \sum\sum \lambda_{ij} x_i x_j - \cdots - \lambda_{12\cdots p} x_1 x_2 \cdots x_p\right].$$

Remark 3. A discretization of BLMP_2 can also be made in terms of conditions (3.1) and (3.2). Under discretization these are to be satisfied for all choices of non-negative integer values of x_1, x_2, y_1 and y_2. The bivariate distribution satisfying BLMP_2 is that of Roy (1993)'s Bivariate Geometric distribution. The local constancy of failure rates and mean residual lives have been reported therein.

Remark 4. An alternative representation of the BLMP_2 is a combination of (3.1) and (3.2) into a single equation as given below:

$$\begin{aligned}(3.11)\qquad & S(x_1+Zy_1, x_2+(1-Z)y_2)S((1-Z)x_1, Zx_2) \\ & = S(x_1,x_2)S((1-Z)x_1+Zy_1, Zx_2+(1-Z)y_2)\end{aligned}$$

for all non-negative values of x_1, x_2, y_1, y_2 and for $Z = 0, 1$.

This representation can be further modified in the lines of univariate modifications of LMP as proposed in Krishnaji (1971). Writing $W(x_1,x_2) = P[X_1 - U \succ x_1, X_2 - U \succ x_2]$ where U is a non-negative random variable independent of (X_1, X_2) we get an equivalent definition of BLMP_2 as

$$\begin{aligned}(3.12)\qquad & W(x_1+Zy_1, x_2+(1-Z)y_2)S((1-Z)x_1, Zx_2) \\ & = S(x_1,x_2)W((1-Z)x_1+Zy_1, Zx_2+(1-Z)y_2)\end{aligned}$$

for all $x_1 \succeq 0$, $y_1 \succeq 0$, $x_2 \succeq 0$, $y_2 \succeq 0$ and $Z = 0, 1$.

For a proof of the above we can simplify (3.12) into

$$(3.13)\qquad W(x_1,x_2)/W(0,x_2) = S(x_1,x_2)/S(0,x_2)$$

from a choice of $Z = 0$ and $y_2 = 0$ and

$$(3.14)\qquad W(x_1,x_2)/W(0,x_2) = S(x_1,x_2)/S(0,x_2)$$

from a choice of $Z = 1$ and $y_1 = 0$. A simplification of (3.12) using (3.13) and (3.14) gives rise to BLMP_2. To prove the converse we start form the LHS of (3.12) and use the result (3.11) to ensure the RHS of (3.12).

Remark 5. It is easy to verify that $\boldsymbol{X}$ follows BLMP_2 if and only if

$$r_1(x_1,x_2)M_1(x_1,x_2) = r_2(x_1,x_2)M_2(x_1,x_2) = 1$$

for all non-negative choices of x_1 and x_2.

The above result covers a bivariate extension of a similar work of Muth (1977) on univariate LMP.

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