Ann. Inst. Statist. Math.
Vol. 54, No. 1, 138–154 (2002)

THE INVERSE GAUSSIAN MODELS: ANALOGUES OF SYMMETRY, SKEWNESS AND KURTOSIS

GOVIND S. MUDHOLKAR[1] AND RAJESHWARI NATARAJAN[2*,**]

[1] *Department of Statistics, University of Rochester, Rochester, NY 14627, U.S.A.*
[2] *Department of Biostatistics, University of Rochester Medical Center, 601 Elmwood Avenue, Box 630, Rochester, NY 14642, U.S.A.*

(Received July 19, 1999; revised May 8, 2000)

Abstract. The inverse Gaussian (IG) family is strikingly analogous to the Gaussian family in terms of having simple inference solutions, which use the familiar χ^2, t and F distributions, for a variety of basic problems. Hence, the IG family, consisting of asymmetric distributions is widely used for modelling and analyzing nonnegative skew data. However, the process lacks measures of model appropriateness corresponding to $\sqrt{\beta_1}$ and β_2, routinely employed in statistical analyses. We use known similarities between the two families to define a concept termed IG-symmetry, an analogue of the symmetry, and to develop IG-analogues δ_1 and δ_2 of $\sqrt{\beta_1}$ and β_2, respectively. Interestingly, the asymptotic null distributions of the sample versions d_1, d_2 of δ_1, δ_2 are exactly the same as those of their normal counterparts $\sqrt{b_1}$ and b_2. Some applications are discussed, and the analogies between the two families, enhanced during this study are tabulated.

Key words and phrases: Contaminated inverse Gaussian distribution, goodness-of-fit tests, IG-scale mixtures.

1. Introduction

The inverse Gaussian (IG) distribution originally introduced in the context of Brownian motion by Schrödinger (1915) and Smoluchowsky (1915), appeared later as the distribution of average sample number in Wald's (1947) monograph on sequential analysis. Contemporaneously, Tweedie (1945), while studying the distributional properties, discovered its striking similarities with the Gaussian distribution and named it the inverse Gaussian distribution. It may be noted that earlier, Etienne Halphen, in search for a distribution which had "exponential decay" in both the tails for modelling hydrological data, invented Halphen's laws, precursors of the inverse Gaussian distribution. Due to religious and political reasons, his work was published by Dugué (1941) under his name; see Seshadri (1997). A review paper by Folks and Chhikara (1978) presented to the Royal Statistical Society, (see also Iyengar and Patwardhan (1988)) highlighted some remarkable analogies between Gaussian and the IG families which, as a discussant Dawid expressed it, "intrigued and baffled" many. It emphasized usefulness of the distribution, advanced research on the subject, and stimulated the use of the distribution in many

*This work is part of the author's Ph.D. Dissertation at the Deprtment of Biostatistics, University of Rochester.

**Now at Department of Statistical Science, Southern Methodist University, Dallas, TX 75275, U.S.A.

areas of applied research.

The inverse Gaussian distribution $IG(\mu, \lambda)$ with p.d.f.

$$f_X(x \mid \mu, \lambda) = \left\{ \frac{\lambda}{2\pi x^3} \right\}^{1/2} \exp\left\{ -\frac{\lambda}{2\mu^2 x}(x-\mu)^2 \right\}, \qquad x > 0, \quad \mu > 0, \ \lambda > 0, \tag{1.1}$$

is now strongly recommended for modelling and analyzing asymmetric data; e.g., see Chhikara and Folks (1989) and Seshadri (1999). Its latest modelling application appears in the emerging and potentially important area of internet communication, see Huberman *et al.* (1998). The quality of IG model fit to internet data is empirically seen to be excellent.

The goal of this paper is to derive indicators of departures from the IG assumptions by pressing known analogies between the Gaussian and inverse Gaussian families. We develop inverse Gaussian analogues of the coefficients of skewness and kurtosis which are commonly employed in evaluating the appropriateness of Gaussian models.

A basic factor underlying the classical morphology of distributions, and the choice of models is the concept of symmetry. In Section 2 we use some moment identities for the IG distributions to define an inverse Gaussian analogue, termed *IG-symmetry*, of the symmetry. In the inverse Gaussian framework, it mimics many common properties of conventional symmetry. For example, scale mixtures of inverse Gaussian distributions are IG-symmetric, similar to the scale mixtures of Gaussian distributions being symmetric. The development of coefficient of IG-kurtosis δ_2 is presented in Section 3, before that of coefficient of IG-skewness δ_1 which appears in Section 4. This is primarily because the definition of δ_2 is a prerequisite for defining δ_1. In Section 5 the use of the new measures is explained and illustrated using several well-known families of distributions with non-negative support. We present the (δ_1, δ_2)-chart as the IG analogue of the classical (β_1, β_2)-chart.

The sample analogues (d_1, d_2) of (δ_1, δ_2) are considered in Section 6. It is shown that, under the IG assumption and as the sample size tends to infinity, the asymptotic distribution of (d_1, d_2) is bivariate normal with independent $N(0, 6/n)$, $N(0, 24/n)$ marginals, i.e., exactly the same as that of $(\sqrt{b_1}, b_2)$ under normality. Subsection 6.1 illustrates the use of the (δ_1, δ_2)-chart for model selection. The final section is given to conclusions and miscellaneous remarks. It also contains a tabulation of how the present work expands the list of analogies between the Gaussian and the inverse Gaussian families.

2. IG-Symmetry, an analogue of symmetry

For any distribution symmetric about zero, all raw moments, μ'_{2r+1}, $r = 1, 2, \ldots$, of odd order vanish. Analogously, for any inverse Gaussian random variable $X \sim IG(\mu, \lambda)$, it can be verified that the following relationship between the moments hold for all positive or negative integers r (in fact real r)

$$E\left[\left(\frac{X}{\mu}\right)^{-r}\right] = E\left[\left(\frac{X}{\mu}\right)^{r+1}\right]. \tag{2.1}$$

Setting aside the issue of proper analogue of symmetry of a random variable, which refers to the geometry of the density function, and also defined in terms of the distribution function, we note that the countable relations at (2.1) are similar to the moments properties of the conventional symmetric distributions in general, and the normal family in particular. In other words, we define the IG-analogue of symmetry by the following:

DEFINITION 2.1. A random variable X with $E(X) = \mu$ and all its moments of negative and positive order $r = \pm 1, \pm 2, \ldots$ finite, is said to be IG-symmetric about μ if the moments satisfy equation (2.1).

PROPOSITION 2.1. *The lognormal distributions satisfy* (2.1). *That is, the lognormal distribution* $LN(\psi, \sigma)$ *is* IG*-symmetric about its mean* $\mu = \exp(\psi + \sigma^2/2)$.

PROOF. The r-th moment of a lognormal variable $X \sim LN(\psi, \sigma)$ is given by $\mu_r' = \exp(r\psi + \frac{1}{2}r^2\sigma^2)$. Therefore, it can be easily seen that, for all $r = 1, 2, \ldots$

$$E\left[\left(\frac{X}{\mu}\right)^{r+1}\right] = \exp\left[\frac{1}{2}\sigma^2 r(r+1)\right] = E\left[\left(\frac{X}{\mu}\right)^{-r}\right]. \tag{2.2}$$

Remark 2.1. Another class of IG-symmetric distributions which contains the lognormal distributions has its origins in Stieltjes (1894); see Shohat and Tamarkin (1943), Heyde (1963), and Mudholkar and Hutson (1998). It is the family with p.d.f.

$$g(x;\epsilon) = \frac{1}{x\sqrt{2\pi}} \exp[-(\log x)^2/2] \times (1 + \epsilon \sin[2\pi(\log x)]), \qquad |\epsilon| < 1. \tag{2.3}$$

It is easy to see that (2.3) reduces to the lognormal p.d.f. when $\epsilon = 0$. Interestingly, this is also a counterexample which illustrates the fact that moments do not always determine a distribution uniquely. The moments $E(X^r)$, $r = \pm 1, \pm 2, \ldots$ of (2.3) are all independent of ϵ, i.e., the same as those of the lognormal variable. Hence (2.3) for all ϵ, $|\epsilon| < 1$, is IG-symmetric.

IG-Scale Mixtures. The family of scale mixtures of normals, also known as the Normal/Independent family, is the family $N(0, Y^2)$ of normal distributions with standard deviation given by a non-negative random variable Y. The normal scale mixtures play an important role in the area of robust inference. The natural IG-analogue of this family may be defined as the distributions of random variables $X \sim IG(\mu, Y)$, where Y is a positive random variable with distribution function H. It is easy to see that the p.d.f. of X is

$$f_X(x;\mu, H) = \int_0^\infty \left(\frac{y}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{y}{2\mu^2 x}(x-\mu)^2\right\} dH(y). \tag{2.4}$$

Remark 2.2. Appropriate scale mixtures of lognormal distributions amongst themselves, and with IG distributions with the same mean, constitute another family of IG scale-mixtures. Additional examples of such distributions may be created by considering mixtures of (2.3) w.r.t. ϵ, i.e., by assuming ϵ to be random.

In particular, the random variable Y which takes the values 1 and λ with probabilities p and $(1-p)$, respectively, defines a *contaminated IG* distribution, i.e., the IG-analogue of a contaminated normal distribution. Its density is given by

$$p\left(\frac{1}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{1}{2\mu^2 x}(x-\mu)^2\right\} + (1-p)\left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}. \tag{2.5}$$

PROPOSITION 2.2. *If the raw moments of all orders for the mixing distribution H exist and are finite, then the IG scale mixtures with p.d.f. given by* (2.4) *satisfy the relations given in* (2.1), *that is, they are IG-symmetric about* μ.

PROOF. If a $r.v$ is distributed according to (2.4), then

$$E\left[\left(\frac{X}{\mu}\right)^{-r}\right]-E\left[\left(\frac{X}{\mu}\right)^{r+1}\right]$$
$$=\int_0^\infty\left[\int_0^\infty\left(\left(\frac{x}{\mu}\right)^{-r}-\left(\frac{x}{\mu}\right)^{r+1}\right)\left\{\frac{y}{2\pi x^3}\right\}^{1/2}\exp\left\{-\frac{y}{2\mu^2 x}(x-\mu)^2\right\}dx\right]dH(y)$$
$$=0.$$

3. The *IG* analogue of kurtosis

The classical coefficient of kurtosis is known in the statistical literature as a measure of "peakedness", and is often used to identify distributions as "platykurtic", "mesokurtic" and "leptokurtic"; see Pearson (1905). Keilson and Steutel (1974) interpret it as a measure of non-normality by establishing its quasi-metric character in the space of scale-mixtures of normals. The coefficient of kurtosis $\beta_2 = \mu_4/\mu_2{}^2$ is commonly used as a quick and rough measure of non-normality. Balanda and MacGillivray (1988) describe it as "vaguely defined as the location- and scale-free movement of probability mass from the shoulders of a distribution into its center and tails" and formalize in many ways, present an overview of the coefficient.

For the purpose of this paper we view β_2 in terms of the asymptotic distribution of the sample variance. Let $X_1,\ldots,X_n$ be a random sample from a population with variance σ^2, and a finite coefficient of kurtosis β_2. Then it is well known that, as $n\to\infty, \sqrt{n}(S^2-\sigma^2)\to^d N(0,(\beta_2-1)\sigma^4)$. Furthermore, in view of the Mann-Wald theorem,

$$\sqrt{n}(\log S^2-\log\sigma^2)\xrightarrow{d} N(0,(\beta_2-1)), \tag{3.1}$$

the convergence being to $N(0,2)$ under normality.

On the other hand, if the population is inverse Gaussian $IG(\mu,\lambda)$ then $n\lambda V = \lambda\sum_{i=1}^n(1/X_i-1/\bar{X})$ is distributed as a chi-square with $n-1$ degrees of freedom, the same as the asymptotic distribution of the variance of a normal sample. More generally, if the four population moments EX^j, $j=\pm1,\pm2$, exist and are finite, then we have, analogous to (3.1) above, the following asymptotic convergence in law result. As $n\to\infty$,

$$\sqrt{n}\,(\log V-\log[\nu-(1/\mu)])\xrightarrow{d} N\left(0,\frac{\eta^2\mu^2}{(\nu\mu-1)^2}\right), \tag{3.2}$$

where $Y=X^{-1}$,

$$\nu=E(Y),\qquad \mu=E(X), \tag{3.3}$$
$$\tau^2=\mathrm{Var}(Y),\qquad \sigma^2=\mathrm{Var}(X),$$

and

$$\eta^2=\tau^2+2(1-\mu\nu)/\mu^2+\sigma^2/\mu^4,$$

which reduces to $N(0,2)$ under the IG assumption. We use the analogy between (3.1) and (3.2) to obtain the following:

DEFINITION 3.1. The IG-kurtosis coefficient δ_2, the analogue of the coefficient of kurtosis, which is asymptotically $\beta_2 = [\mathrm{Var}(S^2)/E^2(S^2)] + 1$, is defined by $\delta_2 = [\mathrm{Var}(V)/E^2(V)] + 1$, or equivalently by:

$$\delta_2 = \frac{\eta^2\mu^2}{(\nu\mu - 1)^2} + 1, \tag{3.4}$$

where the quantities on the right hand side (r.h.s.) are as defined in (3.3).

Remark 3.1. From (3.1), (3.2) and (3.4) we see that $\delta_2 \geq 1$, just as the coefficient of kurtosis has the well-known property $\beta_2 \geq 1$.

4. The IG analogue of skewness

The assumption of symmetry is pivotal in defining notions such as the location parameter (see Bickel and Lehmann (1975)) and in developing various inference methods in robust analysis with normal as the target family (see Mudholkar *et al.* (1991)). Measures of skewness are basic in exploratory data analysis as well as in subsequent statistical analyses in applied research. The best known among such measures is the classical coefficient $\sqrt{\beta_1}$ of skewness which has, from the earliest years had many competitors, e.g. Karl Pearson's $(mean - median)/s.d.$ and more recent measures based on the L-moments and LQ-moments; see Mudholkar and Hutson (1998). In this section we pursue an analogous development in the context of the notion of IG-symmetry introduced in Section 2.

The notion of an analogue of skewness in the context of the unambiguously skewed inverse Gaussian family may appear to be a paradox, or even an oxymoron. However, we start the development by recalling the well-known characteristic independence of the mean $\bar{X}$ and variance $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$ of a random sample $X_1, \ldots, X_n$ from a normal population; see Lukacs and Laha (1964), or Kagan *et al.* (1973). The analogous characterization theorem for inverse Gaussian distribution states that the mean $\bar{X}$ and $\hat{\lambda}^{-1} = V = (1/n)\sum_{i=1}^{n}(1/X_i - 1/\bar{X})$ are independent if and only if the population is IG; see Khatri (1962), or Seshadri (1993).

The above characterization of normality was used by Lin and Mudholkar (1980) to construct the Z-test of normality which is best suited for detecting asymmetric alternatives; see also Mudholkar *et al.* (2001*a*). Given a random sample $X_1, \ldots, X_n$ the Z-statistic of normality is the Fisher transform $Z(G) = \tanh^{-1}(r(G))$, of the correlation coefficient

$$r(G) = \mathrm{Corr}\left\{X_i, \left\{\sum_{j=1, j\neq i}^{n} X_j^2 - \left(\sum_{j\neq i} X_j\right)^2 / (n-1)\right\}^{1/3} ; i = 1, 2, \ldots, n\right\}, \tag{4.1}$$

where G denotes the Gaussian population. Lin and Mudholkar give the sampling distribution of $r(G)$ in the following theorem.

THEOREM 4.1. *Let $X_1, \ldots, X_n$ be a random sample from a population with the first 6 moments finite. Then, as $n \to \infty, r(G)$ is asymptotically normal with mean*

$\mu(G) = -\sqrt{\beta_1}/\sqrt{\beta_2 - 1}$, *and variance* $\sigma^2(G) = \{\mu_6 - 6\mu_4 - \mu_3^2 + 9\}/\{n(\mu_4 - 1)\}$. *Under the null hypothesis of normality, as* $n \to \infty$,

$$\sqrt{n}r(G) \xrightarrow{d} N(0,3). \tag{4.2}$$

Recently, Mudholkar *et al.* (2001*b*) have used the characteristic independence between $\bar{X}$ and V to develop the analogous Z-test for the composite inverse Gaussian goodness-of-fit hypothesis based on an analogue $Z(IG)$ of $Z(G)$. They noted that the test statistic $Z(G)$ was constructed especially to detect asymmetric alternatives and also that its asymptotic mean depends on the coefficient of skewness. They then suggested that the IG analogue $Z(IG)$ may possibly be used to obtain an IG-analogue of the skewness. Such a development follows.

The Z-statistic for the IG hypothesis is the Fisher transform

$$Z(IG) = \frac{1}{2}\log\left\{\frac{1+r(IG)}{1-r(IG)}\right\}, \tag{4.3}$$

where $r(IG)$ is the product moment correlation coefficient given by

$$r(IG) = \text{Corr}\left\{X_i, \sum_{j=1, j\neq i}^{n} (1/X_j - 1/\bar{X}_{-i})/(n-1); i = 1, 2, \ldots, n\right\}. \tag{4.4}$$

The asymptotic distribution of $r(IG)$ is given by the following theorem.

THEOREM 4.2. *Let* $X_1, \ldots, X_n$ *be a random sample from a population with the first four positive moments and the first two negative moments finite. Then as* $n \to \infty$, $r(IG)$ *is asymptotically normal with mean* μ^* *and variance* σ^{*2}, *where* μ^* *is given by*

$$\mu^* = \{\mu_2'/(\mu_1')^2 - \mu_1'\mu_{-1}'\}/\sqrt{\sigma_{22}\sigma_{33}^*}, \tag{4.5}$$

$\sigma_{22} = n\,\text{Var}(\bar{X}), \sigma_{33}^* = n\,\text{Var}(V)$ *given by* (4.7), *and* μ_j' *denotes the* j*-th raw moment of* X. *For the inverse Gaussian population, the asymptotic mean* $\mu^* = 0$, *the asymptotic variance* σ^{*2} *reduces to* $3/n$. *That is, for the IG population, as* $n \to \infty$,

$$\sqrt{n}r(IG) \xrightarrow{d} N(0,3). \tag{4.6}$$

Remark 4.1. For actual (finite sample) goodness-of-fit tests, the Fisher transforms $Z(G)$ and $Z(IG)$ of the correlation coefficients $r(G)$ and $r(IG)$ respectively, are used. Interestingly, the asymptotic null distributions of these two statistics coincide, i.e., both are asymptotically normal with mean zero and variance $3/n$.

Now, from (3.2) we see that

$$n\,\text{Var}(V) = \sigma_{33}^* = \eta^2 = (\delta_2 - 1)(\nu\mu - 1)^2/\mu^2. \tag{4.7}$$

Therefore, equation (4.5) can also be expressed as follows:

$$E(r(IG)) = \mu^* = \frac{\mu_2'/(\mu)^2 - \mu\nu}{\sqrt{\sigma_{22}(\delta_2 - 1)(\nu\mu - 1)^2/\mu^2}}. \tag{4.8}$$

Table 1. The (δ_1, δ_2) values for various distributions.

Label	δ_1	$\delta_2 - 3$	Constraints
$IG(\mu, \lambda)$	0	0	$\mu > 0,\ \lambda > 0$
$RIG(\mu, \lambda)$	$\frac{(-\mu^2/\lambda^2)}{(1+\mu/\lambda)\sqrt{\mu/\lambda+2\mu^2/\lambda^2}}$	$\frac{(\lambda+\mu)^2(\mu^2-\lambda^2)+\lambda^3(\lambda+2\mu)}{\mu\lambda(\lambda+\mu)^2}$	$\mu > 0,\ \lambda > 0$
$LN(\psi, \sigma)$	0	$\exp(\sigma^2) - 1$	$\sigma > 0$
$G(\alpha, \beta)$	$\frac{-1}{\sqrt{\alpha}}$	$\frac{5\alpha-2}{\alpha(\alpha-2)}$	$\alpha > 2$
$RG(\alpha, \beta)$	$\frac{1}{\sqrt{\alpha-3}}$	$\frac{1}{\alpha-3}$	$\alpha > 3$
$P(k, a)$	$\frac{2a-1}{\sqrt{a(a-2)}}$	$\frac{6a^3-4a^2+a+2}{a(a^2-4)}$	$a > 2$
$F(\nu_1, \nu_2)$	$\frac{(\nu_2-2)[(\nu_1^2-4)(\nu_2-2)^2-\nu_2\nu_1^2(\nu_2-4)]}{(\nu_2-4)\nu_1(2\nu_1+2\nu_2-4)^{3/2}}$	*	$\nu_1 > 4,\ \nu_2 > 4$
$B(p, q)$	$-\frac{(2p+q)}{\sqrt{pq(p+q+1)}}$	**	$p > 2,\ q > 0$

*, ** Not displayed due to intricacy.

Hence, by using the definition of δ_2 (3.4) in the analogy between $E(r(G)) = -\sqrt{\beta_1}/\sqrt{\beta_2 - 1}$ and $E(r(IG)) = \delta_1/\sqrt{\delta_2 - 1}$, we get the following:

DEFINITION 4.1. The IG-skewness coefficient δ_1, the analogue of the coefficient $\sqrt{\beta_1}$ of skewness, is given by

$$\delta_1 = \frac{\mu_2'/(\mu^2) - \mu\nu}{(\mu\nu - 1)\sqrt{\mu_2'/\mu^2 - 1}}, \tag{4.9}$$

where the quantities on the r.h.s. are as defined in (3.3), except for $\mu_2' = E(X^2)$.

The first moment $E(X)$ is commonly used to quantify the concept of location and the standardized version of the third central moment defines $\sqrt{\beta_1}$. The coefficient δ_1 defined above may be interpreted in a similar manner. Actually, the numerator of δ_1 is the difference between $E[(X/\mu)^{-1}]$ and $E[(X/\mu)^2]$. In other words it is the difference between the two sides of (2.1) for $r = 1$. The coefficient δ_1 defined above may be interpreted as the standardized version of the first condition of IG-symmetry. It vanishes for an IG-symmetric variable.

5. The (δ_1, δ_2)-chart

The coefficients $\sqrt{\beta_1}$ and β_2 and their sample versions, $\sqrt{b_1}$, b_2 have played a key role in dealing with the problem of model specification. In the earlier decades of the twentieth century a family of distributions selected using $\sqrt{b_1}$ and b_2 from the Pearsonian system was routinely taken as the basic statistical model; see Elderton and Johnson (1969). The coefficients are used to understand the nature of probability distributions and for their comparisons with each other. This is accomplished by using numerical values of the coefficients, or by considering their placement in the well known (β_1, β_2)-chart, see Ord (1972). This could also be accomplished with related graphs for the Pearsonian system of distributions which also include non-Pearsonian distributions.

In this section we examine (δ_1, δ_2) for a spectrum of well-known families of distributions, the notation of density functions follow Johnson *et al.* (1994). The coefficients are tabulated in Table 1. The values of δ_1 and δ_2 are intricate for some distributions

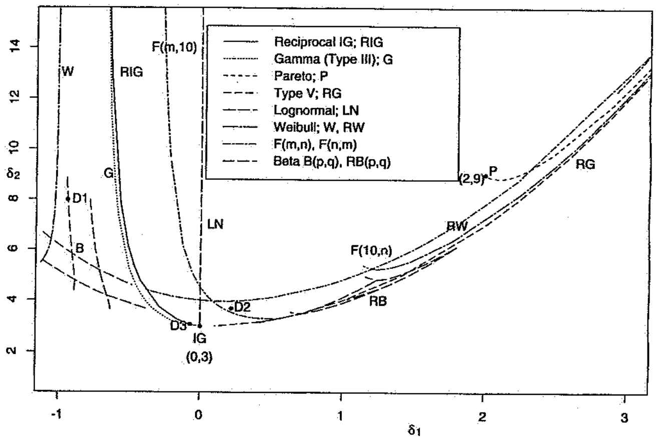

Fig. 1. (δ_1, δ_2)-chart: IG analog of (β_1, β_2)-chart.

and hence not displayed. However, the graphical display of the (δ_1, δ_2)-chart follows as Fig. 1. The coefficients δ_1 and δ_2 are scale invariant. For each of the following distributions such as inverse Gaussian, gamma, beta, Weibull, beta and F, we have considered the reciprocal distributions. For example, $Y = 1/X$, where X follows gamma is termed reciprocal gamma distribution. The labels that appear in Table 1 or Fig. 1 correspond to the following: IG–inverse Gaussian, RIG–reciprocal inverse Gaussian, LN–lognormal, G–gamma, RG–reciprocal gamma, W–Weibull, RW–reciprocal Weibull, P–Pareto, F–F distribution, B–beta, and RB–reciprocal beta.

Remark 5.1. The Weibull line appears to have negative values of δ_1 and the values of δ_1 and δ_2 being intricate, are not displayed in Table 1.

Remark 5.2. For the reciprocal Weibull family of random variables, δ_1, δ_2 for the family exist if $c > 2$. The family is illustrated with label RW in Fig. 1 and appears with positive values of δ_1.

Remark 5.3. Obviously, the coefficients of IG-skewness and IG-kurtosis can be calculated for reciprocal Pareto random variable. However, their values fall out of range in Fig. 1 and thus do not appear in Fig. 1. The calculated values of δ_1 are negative and of δ_2 exceed 3.

Remark 5.4. The F family with two shape parameters lies in the region between Type III and Type V lines in the (δ_1, δ_2)-chart. For fixed ν_1, as $\nu_2 \to \infty$, $F_{\nu_1,\nu_2} \to^d \chi^2_{\nu_1}/\nu_1$ and for fixed ν_2, as $\nu_1 \to \infty$, $F_{\nu_1,\nu_2} \to^d \{\chi^2_{\nu_2}/\nu_2\}^{-1}$ or a reciprocal Gamma variable. These results are illustrated in the (δ_1, δ_2)-chart by using $F_{\nu_1,\nu_2}, \nu_1 = 10, \nu_2 = 4.1(0.01)4.12, 4.2(0.1)4.9, 5(1)100$ and $F_{\nu_1,\nu 2}, \nu_1 = 4.1(0.1)4.9, 5(1)100\, \nu_2 = 10$. Note that the reciprocal of the $F(\nu_1, \nu_2)$ r.v. is the $F(\nu_2, \nu_1)$ random variable.

Remark 5.5. For the reciprocal beta family of random variables, δ_1, δ_2 for the family exist if $p > 2$, $q > 0$. The family is illustrated with label RB in Fig. 1.

Remark 5.6. The discussion of (δ_1, δ_2) in this section can obviously be expanded by considering other families of distributions.

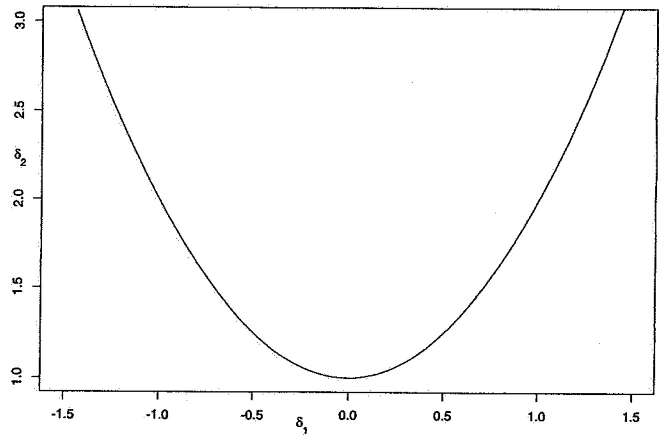

Fig. 2. (δ_1, δ_2)-chart: Region with $\delta_2 < 3$.

5.1 *IG-Kurtosis* $\delta_2 < 3$

From Fig. 1 it is clear that for the families of distributions considered above, the values of the IG-kurtosis exceed or equal 3 i.e., $\delta_2 \geq 3$. From Remark 3.1 we know that $\delta_2 \geq 1$. The case $\delta_2 = 1$ is considered in the fifth remark of Section 7 titled IG-kurtosis $\delta_2 = 1$. To illustrate the existence of distributions with $\delta_2 < 3$, and the case of discrete distributions, we consider the families of two point distributions, all with mean equal to one.

Let the r.v. X take values a and b with probabilities p and $q = 1 - p$ respectively. Then requiring $E(X) = pa + qb = 1$ gives $a = (1 - qb)/p$. The coefficients δ_1 and δ_2 can be computed provided $b \notin \{0, 1/q, 1\}$. Figure 2 shows the representation of these distributions for $b = 1.5$ as the parameter p varies. The figure shows the representation with $\delta_2 < 3$ even though the values of IG-kurtosis of such distributions can exceed 3. Note that the family contains distributions with a broad range of δ_1 and δ_2. Interestingly, graphs of these distributions for different values of b turn out to be remarkably coincident and indistinguishable from each other. The Fig. 2 also shows the fact that for each b there exists a p such that $\delta_2 = 1$.

6. Estimates of δ_1 and δ_2 and null distributions

In this section we consider the sample versions of the IG-analogues δ_1 and δ_2 of the skewness and kurtosis coefficients, respectively, and obtain their large sample distributions under the null hypothesis of IG assumption.

IG-Sample-Skewness d_1. Given a random sample of size n, the obvious sample version of δ_1 is

$$d_1 = \frac{m_2'/\bar{X}^2 - \bar{X}\bar{Y}}{(\bar{X}\bar{Y} - 1)\sqrt{m_2'/\bar{X}^{2-1}}}, \tag{6.1}$$

where $\bar{Y} = \sum_{i=1}^{n}\{1/X_i\}/n$ and $m_2' = \sum_{i=1}^{n} {X_i}^2/n$. In view of the well known consistency of the sample moments, d_1 is a consistent estimator of δ_1. The asymptotic null distribution of d_1 is given by the following:

THEOREM 6.1. *For a random sample of size n from an $IG(\mu, \lambda)$ population,*

$$\sqrt{n} d_1 \xrightarrow{d} N(0, 6). \tag{6.2}$$

PROOF. It is easy to establish using the multivariate version of the central limit theorem (e.g., Cramér (1946)) that for the $IG(\mu, \lambda)$ population, as $n \to \infty$, the vector $(\bar{X}, \bar{Y}, m_2')'$ is asymptotically normally distributed with mean $(\mu, \nu, \mu_2')'$, and covariance matrix $(1/n)\sum = (\sigma_{ij})/n$, where $\sigma_{11} = \text{Var}(X) = \mu^3/\lambda, \sigma_{12} = \text{Cov}(X, Y) = -\mu/\lambda, \sigma_{13} = \text{Cov}(X, X^2) = 3\mu^5/\lambda^2 + 2\mu^4/\lambda, \sigma_{22} = \text{Var}(Y) = 1/(\mu\lambda) + 2/\lambda^2, \sigma_{23} = \text{Cov}(Y, X^2) = -2\mu^2/\lambda - \mu^3/\lambda^2$, and $\sigma_{33} = \text{Var}(X^2) = 4\mu^5/\lambda + 14\mu^6/\lambda^2 + 15\mu^7/\lambda^3$. Hence, by use of the multivariate version of the Mann-Wald theorem (Serfling (1980)), it follows that, as $n \to \infty$, the numerator of d_1 satisfies

$$\sqrt{n}[(m_2'/\bar{X}^2 - \bar{X}\bar{Y}) - (\mu_2'/\mu^2 - \mu\nu)] \xrightarrow{d} N\left(0, 6\frac{\mu^3}{\lambda^3}\right). \tag{6.3}$$

Also, the denominator of d_1 converges in probability to $(\mu/\lambda)^{3/2}$. Hence an appeal to Slutsky's lemma establishes (6.2).

Remark 6.1. Note the analogy implicit in equation (6.2). The asymptotic null distribution of $\sqrt{b_1}$ for a sample from a normal population, see Kendall and Stuart (1969), is exactly the same as that of d_1 for an IG sample.

IG-Sample-Kurtosis d_2. The simplest sample version, d_2 of δ_2 is obtained by estimating the population moments in (3.4) by their sample versions and with some algebraic manipulation it reduces to

$$d_2 = \frac{(m_2'/\bar{X}^2 + m'_{-2}\bar{X}^2 - 3\bar{X}^2\bar{Y}^2 + 2\bar{X}\bar{Y} - 1)}{(\bar{X}\bar{Y} - 1)^2} + 3. \tag{6.4}$$

Alternatively, the sample version of $\delta_2 = [\text{Var}(V)/E^2(V)] + 1$ may be constructed by replacing $E(V)$ and $\text{Var}(V)$ by their jackknife estimates, see Efron (1982). Define the pseudovalues of V by

$$P_{n,i} = nV_n - (n-1)V_{n,i} = \sum_{i=1}^{n} (1/X_i - 1/\bar{X}) - \sum_{j=1, j\neq i}^{n} (1/X_j - 1/\bar{X}_{-i}). \tag{6.5}$$

Then, the alternative sample version of δ_2, defined in terms of empirical quantities, is

$$d_2^* = \frac{S^2(P_{n,i})}{[\bar{P}_n]^2} + 1, \tag{6.6}$$

where $\bar{P}_n$ is the jackknife estimator of V, and $S^2(P_{n,i})$ denotes the variance of the pseudovalues. The version d_2^* given in (6.6) may be more intuitive and can be shown to be asymptotically equivalent to d_2, i.e. $d_2^* = d_2 + O_p(n^{-1})$.

THEOREM 6.2. *If the population is $IG(\mu, \lambda)$, then*

$$\sqrt{n}(d_2 - 3) \xrightarrow{d} N(0, 24). \tag{6.7}$$

PROOF. For the $IG(\mu,\lambda)$ population one can show that, as $n\to\infty$, the asymptotic distribution of the vector $(\bar{X},\bar{Y},m'_2,m'_{-2})'$ is normal with mean $(\mu,\nu,\mu'_2,\mu'_{-2})'$ and covariance matrix $(1/n)\sum = (\sigma_{ij}/n)$ where $\sigma_{11} = \mu^3/\lambda, \sigma_{12} = -\mu/\lambda, \sigma_{13} = 3\mu^5/\lambda^2+2\mu^4/\lambda, \sigma_{14} = -2/\lambda-3\mu/\lambda^2, \sigma_{22} = 1/(\mu\lambda)+2/\lambda^2, \sigma_{23} = -2\mu^2/\lambda-\mu^3/\lambda^2, \sigma_{24} = 12/\lambda^3+9/(\mu\lambda^2)+2/(\mu^2\lambda), \sigma_{33} = 4\mu^5/\lambda+14\mu^6/\lambda^2+15\mu^7/\lambda^3, \sigma_{34} = -4\mu/\lambda-6\mu^2/\lambda^2-3\mu^3/\lambda^3, \sigma_{44} = 4/(\mu^3\lambda)+30/(\mu^2\lambda^2)+87/(\mu\lambda^3)+96/\lambda^4$. Then, using the multivariate version of the Mann-Wald theorem it can be shown that

$$\sqrt{n}(m'_{-2}\bar{X}^2+m'_2/\bar{X}^2-3\bar{X}^2\bar{Y}^2+2\bar{X}\bar{Y}-1)\xrightarrow{d} N\left(0,\frac{24\mu^4}{\lambda^4}\right), \tag{6.8}$$

as $n\to\infty$. The denominator of d_2 converges in probability to μ^2/λ^2. The validity of the null distribution of d_2 given by equation (6.7) can then be confirmed by an appeal to Slutsky's lemma.

Remark 6.2. The asymptotic null distribution of d_2 from an IG sample is exactly the same as the asymptotic sampling distribution of b_2 under the assumption of normality, see Kendall and Stuart (1969).

THEOREM 6.3. *If the population is $IG(\mu,\lambda)$, then as $n\to\infty$, d_1 as defined in* (6.1), *and d_2 as defined in* (6.4), *are asymptotically independent.*

PROOF. We have

$$Numerator(d_1) = \frac{m'_2}{\bar{X}^2} - \bar{X}\bar{Y} \quad \text{and} \tag{6.9}$$

$$Numerator(d_2-3) = (m'_2/\bar{X}^2+m'_{-2}\bar{X}^2-3\bar{X}^2\bar{Y}^2+2\bar{X}\bar{Y}-1). \tag{6.10}$$

For the $IG(\mu,\lambda)$ population it was shown in the proof of Theorem 6.2 that, as $n\to\infty$, the asymptotic distribution of the vector $(\bar{X},\bar{Y},m'_2,m'_{-2})'$ is normal with mean $(\mu,\nu,\mu'_2,\mu'_{-2})'$ and the covariance matrix $(1/n)\sum = (\sigma_{ij}/n)$, where the matrix $\sum$ is given in the proof of Theorem 6.2. Then, a use of the multivariate version of the Mann-Wald theorem yields the asymptotic normality of vector $(m'_{-2}\bar{X}^2, m'_2/\bar{X}^2, \bar{X}\bar{Y}, (\bar{X}\bar{Y})^2)'$. Hence it can be shown that the numerators of d_1 and (d_2-3), which are linear combinations of the components of this vector, are asymptotically independent normal variables. Consequently, by Slutsky's proposition, see Chapter 20 of Cramér (1946), the asymptotic independence of d_1 and d_2 is established.

6.1 *Applications*

In this section we illustrate the use of sample IG-skewness and IG-kurtosis measures in conjunction with the (δ_1,δ_2)-chart for the purpose of parametric model selection. The following three illustrative data sets appear as points labeled D1, D2, and D3 in the (δ_1,δ_2)-chart in Fig. 1. The coefficients $(\sqrt{b_1},b_2)$ and (d_1,d_2) for these data appear in Table 2.

D1. *Rainfall Data.* These data used by Mooley (1973) and appearing in Table 2 give the July rainfall (in millimeters) at Kyoto, Japan over a period of 80 years (1880–1960). Conventionally, such data were analyzed by using a normal fit. Mooley argued that for such data a gamma model would be more appropriate. The IG-skewness and IG-kurtosis for the data are -0.94 and 7.98, respectively. In Fig. 1 the data point D1

Table 2. Comparison of $(\sqrt{b_1}, b_2)$ and (d_1, d_2) for the data sets.

Data Set	$\sqrt{b_1}$	b_2	d_1	d_2
D1	1.00	4.60	−0.94	7.98
D2	4.56	25.44	0.22	3.69
D3	1.96	7.44	−0.07	3.08

Table 3. Rainfall (mm) at Kyoto, Japan for the month of July from 1880–1960.

Rainfall	Observed	Weibull fit	Gamma fit
0–50	5	4.83	3.78
50–100	9	11.05	12.27
100–150	12	13.86	15.45
150–200	18	13.84	14.24
200–250	17	11.93	11.27
250–300	6	9.16	8.14
300–350	5	6.38	5.53
350–400	4	4.07	3.60
400–above	4	4.83	5.72

Source: *World Weather Records*

Smithsonian Institution, Miscellaneous Collections and U.S. Department of Commerce.

Table 4. Consecutive annual flood discharge rates of the Floyd river at James, Iowa.

Years	Flood discharge (ft^3/s)
1935–1944	1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440,
1945–1954	5320, 1400, 3240, 2710, 4520, 4840, 8320, 13900, 71500, 6250,
1955–964	2260, 318, 1330, 970, 1920, 15100, 2870, 20600, 3810, 726,
1965–1973	7500, 7170, 2000, 829, 17300, 4740, 13400, 2940, 5660.

Source: *United States Water Resources Council* (1977).

(−0.94, 7.98) is in the beta region and close to the Weibull line. Since, use of beta for modelling such data is not conventional Weibull is the preferred model. Table 3 also gives expected frequencies corresponding to the two models. The Pearson's chi-square corresponding to the gamma and Weibull models are $\chi^2(gamma) = 7.12$ and $\chi^2(Weibull) = 5.57$, both with six degrees of freedom and p-values of 0.31 and 0.47, respectively. This suggests the superiority of the Weibull model over the gamma model.

D2. *Flood data.* These data, given in Table 4, were used by Mudholkar and Hutson (1996) for illustrating the use of the exponentiated Weibull family for analyzing extremes. The extreme value or the generalized extreme value distributions are conventionally used to model such data. Mudholkar and Hutson (1996) discuss using a member of the exponentiated Weibull family, which is very similar to an inverse Gaussian distribution, for the purpose. Pericchi and Rodriguez-Iturbe (1985) discuss the use of IG for analysis of flood data. The traditional (β_1, β_2)-chart is not usable for model selection in this case

Table 5. Runoff amounts at Jug Bridge, Maryland.

0.17	0.19	0.23	0.33	0.39	0.39	0.40	0.45	0.52	0.56
0.59	0.64	0.66	0.70	0.76	0.77	0.78	0.95	0.97	1.02
1.12	1.24	1.59	1.74	2.92					

Source: *Ang and Tang* (1975), *Probability Concepts in Engineering Planning and Design.*

Table 6. Some well known analogies between the Gaussian and the inverse Gaussian distributions.

Item	Gaussian framework	Inverse Gaussian framework
0.	$X_1, X_2, \ldots, X_n i.i.d N(\mu, \sigma^2)$	$X_1, X_2, \ldots, X_n i.i.d IG(\mu, \lambda)$
1.	If X_i ind. $\sim N(\mu_i, \sigma_i^2)$ then $\sum X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$	If X_i ind. $\sim IG(\mu_i, \lambda_i)$ then $\sum X_i \sim IG(\sum \mu_i, \xi(\sum \mu_i)^2)$; if $\xi = \lambda_i/\mu_i^2\ \forall i$
2.	$\mu = E(X)$; m.l.e. $\hat{\mu} = \bar{X}$	$\mu = E(X)$; m.l.e. $\hat{\mu} = \bar{X}$
3.	m.l.e. $\hat{\sigma}^2 = S^2 = n^{-1}\sum_i (X_i - \bar{X})^2$	m.l.e. $\hat{\lambda}^{-1} = V = n^{-1}\sum_i (X_i^{-1} - \bar{X}^{-1})$
4.	$\bar{X} \sim N(\mu, \sigma^2/n)$, $nS^2/\sigma^2 \sim \chi^2_{n-1}$	$\bar{X} \sim IG(\mu, n\lambda)$, $n\lambda V \sim \chi^2_{n-1}$
5.	$\bar{X}$ and S^2 are independent	$\bar{X}$ and V are independent
6.	$(\bar{X}, S^2)$ complete, sufficient for (μ, σ^2)	$(\bar{X}, V)$ complete, sufficient for (μ, λ)
7.	$(X-\mu)^2/\sigma^2 \sim \chi_1^2$	$\lambda(X-\mu)^2/(\mu^2 X) \sim \chi_1^2$
8.	For $H_0 : \mu = \mu_0$, UMPU t-test	For $H_0 : \mu = \mu_0$, UMPU t-test
9.	$\sum_i (X_i - \mu)^2 = \sum_i (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$	$\sum_i (X_i - \mu)^2/(\mu^2 X_i) = \sum_i (X_i^{-1} - \bar{X}^{-1}) + n(\bar{X} - \mu)^2/(\mu^2 \bar{X})$
10.	For Homogeneity of k means - ANOVA F-test	For Homogeneity of k means - ANORE F-test
11.	$\bar{X}$ and S^2 are independent iff Gaussian	$\bar{X}$ and V are independent iff IG
12.	Saddlepoint approximation for p.d.f. of $\bar{X}$ is exact upto rescaling	Saddlepoint approximation for p.d.f. of $\bar{X}$ is exact upto rescaling
13.	In Bayesian context: Conjugate families for μ, σ^{-2}, and (μ, σ^{-2}) jointly, are normal, gamma, bivariate normal-gamma, respectively	In Bayesian context: Conjugate families for μ^{-1}, λ, and (μ^{-1}, λ) jointly, are truncated normal, gamma, bivariate normal-gamma, respectively

because the values of $\sqrt{b_1}$ and b_2 for these data are well beyond the range covered by the (β_1, β_2)-chart. In Fig. 1, the point D2 is in the variance-ratio F region and also close to the IG point. Since F modelling for such data would be unconventional, it is reasonable to use the inverse Gaussian distribution in this context, in agreement with the proposal of Pericchi and Rodriguez-Iturbe (1985).

D3. *Runoff amounts.* Folks and Chhikara (1978) use data on runoff amounts at Jug Bridge, Maryland from Ang and Tang (1975) to exhibit the inverse Gaussian distribution as a contender to the lognormal model suggested by Ang and Tang. They use the Kolmogorov-Smirnov test statistic and Q-Q plot to illustrate the fit. The data are given in Table 5 with the (d_1, d_2) point very close to the IG (0,3) point in Fig. 1. It is also close to the lognormal line, providing support for the model used by Ang and Tang (1975).

7. Conclusions and miscellaneous remarks

Inverse Gaussian models are used in various areas of applied research but lack measures of fit. We have used known similarities between the Gaussian and inverse Gaussian families to construct the IG-analogues δ_1 and δ_2 of the coefficients $\sqrt{\beta_1}$ and β_2, which are commonly used as simple measures of appropriateness for Gaussian models. Some

Table 7. Recent additions to the analogies in Table 6.

Item	Gaussian framework	Inverse Gaussian framework
14.	g-of-f test based on item 11, statistic: $Z(G) = \tanh^{-1}(r(G))$	g-of-f test based on item 11, statistic: $Z(IG) = \tanh^{-1}(r(IG))$
15.	$r(G) = \text{Corr}(X_i, U_i),\ i = 1, 2, \ldots, n$ $U_i = \{\sum_{j\neq i} X_j^2 - (\sum_{j\neq i} X_j)^2/(n-1)\}^{1/3}$	$r(IG) = \text{Corr}(X_i, V_{-i}),\ i = 1, 2, \ldots, n$ $V_{-i} = \frac{1}{n-1}\sum_{j=1,\ j\neq i}^{n}(X_j^{-1} - \bar{X}_{-i}^{-1})$
16.	$E(r(G)) = -\sqrt{\beta_1}/\sqrt{(\beta_2 - 1)}$	$E(r(IG)) = \delta_1/\sqrt{(\delta_2 - 1)}$
17.	Asymptotic Null Distribution: $\sqrt{n}r(G) \to N(0,\ 3)$	Asymptotic Null Distribution: $\sqrt{n}r(IG) \to N(0,\ 3)$
18.	Asymptotic Null Distribution: $\sqrt{n}Z(G) \to N(0,\ 3)$	Asymptotic Null Distribution: $\sqrt{n}Z(IG) \to N(0,\ 3)$
19.	Symmetry about $\mu = 0$: All Odd order moments = 0	IG-symmetry about μ: For $r = 1, 2, \ldots$ $E[(X/\mu)^{r+1}] = E[(X/\mu)^{-r}]$
20.	Z-test suited for skew alternatives	Z-test suited for IG-skew alternatives
21.	Contaminated Gaussian distributions	Contaminated IG distributions
22.	Scale mixtures of normals are symmetric about μ	Scale mixtures of IG are IG-symmetric about μ
23.	Coefficient of skewness $\sqrt{\beta_1}$	Coefficient of IG-skewness δ_1
24.	Coefficient of kurtosis β_2	Coefficient of IG-kurtosis δ_2
25.	$\beta_2 \geq 1$, with equality for symmetric two point distributions	$\delta_2 \geq 1$, with equality for IG-symmetric two point distributions
26.	Pearson's $(\beta_1,\ \beta_2)$-Chart	$(\delta_1,\ \delta_2)$-Chart
27.	Sample versions $(\sqrt{b_1},\ b_2)$	Sample versions $(d_1,\ d_2)$
28.	Asymptotic Null Distribution: $\sqrt{nb_1} \to N(0,\ 6)$	Asymptotic Null Distribution: $\sqrt{n}d_1 \to N(0,\ 6)$
29.	Asymptotic Null Distribution: $\sqrt{n}(b_2 - 3) \to N(0,\ 24)$	Asymptotic Null Distribution: $\sqrt{n}(d_2 - 3) \to N(0,\ 24)$
30.	$\sqrt{b_1}$ and b_2 asymptotically independent under normality.	d_1 and d_2 asymptotically independent under IG assumption.

general observations follow.

1. *Expanded analogy.* Table 6 gives a list of properties in terms of which the inverse Gaussian family is well-known to resemble the Gaussian family. Table 7 contains the list augmented as a result of our recent effort.

2. *Interpretation.* Since their inception, much work has gone into understanding the meaning of $\sqrt{\beta_1}$ and β_2. The coefficients δ_1 and δ_2 developed here share some similarities with $\sqrt{\beta_1}$ and β_2. Obviously considerable work is needed before their meaning and usefulness can be adequately appreciated. Alternative derivations of δ_1 and δ_2 may be helpful in this respect.

3. *Goodness-of-fit tests.* $\sqrt{b_1}$ and b_2 have been employed individually and jointly to test the composite goodness-of-fit hypothesis of normality; e.g., see Bowman and Shenton (1975) or D'Agostino and Stephens (1986). Similar applications of d_1 and d_2 are under development.

4. *IG-skew distributions with* $\delta_1 = 0$. Families of asymmetric distributions with $\sqrt{\beta_1} = 0$ are shown in Fig. 2 in Freimer *et al.* (1988), see also MacGillivray (1986). We have seen that both the IG scale mixtures and lognormal distributions are IG-symmetric. The distributions that have $\delta_1 = 0$ but are IG-skewed may be interesting.

5. *IG-Kurtosis* $\delta_2 = 1$. It was noted in Remark 3.1 that analogous to the property $\beta_2 \geq 1$, we have $\delta_2 \geq 1$. It is easy to verify that for any two point distribution, with equal probability mass at each point, $\beta_2 = 1$. The coefficient of IG-kurtosis exhibits a remarkably similar behavior. For any two point IG-symmetric probability distribution, $\delta_2 = 1$. To see this, consider a r.v. X with probability masses p and $(1-p)$ at a and b, respectively, and $E(X) = 1$. Then, it can be shown that the distribution is IG-symmetric if $ab = 1$ and $p = b/(1+b)$. For this family of r.v.'s X, which is IG-symmetric about 1, the coefficient of IG-kurtosis $\delta_2 = 1$. Similar families of distributions that are IG-symmetric about arbitrary μ are easy to construct.

6. *Scale-Mixtures.* Efron and Olshen (1978) had raised the question "How broad is the class of normal scale mixtures ?" The family is commonly used in robustness studies of normal theory methods. The parallel question in the IG context can be of interest since the scale mixtures of the IG family could play a similar role in studying robustness of IG-theory methods.

7. *IG-Goodness-of-fit.* Box (1953) in his pioneering neo-robustness study argues that Bartlett's (1937) test for homogeneity of variances applied to k samples obtained by splitting a random sample into k subsamples is as sensitive for testing normality as the goodness-of-fit test based on b_2. It is possible that the likelihood ratio test for the homogeneity of λ's of the k similar subsamples of a random sample can serve as a goodness-of-fit test of the IG hypothesis.

Acknowledgement

The authors are thankful to Professor M. P. McDermott and the referees for their detailed constructive comments on an earlier draft of this manuscript.

References

Ang, A. H-S. and Tang, W. H. (1975). *Probability Concepts in Engineering Planning and Design*, Vol. I, Wiley, New York.

Balanda, K. P. and MacGillivray, H. L. (1988). Kurtosis: A critical review, *Amer. Statist.*, **42**, 111–119.

Bartlett, M. S. (1937). Properties of sufficiency and statistical tests, *Proceedings in Royal Statistical Society, A*, **160**, 268–282.

Bickel, P. J. and Lehmann, E. L. (1975). Descriptive statistics for nonparametric models. II: Location, *Ann. Statist.*, **3**, 1045–1069.

Bowman, K. O. and Shenton, L. R. (1975). Omnibus test contours for departures from normality based on $\sqrt{b_1}$ and b_2, *Biometrika*, **62**, 243–250.

Box, G. E. P. (1953). Non normality and tests on variances, *Biometrika*, **40**, 318–335.

Chhikara, R. S. and Folks, J. L. (1989). *The Inverse Gaussian Distribution*, Marcel Dekker, New York.

Cramér, H. (1946). *Mathematical Methods of Statistics*, Princeton University Press, Princeton, New Jersey.

D'Agostino, R. B. and Stephens, M. A. (1986). *Goodness-of-fit Techniques*, Marcel Dekker, New York.

Dugué, D. (1941). Sur un nouveau type de courbe de fréquence, *Comptes Rendes de l'Academie des Sciences*, **213**, 634–635.

Efron, B. (1982). The jackknife, the bootstrap and other resampling plans, *CBMS Regional Conf. Ser. in Appl. Math.*, **38**.

Efron, B. and Olshen, R. A. (1978). How broad is the class of normal scale mixtures?, *Ann. Statist.*, **6**, 1159–1164.

Elderton, W. P. and Johnson, N. L. (1969). *Systems of Frequency Curves*, Cambridge University Press, London-New York.

Folks, J. L. and Chhikara, R. S. (1978). The Inverse Gaussian distribution and its Statistical Application-A Review, *J. Roy. Statist. Soc. Ser. B*, **40**, 263–289.

Freimer, M., Mudholkar, G. S., Kollia, G. and Lin, C. T. (1988). A study of the generalized Tukey lambda family, *Comm. Statist. Theory Methods*, **17** (10), 3547–3567.

Heyde, C. C. (1963). On a property of the lognormal distribution, *J. Roy. Statist. Soc., Ser. B*, **25**, 392–393.

Huberman, B. A., Pirolli, P. L. T., Pitkow, J. E. and Lukose R. M. (1998). Strong regularities in World Wide Web Surfing, *Science*, **280**, 95–97.

Iyengar, S. and Patwardhan, G. (1988). Recent developments in the inverse Gaussian distribution, *Handbook of Statist.* (eds. P. R. Krishnaiah and C. R. Rao), **7**, 479–480.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, 2nd ed., Wiley, New York.

Kagan, A. M., Linnik, Y. and Rao, C. R. (1973). *Characterization Problems in Mathematical Statistics*, Wiley, New York.

Keilson, J. and Steutel, F. W. (1974). Mixtures of distributions: Moment inequality measures of exponentiality and normality, *Ann. Probab.*, **2**, 112–130.

Kendall, M. G. and Stuart, A. (1969). *Kendall's Advanced Theory of Statistics*, Vol. I, Carles Griffin, London.

Khatri, C. G. (1962). A characterization of the inverse Gaussian distribution, *Ann. Math. Statist.*, **33**, 800–803.

Lin, C. C. and Mudholkar, G. S. (1980). A simple test for normality against asymmetric alternatives, *Biometrika*, **67**, 455–461.

Lukacs, E. and Laha, R. G. (1964). *Applications of Characteristic Functions*, Hafner, New York.

MacGillivray, H. L. (1986). Skewness and asymmetry: Measures and ordering, *Ann. Statist.*, **14**, 994–1011.

Mooley, D. A. (1973). Gamma distribution probability model for Asian summer monsoon monthly rainfall, *Monthly Weather Review*, **101**, 160–176.

Mudholkar, G. S. and Hutson, A. (1996). The exponentiated Weibull family: Some properties and a flood data application, *Comm. Statist.-Theory Methods*, **25** (12), 3059–3083.

Mudholkar, G. S. and Hutson, A. (1998). LQ-moments: Analogs of L-moments, *J. Statist. Plann. Inference*, **71**, 191–208.

Mudholkar, A. Mudholkar, G. S. and Srivastava, D. K. (1991). A construction and appraisal of pooled trimmed-*t* statistics, *Comm. Statist. Ser. A*, **20**, 1345–1359.

Mudholkar, G. S., Marchetti, C. , and Lin, C. T. (2001*a*). Independence characterization and testing normality against restricted skewness and kurtosis alternatives, *J. Statist. Plann. Inference* (to appear).

Mudholkar, G. S., Natarajan, R. and Chaubey, Y. P. (2001*b*). Independence characterization and inverse Gaussian goodness-of-fit composite hypothesis, *Sankhyā, Ser. B* (to appear).

Ord, J. K. (1972). *Families of Frequency Distributions*, Hafner, New York.

Pearson, K. R. (1905). Skew variation, a rejoinder, *Biometrika*, **4**, 169–212.

Perricchi, L. R. and Rodriguez-Iturbe, I. (1985). On the statistical analysis of floods, *The ISI Centenary volume* (eds. A. C. Atkinson and S. E. Feinberg), 511–541, Springer, New York.

Schrödinger, E. (1915). Zur theorie der fall-und steigversuche an teilchenn mit Brownsche bewegung, *Physikalische Zeitschrift*, **16**, 289–295.

Serfling, R. J. (1980). *Approximation Thorems of Mathematical Statistics*, Wiley, New York.

Seshadri, V. (1993). The Inverse Gaussian distribution: A case study in exponential families, Clarendon Press, Oxford.

Seshadri, V. (1997). *Halphen's Law in Encyclopedia of Statistical Science*, Update Vol. 1, 302–306, Wiley, New York.

Seshadri, V. (1999). *The Inverse Gaussian distribution-Statistical Theory and Applications*, Lecture Notes in Statist., No. 137, Springer, New York.

Shohat, J. A. and Tamarkin, J. D. (1943). *The Problem of Moments*, American Mathematical Society, New York.

Smithsonian Institution, Miscellaneous Collections (1927). *World Weather Records*, **79**, Washington, D. C., p. 323.

Smithsonian Institution, Miscellaneous Collections (1934). *World Weather Records, 1921–1930*, **90**, Washington, D. C., p. 146.

Smithsonian Institution, Miscellaneous Collections (1947). *World Weather Records, 1931–1940*, **105**, Washington, D. C., p. 145.

Smoluchowsky, M. V. (1915). Notizüber die Berechning der Brownshen Molkular-bewegung bei des Ehrenhaft-millikanchen Versuchsanordnung, *Physikalische Zeitschrift*, **16**, 318–321.

Stieltjes, T. J. (1894). Recherches sur les fractions continues, *Annales de la Faculté des Sciences de Toulouse*, **8**, 1–122 .

Tweedie, M. C. K. (1945). Inverse statistical variates, *Nature*, **155**, p. 453.

U. S. Department of Commerce, Weather Bureau (1959). *World Weather Records, 1941–1950*, Washington D. C., p. 398.

U. S. Department of Commerce, Weather Bureau (1967). *World Weather Records, 1951–1960*, **4**, Asia, Washington D. C., p. 345.

U. S. Water Resources Council, Washington, Hydrology Comittee (1977). *Guidelines for determining flood flow frequency*, United States. Water Resources Council, Washington D. C.

Wald, A. (1947). *Sequential Analysis*, Wiley, New York.