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A NOTION OF α-MONOTONICITY WITH GENERALIZED MULTIPLICATIONS

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Abstract. The multiplications of van Harn *et al.* (1982, *Z. Wahrsch. Verw. Gebiete*, **61**, 97–118) are used to generalize the definition of α-monotonicity of Olshen and Savage (1970, *J. Appl. Probab.*, **7**, 21–34) and Steutel (1988, *Statist. Neerlandica*, **42**, 137–140) for distributions with support in $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$. Several characterizations are offered and a convolution property is established. Some relevant stability equations are solved and a relationship with the important concept of self-decomposability is noted. Poisson mixtures are used to deduce results for the $\boldsymbol{R}_+$-case from those for the $\boldsymbol{Z}_+$-case.

Key words and phrases: Semigroup, monotonicity, self-decomposability, Poisson mixtures.

1. Introduction

Olshen and Savage (1970) define a real-valued random variable (rv) X to have an α-unimodal distribution about 0 for some $\alpha > 0$ if X has the following representation:

$$X \stackrel{d}{=} W^{1/\alpha} Y, \tag{1.1}$$

where W and Y are independent rv's and W is uniform (0,1). The case $\alpha = 1$ corresponds to the classical concepts of unimodality. If Y in (1.1) is $\boldsymbol{R}_+$-valued ($\boldsymbol{R}_+ = [0, \infty)$), then the distribution of X is said to be α-monotone. Among other characterizations, a distribution on the real line is α-unimodal about 0 if and only if its restrictions to the positive half-line and to the negative half-line are α-monotone (for example, see Hansen (1990), p. 47).

Discrete α-unimodality (also about 0) for distributions on the integers was introduced by Abouammoh (1987) (see also Alamatsaz (1993) and Wu and Dharmadhikari (1999)). Using the binomial thinning operator $\circ$ of Steutel and van Harn (1979), Steutel (1988) defined α-monotonicity of a $\boldsymbol{Z}_+$-valued ($\boldsymbol{Z}_+ = \{0, 1, 2, \dots\}$) rv X similarly to (1.1):

$$X \stackrel{d}{=} W^{1/\alpha} \circ Y, \tag{1.2}$$

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where W and Y are as in (1.1) with Y $\boldsymbol{Z}_+$-valued. Steutel (1988) showed that if $(p_n, n \geq 0)$ is the probability distribution function (pdf) of X then (1.2) is equivalent to

$$(1.3) \qquad (n+1)p_{n+1} \leq (n+\alpha)p_n, \qquad \text{for any} \quad n \geq 0.$$

As in the continuous case, discrete α-unimodality about 0 is equivalent to α-monotonicity at both sides of 0.

The purpose of this paper is to present a generalized notion of α-monotonicity for distributions on $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$. We arrive at the right definitions by using the generalized multiplications $\odot_F$ and $\odot_C$ in lieu of the standard multiplication in (1.1). These operations were introduced by van Harn *et al.* (1982) and van Harn and Steutel (1993) who used them to extend the classical concepts of self-decomposability and stability to distributions on $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$. They were also used by Bouzar (1999) to define a concept of geometric stability. We obtain various characterizations of the $\odot_F$ and $\odot_C$-monotonicity, thus generalizing the work of several authors. A convolution property is established and some relevant stability equations are solved. We explore the relationship between self-decomposability and monotonicity and offer some useful characterizations. Poisson mixtures are used to deduce results for the $\boldsymbol{R}_+$-case from those for the $\boldsymbol{Z}_+$-case. Several examples are mentioned.

The paper is organized as follows. In Section 2 the notion of (discrete) $[F;\alpha]$-monotonicity is introduced and its properties are examined. The $\boldsymbol{R}_+$-valued case is studied in Section 3 through Poisson mixtures.

In the rest of this section we briefly recall some definitions and results that are needed in the sequel. For proofs and further details we refer to van Harn *et al.* (1982) and van Harn and Steutel (1993).

The following notation will be used. P_X is the probability generating function (pgf) of the $\boldsymbol{Z}_+$-valued rv X, ϕ_Y is the Laplace-Stieltjes transform (LST) of the $\boldsymbol{R}_+$-valued rv Y. $F = (F_t; t \geq 0)$ is a continuous composition semigroup of pgf's F_t such that $F_t \not\equiv 1$ and $\delta_F = -\ln F_1'(1) \geq 0$. $C = (C_t; t \geq 0)$ is a continuous composition semigroup of cumulant generating functions (cgf's), $C_t = -\ln L_t$ where L_t is the LST of an infinitely divisible rv such that $\delta_C = -\ln(-L_1'(0)) \geq 0$. We denote by U_F and U_C the infinitesimal generators of the semigroups F and C, respectively. The related A-functions defined by

$$(1.4) \qquad A_F(z) = \exp\left\{-\int_0^z (U_F(x))^{-1}dx\right\}, \qquad A_C(\tau) = \exp\left\{\int_\tau^1 (U_C(x))^{-1}dx\right\},$$

$z \in [0,1)$, $\tau \geq 0$ satisfy

$$(1.5) \qquad A_F(F_t(z)) = e^{-t}A_F(z), \qquad A_C(C_t(\tau)) = e^{-t}A_C(\tau), \quad t \geq 0.$$

Moreover, there exists a constant $a > 0$ and a pgf $H(z)$ given by

$$(1.6) \qquad H(z) = \sum_{n\geq 0} h_n z^n,$$

with $h_1 = 0$, $H'(1) < \infty$, and such that

$$(1.7) \qquad U_F(z) = a\{H(z) - z\}, \qquad |z| \leq 1.$$

2. A generalized notion of discrete monotonicity

Let X be a Z_+-valued rv and $\nu \in (0,1)$. The generalized multiplication $\nu \odot_F X$ is defined by (van Harn *et al.* (1982))

$$\nu \odot_F X \stackrel{d}{=} \sum_{i=1}^{X} Y_i, \tag{2.1}$$

where $(Y_i, i \geq 1)$ is a sequence of iid rv's independent of X, with common pgf F_t, $t = -\ln \nu$. In terms of pgf's (2.1) is equivalent to

$$P_{\nu \odot_F X}(z) = P_X(F_t(z)), \qquad t = -\ln \nu. \tag{2.2}$$

DEFINITION 2.1. A Z_+-valued rv X is said to be $[F;\alpha]$-monotone (or to have an $[F;\alpha]$-monotone distribution) for some $\alpha > 0$ if

$$X \stackrel{d}{=} W^{1/\alpha} \odot_F Y, \tag{2.3}$$

where W is uniformly distributed over $(0,1)$, Y is a Z_+-valued rv, and Y and W are independent.

It follows easily from (2.2) that X is $[F;\alpha]$-monotone if and only if for any $|z| \leq 1$,

$$P_X(z) = \int_0^1 \alpha x^{\alpha-1} Q \circ F_{-\ln x}(z) dx = \alpha \int_0^\infty Q \circ F_t(z) e^{-\alpha t} dt \tag{2.4}$$

for some pgf Q.

The following proposition gives several characterizations of discrete $[F;\alpha]$-monotonicity.

PROPOSITION 2.2. *Let X be a Z_+-valued rv with pdf $(p_j, j \geq 0)$. The following assertions are equivalent.*

(i) *X is $[F;\alpha]$-monotone;*

(ii) *There exists a pgf Q such that for any $|z| \leq 1$,*

$$P_X(z) = \alpha [A_F(z)]^{-\alpha} \int_z^1 Q(v) A_F(v)^\alpha [U_F(v)]^{-1} dv. \tag{2.5}$$

(iii) *For any $n \geq 0$,*

$$\sum_{i=1}^{n+1} i p_i h_{n-i+1} \leq (n + \alpha a^{-1}) p_n, \tag{2.6}$$

where $(h_n, n \geq 0)$ and $a > 0$ are as in (1.6) and (1.7).

PROOF. (i) ⇔ (ii): Assume that (i) holds. We show that (ii) holds for $z \in [0,1)$ and the general result will follow by analytic continuation. Let $z \in (0,1)$. It can be shown by (1.4) and (1.5) and the fact that $\int_z^{F_t(z)}[U(x)]^{-1}dx = t$, that $F_t(z)$, as a function of t, admits

$$\varphi_z(v) = \int_z^v [U_F(x)]^{-1}dx, \qquad v \in [0,1]$$

as its inverse. By making the change of variable $v = F_t(z)$ in equation (2.4) we obtain

$$P_X(z) = \alpha \int_z^1 Q(v)[U_F(v)]^{-1}\exp\{-\alpha\varphi_z(v)\}dv. \tag{2.7}$$

Combining (2.7) with the equation $\exp\{-\alpha\varphi_z(v)\} = A_F(v)^\alpha[A_F(z)]^{-\alpha}$ yields (2.5). (ii) $\Rightarrow$ (i) follows from the change of variable $t = \varphi_z(v)$ in (2.5) which implies (2.4) and hence (i).

(ii) $\Leftrightarrow$ (iii): Assume (ii) holds. Differentiating $P_X(z)$ in (2.5) yields for any $|z| \le 1$,

$$P'_X(z) = -\alpha\left[\frac{A'_F(z)}{A_F(z)}P_X(z) + \frac{Q(z)}{U_F(z)}\right].$$

By (1.4), $\frac{A'_F(z)}{A_F(z)} = -\frac{1}{U_F(z)}$ which implies that

$$Q(z) = -\alpha^{-1}P'_X(z)U_F(z) + P_X(z). \tag{2.8}$$

Using (1.6) and (1.7) and equating the coefficients of z^n in (2.8) yields

$$\alpha q_n = (\alpha + na)p_n - a\sum_{i=1}^{n+1} ip_i h_{n-i+1}, \tag{2.9}$$

where $(q_n, n \ge 0)$ is the pdf of Q. This in turn implies (iii). To prove that (iii) $\Rightarrow$ (ii), it is enough to show that $(q_n, n \ge 0)$ defined by equation (2.9) is a pdf. Straightforward manipulations yield

$$\alpha\sum_{n=0}^{N-1} q_n = \alpha\sum_{n=0}^{N-1} p_n + aNb_N - a\sum_{i=1}^{N-1} iK_i p_{N-i}, \tag{2.10}$$

where $K_i = \sum_{k=i+1}^{\infty} h_k$ and $b_N = \sum_{i=1}^{N-1} K_i p_{N-i} - p_N h_0$. Since $H'(1) < \infty$, $\lim_{i\to\infty} iK_i = 0$ and $\sum_{i\ge 0} K_i < \infty$, it follows that $\sum_{N\ge 1}|b_N| < \infty$. Moreover, noting that $\lim_{N\to\infty} p_{N-i} = 0$ for each $i \ge 1$, and $\sum_{i=0}^{N} p_{N-i} \le 1$ for each $N \ge 0$, we have by Toeplitz' theorem (Knopp (1990)),

$$\lim_{N\to\infty}\sum_{i=1}^{N-1} iK_i p_{N-i} = 0. \tag{2.11}$$

Since $q_n \ge 0$, the left-hand side of equation (2.10) has a limit. By (2.11) and the fact that $\sum_{N\ge 1}|b_N|$ converges, that limit must be finite and $\lim_{N\to\infty} Nb_N = 0$. □

Next, we give a characterization of $[F;\alpha]$-monotonicity in terms of monotonicity of some special functions.

PROPOSITION 2.3. *A $\boldsymbol{Z}_+$-valued rv X is $[F;\alpha]$-monotone if and only if for every nonnegative function f defined on $\boldsymbol{Z}_+$, the function*

$$S_F(\alpha,\nu,f)=\nu^\alpha E[f(\nu\odot_F X)] \tag{2.12}$$

is nondecreasing in $\nu\in(0,1)$.

PROOF. Assuming (2.3) and noting that $u\odot_F(v\odot_F Y)\overset{d}{=}(uv)\odot_F Y$ for $u,v\in(0,1)$, we have

$$S_F(\alpha,\nu,f)=\nu^\alpha\int_0^1 E(f(\nu w\odot_F Y))\alpha w^{\alpha-1}dw=\alpha\int_0^\nu E(f(w\odot_F Y))w^{\alpha-1}dw. \tag{2.13}$$

Conversely, if $(p_n(t),n\geq 0)$ denotes the pdf of $e^{-t}\odot_F X$ for $t>0$, then $S_F(\alpha,\nu,f)$ is nondecreasing in $\nu\in(0,1)$ for any $f\geq 0$ if and only if $\nu^\alpha p_n(-\ln\nu)$ is nondecreasing for any $n\geq 0$. Let $(p_{ij}(t),j\geq 0)$ be the pdf of $\{F_t(z)\}^i$, $t>0$. Then by (2.2)

$$p_j(t)=\sum_{n\geq 0}p_{nj}(t)p_n,\qquad j\geq 0,\quad t\geq 0. \tag{2.14}$$

Since the $p_{ij}(t)$'s form the transition matrix of a continuous-time, subcritical Markov branching process determined by $H(z)$ and $a>0$ of (1.6) and (1.7) (see Athreya and Ney (1972), Chapter III), we deduce from (2.14) and the Kolmogorov forward equation that

$$\frac{\partial}{\partial t}p_n(t)=-nap_n(t)+a\sum_{i=1}^{n+1}ip_i(t)h_{n-i+1}. \tag{2.15}$$

It follows that

$$\frac{\partial}{\partial\nu}\nu^\alpha p_n(-\ln\nu)=\nu^{\alpha-1}\left[(\alpha+na)p_n(-\ln\nu)-a\sum_{i=1}^{n+1}ip_i(-\ln\nu)h_{n-i+1}\right]. \tag{2.16}$$

This implies that for any $n\geq 0$, $\nu^\alpha p_n(-\ln\nu)$ is nondecreasing if and only if $\nu\odot_F X$ is $[F;\alpha]$-monotone. By continuity of the semigroup F and (2.14), $\lim_{t\downarrow 0}p_n(t)=p_n$ for any $n\geq 0$. Therefore, (2.6) is obtained for p_n by letting $\nu\uparrow 1$. □

COROLLARY 2.4. (i) *If X is $[F;\alpha]$-monotone, then it is $[F;\alpha']$-monotone for any $\alpha'>\alpha$.*

(ii) *Discrete mixtures of $[F;\alpha]$-monotone distributions on $\boldsymbol{Z}_+$ are also $[F;\alpha]$-monotone.*

(iii) *Let X be an $[F;\alpha]$-monotone rv and let W_1 be a rv distributed on $(0,1)$, independent of X (and possibly degenerate), then $W_1\odot_F X$ is $[F;\alpha]$-monotone.*

(iv) *If X is $[F;\alpha]$-monotone, then $S_F(\alpha,\nu,f)$ is differentiable over $(0,1)$ and*

$$\frac{\partial}{\partial\nu}S_F(\alpha,\nu,f)=\alpha\nu^{\alpha-1}E(f(\nu\odot_F Y)), \tag{2.17}$$

for some $\boldsymbol{Z}_+$-valued rv Y.

PROOF. (i) and (ii) directly follow from (2.6) and (iv) from (2.13). (iii) follows readily from (2.4). If $K(u)$ is the (cumulative) distribution of $-\ln W_1$, then by (2.2) and (2.4),

$$P_{W_1 \odot_F X}(z) = \int_0^\infty P_X \circ F_u(z) dK(u) = \alpha \int_0^\infty \left(\int_0^\infty Q \circ F_u \circ F_t(z) dK(u) \right) e^{-\alpha t} dt$$

for some pgf Q. The conclusion follows by noting that $\int_0^\infty Q \circ F_u(z) d\psi(u)$ is a pgf (it is in fact the pgf of $W_1 \odot_F Y'$ where Y' has pgf Q) and by applying (2.4) again. □

Next, we establish a convolution property.

PROPOSITION 2.5. *Let $\alpha, \beta > 0$. The convolution of an $[F;\alpha]$-monotone distribution and an $[F;\beta]$-monotone distribution is $[F;\alpha+\beta]$-monotone.*

PROOF. Assume $(p_n, n \geq 0)$ (resp. $(q_n, n \geq 0)$) $[F;\alpha]$-monotone (resp. $[F;\beta]$-monotone). Let $(p * q)_n = \sum_{i=0}^n p_i q_{n-i}$ be the convolution of $\{p_n\}$ and $\{q_n\}$. Then by interchanging sums and by applying (2.6) twice we have

$$\sum_{i=1}^{n+1} i(p*q)_i h_{n-i+1} \leq \sum_{j=0}^{n} (n-j+\beta a^{-1}) p_j q_{n-j} + \sum_{j=0}^{n} (n-j+\alpha a^{-1}) p_{n-j} q_j. \tag{2.18}$$

A change in subscripts in (2.18) implies that (2.6) holds for $(p*q)_n$. □

van Harn *et al.* (1982) give some rich examples of semigroups from which one can generate $[F;\alpha]$-monotone distributions. We mention the standard semigroup $F_t(z) = 1 - e^{-t} + e^{-t}z$ (for which $\delta_F = 1$, $U(z) = A_F(z) = 1 - z$, $H(z) = 1$, $a = 1$) which yields the binomial thinning operator of Steutel and van Harn (1979) and thus (1.2). In this case $[F;\alpha]$-monotonicity is the α-monotonicity of Steutel (1988) and Hansen (1990) (cf. also Alzaid and Al-Osh (1990)). Propositions 2.2 and 2.3 generalize the results of these authors, yielding for example (1.3) (from (2.6)). Proposition 2.5 applied to the standard semigroup gives Alamatsaz's result (1993).

In general, $[F;\alpha]$-monotone distributions are not necessarily monotone in the usual sense as the following example shows. Let $(p_n, n \geq 0)$ be such that $p_0 = 1/4$, $p_1 = 3/8$, $p_2 = 10/56$, $p_3 = 11/56$, and $p_n = 0$, $n \geq 4$. It is easy to verify by way of (1.3) that $(p_n, n \geq 0)$ is an $[F;2]$-monotone distribution (for the standard semigroup) that is not monotone.

We next identify two important classes of $[F;\alpha]$-monotone distributions. We recall that a $\boldsymbol{Z}_+$-valued rv X is said to be F-self-decomposable (van Harn *et al.* (1982)) if for any $v \in (0,1)$, there exists a rv X_v such that $X =^d v \odot_F X + X_v$. X is said to be F-stable with exponent $\gamma > 0$ if there exits a sequence of iid rv's $(X_n, n \geq 0)$, $X_i =^d X$ for all i, such that for all $n > 0$, $X =^d n^{-1/\gamma} \odot_F \sum_{i=1}^n X_i$. F-self-decomposable distributions are infinitely divisible and F-stable distributions are necessarily F-self-decomposable and exist only when $0 < \gamma \leq \delta_F$; cf. van Harn *et al.* (1982). Moreover, van Harn and Steutel (1993) show (cf. their Theorem 3.2) that X is F-self-decomposable if and only if it satisfies the stability equation

$$X \stackrel{d}{=} W^{1/\alpha} \odot_F (X + S) \tag{2.19}$$

for some $\alpha > 0$ and some rv S. We can summarize as follows.

PROPOSITION 2.6. *Any F-self-decomposable distribution is $[F;\alpha]$-monotone for some $\alpha > 0$. Moreover, if $\delta_F > 0$, then any F-stable rv is $[F;\alpha]$-monotone for some $\alpha > 0$.*

van Harn and Steutel (1993) studied an important special case of (2.19). We state a version of their result without proof.

PROPOSITION 2.7. *Assume $\delta_F > 0$. Let $Y_1, \ldots, Y_M$ be independent, $W, Y_1', \ldots, Y_N'$ be independent, $\mathbf{Z}_+$-valued. Further, assume $M < N$, W is uniform over $(0,1)$, and $Y_1, \ldots, Y_M, Y_1', \ldots, Y_N'$ identically distributed. Then*

$$\sum_{i=1}^{M} Y_i \stackrel{d}{=} W^{1/\alpha} \odot_F \sum_{i=1}^{N} Y_i', \tag{2.20}$$

for some $\alpha > 0$, if and only if $\gamma = \alpha(N-M)/M \leq \delta_F$ and the pgf of Y_1 has the form

$$P(z) = (1 + dA_F(z)^\gamma)^{-1/(N-M)}, \qquad \text{for some} \quad d > 0. \tag{2.21}$$

The following proposition solves a related stability equation and can be seen as an extension of a result of Huang and Chen (1989) to the discrete case .

PROPOSITION 2.8. *Assume $\delta_F > 0$. Let $Y_1, \ldots, Y_M$ be independent, $(W_1, Y_1'), \ldots, (W_N, Y_N')$ be independent, all $\mathbf{Z}_+$-valued. Further, assume $M < N$, W_i is uniform over $(0,1)$ and independent of Y_i', and $Y_1, \ldots, Y_M, Y_1', \ldots, Y_N'$ identically distributed. Then*

$$\sum_{i=1}^{M} Y_i \stackrel{d}{=} \sum_{i=1}^{N} W_i^{1/\alpha} \odot_F Y_i', \tag{2.22}$$

for some $\alpha > 0$, if and only if $\gamma = \alpha(N-M)/M \leq \delta_F$ and the pgf of Y_1 has the form

$$P(z) = (1 + dA_F(z)^\gamma)^{-N/(N-M)}, \qquad \text{for some} \quad d > 0. \tag{2.23}$$

PROOF. Assume (2.22) holds. It follows from Proposition 2.2 (ii) that

$$P^{M/N}(z) = \alpha A_F(z)^{-\alpha} \int_z^1 P(v) A_F(v)^\alpha [U_F(v)]^{-1} dv.$$

Taking derivatives and using $A_F'(z)/A_F(z) = -[U_F(z)]^{-1}$, we have

$$MP'(z) = \alpha N [U_F(z)]^{-1} P(z)(1 - P^{(N-M)/N}(z)). \tag{2.24}$$

Using the substitution $P(z) = (1 + f(z))^{-N/(N-M)}$ for some function f, (2.24) becomes

$$\frac{f'(z)}{f(z)} = -\frac{\gamma}{U_F(z)}, \qquad \gamma = \alpha(N-M)/M,$$

whose solution is $f(z) = dA_F(z)^\gamma$ for some $d > 0$. This implies (2.23). By Lemma 4.2 in van Harn and Steutel, $P(z)$ of (2.23) is a pgf only if $\gamma \leq \delta_F$. Conversely, if $P(z)$ satisfies (2.23), then direct calculations show that (2.22) holds. □

In the binomial thinning case, the solutions to the stability equations in Propositions 2.7 and 2.8 are from the family of discrete Linnik distributions with pgf's of the form $P(z) = (1 + d(1-z)^\alpha)^{-r}$, $0 < \alpha \leq 1$, $d, r > 0$ (see Devroye (1993)).

It is clear that one can use (2.6) to extend the definition of $[F;\alpha]$-monotonicity to any sequence $(p_n, n \geq 0)$ of nonnegative real numbers and $\alpha \in \boldsymbol{R}$. We proceed to show that such sequences do arise in the context of a generalized notion of self-decomposability due to Hansen (1989).

A $\boldsymbol{Z}_+$-valued rv X with pgf $P(z)$ is said to have an $[F;\alpha]$-self-decomposable distribution for some $\alpha \in \boldsymbol{R}$ if for every $t > 0$, there exists a pgf P_t such that

$$P(z) = P^{e^{-\alpha t}}(F_t(z))P_t(z), \qquad |z| \leq 1. \tag{2.25}$$

The special case $\alpha = 0$ in (2.28) yields the notion of F-self-decomposability recalled above. Hansen (1989) shows that $[F;\alpha]$-self-decomposable distributions exist only for $\alpha \geq -\delta_F$ and are infinitely divisible with their pgf's characterized by the equation

$$\ln P(z) = A_F(z)^{-\alpha}\left(\int_z^1 A_F(v)^a [U_F(v)]^{-1} \ln Q(v)dv - \kappa\right), \tag{2.26}$$

where Q is a unique infinitely divisible pgf and $\kappa \geq 0$ (with $\kappa = 0$ if $\alpha \geq 0$). For example, F-stable distributions with exponent γ, $0 < \gamma \leq \delta_F$ are $[F;\alpha]$-self-decomposable for any $\alpha \geq -\gamma$.

We recall that the Lévy measure $(r_n, n \geq 0)$ of an infinitely divisible pdf $(p_n, n \geq 0)$ with pgf $P(z)$ is the sequence of the nonnegative coefficients of the power series expansion of $R(z) = \frac{\partial}{\partial z} \ln P(z)$ and satisfies $\sum_{n=0}^\infty r_n(n+1)^{-1} < \infty$.

The next proposition extends a result obtained by Hansen (1990) in the binomial thinning case.

PROPOSITION 2.9. *Let $(p_n, n \geq 0)$ be the pdf of an infinitely divisible distribution on $\boldsymbol{Z}_+$ with Lévy measure $(r_n, n \geq 0)$. Let $\alpha \geq -\delta_F$. Then $(p_n, n \geq 0)$ is $[F;\alpha]$-self-decomposable if and only if the sequence $(n^{-1}r_{n-1}, n \geq 1)$ is $[F;\alpha]$-monotone.*

PROOF. If $(p_n, n \geq 0)$ is $[F;\alpha]$-self-decomposable, then by (2.26) its pgf $P(z)$ satisfies

$$\ln Q(z) = -U_F(z)\frac{\partial}{\partial z}\ln P(z) + \alpha \ln P(z) \tag{2.27}$$

for some infinitely divisible pgf Q. If $(r'_n, n \geq 0)$ denotes the Lévy measure of $Q(z)$, then by equating the coefficients of z^n in (2.30), we obtain

$$n^{-1}r'_{n-1} = (a + \alpha n^{-1})r_{n-1} - a\sum_{i=1}^{n+1} r_{i-1}h_{n-i+1}, \qquad n \geq 1. \tag{2.28}$$

Since $r'_n \geq 0$, (2.28) is equivalent to the $[F;\alpha]$-monotonicity of $(n^{-1}r_{n-1}, n \geq 1)$. Conversely, define r'_n through (2.28) and $Q(z)$ through $\frac{\partial}{\partial z}\ln Q(z) = \sum_{n=0}^\infty r'_n z^n$. Then $Q(z)$ satisfies (2.27). Now $\lim_{z\uparrow 1} U(z)\frac{\partial}{\partial z}\ln P(z) = 0$ (see van Harn *et al.* (1982)) implies $\lim_{z\uparrow 1}\ln Q(z) = 0$. Hence, $Q(z)$ is an infinitely divisible pgf and $P(z)$ satisfies (2.26). □

Next, we establish a simple characterization of $[F;\alpha]$-monotonicity for $[F;\beta]$-self-decomposable distribution, $-\delta_F \leq \beta \leq 0$. Our proposition generalizes a result obtained by van Harn and Steutel (1993) in the binomial thinning case and $\beta = 0$.

PROPOSITION 2.10. *Assume $-\delta_F \le \beta \le 0$. For $\alpha > 0$ the pdf $(p_n, n \ge 0)$ of an $[F;\beta]$-self-decomposable rv X is $[F;\alpha]$-monotone if and only if $p_1 h_0/p_0 \le \alpha a^{-1}$.*

PROOF. The necessary part follows from (2.6) applied to $n = 0$. Conversely, since $(p_n, n \ge 0)$ is $[F;\beta]$-self-decomposable, it is infinitely divisible and is therefore related to its Lévy measure $(r_n, n \ge 0)$ by (see Steutel and van Harn (1979))

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \qquad n \ge 0. \tag{2.29}$$

Hence,

$$\sum_{i=1}^{n+1} i p_i h_{n-i+1} = \sum_{i=1}^{n+1} \left(\sum_{k=0}^{i-1} p_k r_{i-1-k} \right) h_{n-i+1} = \sum_{k=0}^{n} \left(\sum_{i=1}^{n-k+1} r_{i-1} h_{n-k-i+1} \right) p_k. \tag{2.30}$$

Since $(p_n, n \ge 0)$ is $[F;\beta]$-self-decomposable, by Proposition 2.9

$$\sum_{i=1}^{n-k+1} r_{i-1} h_{n-k-i+1} \le (n - k + \beta a^{-1}) \frac{r_{n-k-1}}{n-k}, \qquad 0 \le k < n. \tag{2.31}$$

Combining (2.29)–(2.31) with $-1 < \beta \le 0$ yields

$$\sum_{i=1}^{n+1} i p_i h_{n-i+1} \le (n + r_0 h_0) p_n + \beta a^{-1} \sum_{k=0}^{n-1} \frac{r_{n-k-1}}{n-k} p_k \le (n + r_0 h_0) p_n.$$

Noting that $r_0 = p_1/p_0$, (2.6) follows if $p_1 h_0 / p_0 \le \alpha a^{-1}$. □

3. The $\boldsymbol{R}_+$-valued case

Let X be an $\boldsymbol{R}_+$-valued rv and $\nu \in (0,1)$. The generalized multiplication $\nu \odot_C X$ (van Harn and Steutel (1993)) is defined by

$$\nu \odot_C X \overset{d}{=} Y^{(\nu)}(X), \tag{3.1}$$

where $Y^{(\nu)}(\cdot)$ is an $\boldsymbol{R}_+$-valued process with stationary, independent increments (sii), independent of X with

$$\phi_{Y^{(\nu)}(1)}(\tau) = e^{-C_t(\tau)}, \qquad t = -\ln \nu. \tag{3.2}$$

In terms of LST's, (3.1) is equivalent to

$$\phi_{\nu \odot_C X}(\tau) = \phi_X(C_t(\tau)), \qquad t = -\ln \nu. \tag{3.3}$$

DEFINITION 3.1. An $\boldsymbol{R}_+$-valued rv X is said to be $[C;\alpha]$-monotone (or to have a $[C;\alpha]$-monotone distribution) for some $\alpha > 0$ if

$$X \overset{d}{=} W^{1/\alpha} \odot_C Y, \tag{3.4}$$

where W is uniformly distributed over $(0,1)$, Y is an $\boldsymbol{R}_+$-valued rv and Y and W are independent.

It is easy to see that X is $[C;\alpha]$-monotone if and only if its LST ϕ_X satisfies

$$\phi_X(\tau) = \alpha \int_0^\infty \psi \circ C_t(\tau) e^{-\alpha t} dt, \tag{3.5}$$

for some LST ψ.

As in the $\boldsymbol{Z}_+$-case, one can use (3.5) as a starting point to characterize $[C;\alpha]$-monotonicity. Instead, we will use an approach due to van Harn and Steutel (1993) to extend the results of Section 2 to the $\boldsymbol{R}_+$-valued case via Poisson mixtures.

Let $N_\lambda(\cdot)$ be a Poisson process of intensity λ and T be an $\boldsymbol{R}_+$-valued rv independent of $N_\lambda(\cdot)$. The $\boldsymbol{Z}_+$-valued rv $N_\lambda(T)$ is called a Poisson mixture. Its pgf is given by

$$P_{N_\lambda(T)}(z) = \phi_T(\lambda(1-z)). \tag{3.6}$$

For every $\lambda > 0$, $F^{(\lambda)} = (F_t^{(\lambda)}; t \geq 0)$ with $F_t^{(\lambda)}(z) = 1 - \lambda^{-1} C_t(\lambda(1-z))$ is a continuous composition semigroup of pgf's with $\delta_{F^{(\lambda)}} = \delta_C$ (cf. van Harn and Steutel (1993)). Moreover, the U and A-functions of $F^{(\lambda)}$ are respectively

$$U_\lambda(z) = -U_C(\lambda(1-z))/\lambda \quad \text{and} \quad A_\lambda(z) = A_C(\lambda(1-z))/A_C(\lambda). \tag{3.7}$$

PROPOSITION 3.2. *Let X be an $\boldsymbol{R}_+$-valued rv. The following assertions are equivalent.*

(i) *X is $[C;\alpha]$-monotone;*
(ii) *For any $\lambda > 0$, $N_\lambda(X)$ is $[F^{(\lambda)};\alpha]$-monotone;*
(iii) *There exists an LST ψ such that for any $\tau > 0$,*

$$\phi_X(\tau) = -\alpha[A_C(\tau)]^{-\alpha} \int_0^\tau \psi(v) A_C(v)^\alpha [U_C(v)]^{-1} dv. \tag{3.8}$$

(iv) *For every nonnegative, bounded, Borel measurable function f on $\boldsymbol{R}_+$, the function*

$$S_C(\alpha,\nu,f) = \nu^\alpha E[f(\nu \odot_C X)] \tag{3.9}$$

is nondecreasing in $\nu \in (0,1)$.

PROOF. (i)⇒(ii): If X satisfies (3.4), then it follows from (3.5), (3.6), and (2.4) that

$$N_\lambda(X) \stackrel{d}{=} W^{1/\alpha} \odot_{F^{(\lambda)}} N_\lambda(Y), \tag{3.10}$$

for any $\lambda > 0$, implying (ii) by (2.3).

(ii)⇒(iii): Let $\lambda > 0$ and $0 \leq \tau \leq \lambda$. Applying (2.5) to $N_\lambda(X)$ for $z = 1 - (\tau/\lambda)$ and using (3.7) leads to

$$\begin{aligned}\phi_X(\tau) &= P_{N_\lambda(X)}\left(1 - \frac{\tau}{\lambda}\right)\\ &= -\alpha\lambda[A_C(\tau)]^{-\alpha} \int_{1-\tau/\lambda}^1 Q_\lambda(u) A_C(\lambda(1-u))^\alpha [U_C(\lambda(1-u))]^{-1} du,\end{aligned}$$

for some pgf Q_λ. The change of variable $v = \lambda(1-u)$ yields

$$\phi_X(\tau) = -\alpha[A_C(\tau)]^{-\alpha} \int_0^\tau Q_\lambda(1-(v/\lambda))A_C(v)^\alpha [U_C(v)]^{-1} dv. \tag{3.11}$$

By taking derivatives, it follows from (3.11) that for any $0 < \lambda_1 < \lambda_2$,

$$Q_{\lambda_1}\left(1-\frac{\tau}{\lambda_1}\right) = Q_{\lambda_2}\left(1-\frac{\tau}{\lambda_2}\right), \qquad \text{for any} \quad 0 \le \tau \le \lambda_1 < \lambda_2.$$

Hence, (3.8) holds for $\psi(\tau) = \lim_{\lambda\to\infty} Q_\lambda(1-(\tau/\lambda))$. The convergence being uniform over finite intervals and Q_λ being a pgf implies that ψ is completely monotone. Since $\psi(0) = 1$, ψ is an LST.

(iii)$\Rightarrow$(i): By (3.7), (3.8), and a suitable change of variable, we have

$$P_{N_\lambda(X)}(z) = \alpha A_\lambda(z)^{-\alpha} \int_z^1 \psi(\lambda(1-v))A_\lambda(v)^\alpha U_\lambda(v)^{-1} dv.$$

By Proposition 2.2, $N_\lambda(X) \overset{d}{=} W^{1/\alpha} \odot_{F^{(\lambda)}} N_\lambda(Y)$ for an $\boldsymbol{R}_+$-valued rv Y with LST ψ. By (2.4), the definition of $F^{(\lambda)}$, and (3.7), for any $\lambda > 0$ and $0 \le z \le 1$, $\phi_X(\lambda(1-z)) = \alpha \int_0^\infty \psi \circ C_t(\lambda(1-z))e^{-\alpha t} dt$. This proves (3.5) since λ and z are arbitrary.

(i)$\Leftrightarrow$(iv): As in the discrete case (Proposition 2.2), one can show that if (3.4) holds then

$$S_C(\alpha,\nu,f) = \nu^\alpha \int_0^1 E(f(\nu w \odot_C Y))\alpha w^{\alpha-1} dw = \alpha \int_0^\nu E(f(w \odot_C Y))w^{\alpha-1} dw, \tag{3.12}$$

which implies (iv). Assuming (iv), we need to show that $N_\lambda(X)$ is $[F^{(\lambda)};\alpha]$-monotone for any $\lambda > 0$. By a corollary to the proof of Propositon 2.2 ((i)$\Leftrightarrow$(iv)), it is enough to show that for every $n \ge 0$, $\nu^\lambda P(\nu \odot_{F^{(\lambda)}} N_\lambda(X) = n)$ is an increasing function of $\nu \in (0,1)$. By Proposition 5.1 in van Harn and Steutel (1993),

$$\nu^\alpha P(\nu \odot_{F^{(\lambda)}} N_\lambda(X) = n) = \nu^\alpha P(N_\lambda(\nu \odot_C X) = n) = \nu^\alpha E(f(\nu \odot_C X)), \tag{3.13}$$

where $f(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$ which is nonnegative, bounded, and Borel measurable. □

The properties of $[F;\alpha]$-monotonicity stated in Corollary 2.4 and Proposition 2.5 extend verbatim to $[C;\alpha]$-monotone distributions. A Poisson mixture argument can be applied to derive most of these properties from their discrete counterparts, via Proposition 3.2. For example, if X is $[C;\alpha]$-monotone and Y is $[C;\beta]$-monotone, X, Y independent, then for any $\lambda > 0$, $N_\lambda(X+Y) =^d N_\lambda(X) + N_\lambda(Y)$. Proposition 2.5 implies $N_\lambda(X+Y)$ is $[F^{(\lambda)};\alpha+\beta]$-monotone. Hence, by Proposition 3.2 ((i)$\Leftrightarrow$(ii)), $X+Y$ is $[C;\alpha+\beta]$-monotone.

The concepts of C-self-decomposability and C-stability for $\boldsymbol{R}_+$-valued rv's are defined (with $\odot_C$) in a manner completely analogous to their F-counterparts; see van Harn and Steutel (1993). The analogue of (2.19) holds with $\odot_C$, implying the $\boldsymbol{R}_+$-version of Proposition 2.6. Also, by way of a Poisson mixture argument, it can be shown that Propositions 2.7 and 2.8 carry over as well to the $\boldsymbol{R}_+$-case.

An $\boldsymbol{R}_+$-valued rv X with LST $\phi(\tau)$ is said to have a $[C;\alpha]$-self-decomposable distribution (Hansen (1989)) for $\alpha \in \boldsymbol{R}$, $\alpha \geq -\delta_C$ necessarily, if for every $t > 0$ there exists an LST ϕ_t such that

$$\phi(\tau) = \phi^{e^{-\alpha t}}(C_t(\tau))\phi_t(\tau), \qquad \tau \geq 0. \tag{3.14}$$

The following proposition is proved along the same lines as Proposition 3.2 (by making use of (2.25) and (2.26)). The details are omitted.

PROPOSITION 3.3. *Assume $\alpha \geq -\delta_C$. Let X be an $\boldsymbol{R}_+$-valued rv with LST ϕ. The following assertions are equivalent.*

(i) *X is $[C;\alpha]$-self-decomposable;*

(ii) *For any $\lambda > 0$, $N_\lambda(X)$ is $[F^{(\lambda)};\alpha]$-self-decomposable;*

(iii) *X is infinitely divisible and there exists a unique infinitely divisible LST ψ and $\kappa \geq 0$ (with $\kappa = 0$ if $\alpha \geq 0$), such that for any $\tau \geq 0$,*

$$\ln \phi(\tau) = -A_C(\tau)^{-\alpha}\left(\int_0^\tau A_C(v)^\alpha [U_C(v)]^{-1} \ln \psi(v)dv - \kappa\right). \tag{3.15}$$

We conclude with the description of a family of semigroups of cgf's that can be used to generate $[C;\alpha]$-monotone distributions. For $\theta \in [0,1)$, let

$$C_t^{(\theta)}(\tau) = \frac{\bar{\theta}e^{-\bar{\theta}t}\tau}{\bar{\theta} + \theta(1 - e^{-\bar{\theta}t})\tau}, \qquad t,\tau \geq 0, \quad \bar{\theta} = 1 - \theta. \tag{3.16}$$

It is easy to verify that $C_t^{(\theta)}(\tau)$ has a completely monotone drivative and hence is a cgf (since $C_t(0) = 0$). In this case

$$U_C^{(\theta)}(\tau) = -\tau(\bar{\theta} + \theta\tau), \qquad A_C^{(\theta)}(\tau) = \left(\frac{\tau}{\bar{\theta} + \theta\tau}\right)^{1/\bar{\theta}}, \quad \delta_C^{(\theta)} = \bar{\theta}. \tag{3.17}$$

The special case $\theta = 0$ corresponds to the ordinary multiplication. Therefore, the results in this section can be seen as generalizations of results obtained by Olshen and Savage (1970) and Huang and Chen (1989).

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