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ON GEOMETRIC INFINITE DIVISIBILITY AND STABILITY

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Abstract. The purpose of this paper is to study geometric infinite divisibility and geometric stability of distributions with support in $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$. Several new characterizations are obtained. We prove in particular that compound-geometric (resp. compound-exponential) distributions form the class of geometrically infinitely divisible distributions on $\boldsymbol{Z}_+$ (resp. $\boldsymbol{R}_+$). These distributions are shown to arise as the only solutions to a stability equation. We also establish that the Mittag-Leffler distributions characterize geometric stability. Related stationary autoregressive processes of order one ($AR(1)$) are constructed. Importantly, we will use Poisson mixtures to deduce results for distributions on $\boldsymbol{R}_+$ from those for their $\boldsymbol{Z}_+$-counterparts.

Key words and phrases: Geometric infinite divisibility, geometric stability, compound-geometric, compound-exponential, Mittag-Leffler, Poisson mixtures, Lévy process.

1. Introduction

Klebanov *et al.* (1984) introduced a special form of infinite divisibility (i.d.) as follows. A real-valued random variable (rv) X is said to have a geometrically infinitely divisible (g.i.d.) distribution if for any $p \in (0,1)$, there exits a sequence of iid, real-valued rv's $\{X_i^{(p)}\}$ such that

$$X \stackrel{d}{=} \sum_{i=1}^{N_p} X_i^{(p)}, \tag{1.1}$$

where N_p has the geometric distribution

$$P(N_p = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \dots, \tag{1.2}$$

and N_p and $\{X_i^{(p)}\}$ are independent. Klebanov *et al.* (1984) obtained several characterizations of the g.i.d. property in terms of characteristic functions, including analogues of the Lévy-Khinchin and Lévy representations in classical infinite divisibility. They also introduced and characterized the related concept of geometric strict stability (g.s.s.) for real-valued rv's (recalled in Section 4 below.) The exponential and geometric distributions are examples of distributions that possess the g.i.d. and the g.s.s. properties. More generally, the Mittag-Leffler distributions (cf. Pillai (1990) and Pillai and Jayakumar (1995)) are g.i.d. Using Bernstein functions, Fujita (1993) constructed an even larger class of g.i.d. distributions with support on the nonnegative half-line.

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The purpose of this paper is to study geometric infinite divisibility and stability of distributions on $\boldsymbol{Z}_+ := \{0, 1, 2, \ldots\}$ and $\boldsymbol{R}_+ := [0, \infty)$. Several new characterizations are obtained. We will prove in particular that compound-geometric (resp. compound-exponential) distributions form the class of g.i.d. distributions on $\boldsymbol{Z}_+$ (resp. $\boldsymbol{R}_+$). These distributions are shown to arise as the only solutions to a stability equation and also as the marginals of special Lévy processes. We will establish that the Mittag-Leffler distributions characterize geometric stability. Related autoregressive processes of order one ($AR(1)$) are presented. Importantly, we will use Poisson mixtures to deduce results for distributions on $\boldsymbol{R}_+$ from those for their Z_+-counterparts. Generalized notions of infinite divisibility and stability are also discussed. The results of this paper extend several aspects of the work of Pillai (1990), Pillai and Jayakumar (1995), and Fujita (1993).

2. Geometric infinite divisibility: the discrete case

We start out by noting that a $\boldsymbol{Z}_+$-valued rv X with probability generating function (pgf) P is g.i.d. if and only if for any $p \in (0, 1)$ there exists a pgf $G^{(p)}$ such that

$$P(z) = \frac{pG^{(p)}(z)}{1 - qG^{(p)}(z)}, \qquad |z| \le 1, \qquad q = 1 - p. \tag{2.1}$$

In the sequel a pgf that satisfies (2.1) will be said to be g.i.d. We also recall that the pgf of a compound-geometric distribution is given by

$$P(z) = \{1 + c(1 - Q(z))\}^{-1}, \tag{2.2}$$

for some constant $c > 0$ and some pgf $Q(z)$ (where, *w.l.o.g.*, Q(0)=0).

The following result gives several characterizations of the discrete g.i.d. property and constitutes the main result of the section.

PROPOSITION 2.1. *Let X be a $\boldsymbol{Z}_+$-valued rv with pgf $P(z)$, $0 < P(0) < 1$. The following assertions are equivalent.*

(i) *X has a g.i.d. distribution;*

(ii) *$H(z) = \exp\{1 - \frac{1}{P(z)}\}$ is the pgf of an i.d. distribution.*

(iii) *X has a compound-geometric distribution;*

(iv) *X satisfies the stability equation*

$$X \overset{d}{=} B\,(X + S), \tag{2.3}$$

for some $\boldsymbol{Z}_+$-valued rv S and some mixed Bernoulli variable B with mixing variable W taking values in $(0, 1)$ and with mean $E(W) = \frac{c}{1+c}$, $c > 0$. The rv's X, B, and S are assumed independent.

PROOF. Let $P(z)$ be the pgf of X. A mere adaptation of characteristic function arguments due to Klebanov *et al.* (1984) shows (i)$\Rightarrow$(ii). Next, we assume (ii) holds. Then there exists a pgf $Q(z)$ and $c > 0$ such that $\exp\{1 - \frac{1}{P(z)}\} = e^{-c(1-Q(z))}$ (cf., for example, Steutel and Van Harn (1979)) which implies (2.2), and hence (iii). If the latter holds, then $P(z)$ satisfies (2.2) and (2.1) can be easily shown to hold for any $p \in (0, 1)$

with the pgf $G^{(p)}(z)$ given by $G^{(p)}(z) = \{1 + cp(1 - Q(z))\}^{-1}$, implying (i). Finally, we show (iii)$\Leftrightarrow$(iv). The pgf version of (2.3) is given by

$$P(z) = \int_0^1 (1 - p + pP(z)Q(z))\, dF(p), \tag{2.4}$$

where $Q(z)$ is the pgf of S and $F(p)$ is the distribution of the mixing rv W. Solving for $P(z)$, we obtain (2.2). Conversely, if $P(z)$ is as in (2.2) then it can be easily shown to satisfy (2.4). □

COROLLARY 2.2. *Any g.i.d. discrete distribution is i.d.*

PROOF. By Proposition 2.1, $P(z) = (P_n(z))^n$ where $P_n(z) = (1 + c(1 - Q(z)))^{1/n}$, and $P_n(z)$ is the pgf of a compound-negative binomial distribution. □

In the following proposition we show that a compound-geometric distribution, and more generally a compound-negative binomial, can arise as the marginal distribution of a Lévy process.

PROPOSITION 2.3. *Let $X(\cdot)$ be a Z_+-valued Lévy process.*
(i) *For some $a > 0$, $X(a)$ satisfies a stability relation of the form*

$$X(a) \stackrel{d}{=} B\,(X(a) + S), \tag{2.5}$$

where B is a mixed Bernoulli variable with mixing rv W taking values in $(0, 1)$ and with mean $E(W) = \frac{c}{1+c}$, $c > 0$, and S is Z_+-valued, and independent of B, if and only if, for all $t > 0$, $X(t)$ has a pgf of the form

$$P_t(z) = [1 + c(1 - Q_S(z))]^{-t/a}, \tag{2.6}$$

where $Q_S(z)$ is the pgf of S.
(ii) *Moreover in this case, $X(\cdot)$ can be represented as subordinated to a gamma process $T(\cdot)$, in the sense that $X(t)$ can be written in the form*

$$X(t) \stackrel{d}{=} Y[T(t)], \tag{2.7}$$

where for all $t > 0$, $T(t)$ has a $\Gamma(t/a, c^{-1})$ distribution, and $Y(t)$ is a Lévy process with pgf $\exp\{-t[1 - Q_S(z)]\}$. Equivalently, $X(t)$ can be written in the forms

$$X(t) \stackrel{d}{=} \sum_{i=1}^{N[T(t)]} S_i \stackrel{d}{=} \sum_{i=1}^{N^*(t)} S_i, \tag{2.8}$$

where $N(\cdot)$ is a Poisson process with intensity $\lambda = 1$, $N^*(\cdot)$ is a Lévy process with a negative binomial (t/a, $(1 + c)^{-1}$) *marginal at time $t > 0$, and $\{S_i\}$ is iid, $S_i =^d S$.*

PROOF. Part (i): by Proposition 2.1 ((i)$\Rightarrow$(iv)), if $X(a)$ statisfies (2.5) then $P_a(z) = [1 + c(1 - Q_S(z))]^{-1}$ which implies $P_1(z) = [1 + c(1 - Q_S(z))]^{-1/a}$ which in turn implies (2.6). The converse is a straightforward consequence of Proposition 2.1 ((iv)$\Rightarrow$(i)) by taking $t = a$ in (2.6). To prove (ii), let $Y(\cdot)$ and $T(\cdot)$ be as in the statement of the proposition. Then the pgf of $Y[T(1)]$ is given by
(2.9)

$$E(z^{Y[T(1)]}) = \int_0^\infty e^{-(1-Q_S(z))s} \frac{1}{\Gamma(1/a)c^{1/a}} s^{1/a-1} e^{-s/c}\, ds = (1 + c(1 - Q_S(z)))^{-1/a}.$$

It follows that $Y[T(t)]$, $t > 0$, has pgf (2.6), implying the representation (2.7). The representations in (2.8) are established similarly. □

3. Discrete geometric stability

In this section we introduce a new concept of discrete geometric stability. First, we recall the definition of the binomial thinning operator $\odot$ introduced by Steutel and Van Harn (1979). For a $\boldsymbol{Z}_+$-valued rv X and $\alpha \in (0,1)$, let

$$\alpha \odot X = \sum_{i=1}^{X} X_i, \tag{3.1}$$

where $\{X_i\}$ is a sequence of iid Bernoulli(α) rv's independent of X.

Steutel and Van Harn (1979) viewed the operator $\odot$ of (3.1) as the discrete analogue of the ordinary multiplication and used it to define notions of self-decomposability and stability for distributions on $\boldsymbol{Z}_+$. We use their operator to define the discrete counterpart of the notion of geometric stability introduced by Klebanov *et al.* (1984) in the continuous case (cf. Definition 4.1 below.)

DEFINITION 3.1. A $\boldsymbol{Z}_+$-valued rv X is said to be discrete geometrically strictly stable (or discrete g.s.s.) if for any $p \in (0,1)$, there exists $\alpha(p) \in (0,1)$ such that

$$X \stackrel{d}{=} \alpha(p) \odot \sum_{i=1}^{N_p} X_i, \tag{3.2}$$

where $\{X_i\}$ is a sequence of iid rv's, $X_i =^d X$, N_p has the geometric distribution with parameter p, and $\{X_i\}$ and N_p are independent.

It is easy to see that a $\boldsymbol{Z}_+$-valued rv X with pgf $P(z)$ is discrete g.s.s. if and only if for any $p \in (0,1)$ there exists $\alpha(= \alpha(p)) \in (0,1)$ such that

$$P(z) = \frac{pP(1-\alpha+\alpha z)}{1-qP(1-\alpha+\alpha z)}. \tag{3.3}$$

It follows then from (2.1) and (3.3) that any discrete g.s.s. distribution is necessarily g.i.d.

PROPOSITION 3.2. *Let X be a $\boldsymbol{Z}_+$-valued rv with pgf $P(z)$, $0 < P(0) < 1$. The following assertions are equivalent.*

(i) *X has a discrete g.s.s. distribution;*

(ii) *$H(z) = \exp\{1 - \frac{1}{P(z)}\}$ is the pgf of a discrete stable distribution;*

(iii) *There exist $0 < \gamma \le 1$, and $c > 0$ such that*

$$P(z) = (1 + c(1-z)^\gamma)^{-1}. \tag{3.4}$$

PROOF. (i)$\Rightarrow$(ii)$\Rightarrow$(iii). If X is discrete g.s.s., then, by Proposition 2.1, $H(z) = \exp\{1 - \frac{1}{P(z)}\}$ is an i.d. pgf. Using (3.3), it follows that any $p \in (0,1)$ there exists $\alpha(p) \in (0,1)$ such that $H(z) = (H(1-\alpha(p)+\alpha(p)z))^{1/p}$. This implies $H(z)$ is the pgf of a discrete stable distribution in the sense of Steutel and Van Harn (1979), and by their Theorem 3.2, there exist $0 < \gamma \le 1$, and $c > 0$ such that $H(z) = \exp\{-c(1-z)^\gamma\}$, from which (3.4) follows. (iii)$\Rightarrow$(i): if $P(z)$ has the form (3.4), then (3.3) holds with $\alpha(p) = p^{1/\gamma}$ for any $p \in (0,1)$. $\square$

The distribution with pgf (3.4), called the discrete Mittag-Leffler distribution, has been studied extensively by several authors (cf., for example, Pillai and Jayakumar (1995)). Proposition 3.2 characterizes (3.4) and adds to other characterizations obtained by Alzaid and Al-Osh (1990) and van Harn and Steutel (1993). Moreover, Pillai and Jayakumar (1995) constructed a Z_+-valued stationary $AR(1)$ processes with Mittag-Leffler marginal (3.4). Their model extends the geometric $AR(1)$ process of McKenzie (1986). We propose, next, to further extend these models by constructing a compound-negative binomial stationary $AR(1)$ process with the marginal pgf

$$P(z) = \{1 + c(1-z)^\gamma\}^{-r}, \qquad r > 0, \quad c > 0, \ 0 < \gamma \leq 1. \tag{3.5}$$

We recall that a Z_+-valued $AR(1)$ process $\{X_n\}$ is defined by the equation

$$X_{n+1} = \alpha \odot X_n + \epsilon_n, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3.6}$$

where $\odot$ is as in (3.1), $\{\epsilon_n\}$ is an iid sequence of Z_+-valued rv's, the innovation sequence, and $0 < \alpha < 1$. The pgf version of (3.6) is

$$P_{n+1}(z) = P_n(1 - \alpha + \alpha z) P_\epsilon(z), \tag{3.7}$$

where P_n (resp. P_ϵ) is the pgf of the X_n (resp. ϵ_n).

Assuming $\{X_n\}$ of (3.6) stationary with marginal pgf (3.5) and solving (3.7) for P_ϵ we obtain

$$P_\epsilon(z) = \left(\frac{1 + c\alpha^\gamma (1-z)^\gamma}{1 + c(1-z)^\gamma} \right)^r. \tag{3.8}$$

It can be shown by a straightforward pgf argument that $\{\epsilon_n\}$ has the representation:

$$\epsilon_n \stackrel{d}{=} \sum_{i=1}^{N} (\alpha^{U_i}) \odot W_i, \tag{3.9}$$

where $\{W_i\}$ are iid rv's with common pgf (3.4), $\{U_i\}$ are iid uniform $(0,1)$ rv's, and N is Poisson with mean $-r\gamma \ln \alpha$, with all these variables independent. This leads to a shot-noise interpretation of the process similar to the one given by Lawrance (1982) for the gamma $AR(1)$ process (cf. also McKenzie (1987) for the case of the negative binomial $AR(1)$ process). A shot-noise process is defined by

$$X(t) = \sum_{m=N(-\infty)}^{N(t)} \alpha^{t-\tau_m} \odot W_m, \tag{3.10}$$

where $\{W_m\}$ are Z_+-valued iid rv's (amplitudes of the shots) and $N(t)$ is a Poisson process with occurence times at τ_m. If the $\{W_m\}$'s have their common pgf given by (3.4), then $X(t)$ sampled at $n = 0, \pm 1, \pm 2, \ldots$ is another representation of the stationary $AR(1)$ process (3.6) with marginal pgf (3.5). The proof of this fact is an adaptation of Lawrance's (1982) argument and the details are omitted.

4. The $\boldsymbol{R}_+$-valued case via Poisson mixtures

First, we recall a definition due to Klebanov *et al.* (1984).

DEFINITION 4.1. An $\boldsymbol{R}_+$-valued rv X is said to be g.s.s. if for any $p \in (0,1)$, there exists $\alpha(p) \in (0,1)$

$$X \stackrel{d}{=} \alpha(p) \sum_{i=1}^{N_p} X_i, \tag{4.1}$$

where $\{X_i\}$ is a sequence of iid rv's, $X_i \stackrel{d}{=} X$, N_p has the geometric distribution with parameter p, and $\{X_i\}$ and N_p are independent.

An $\boldsymbol{R}_+$-valued rv X with LST $\phi(u)$ is g.s.s. if and only if for any $p \in (0,1)$ there exists α $(= \alpha(p)) \in (0,1)$ such that

$$\phi(u) = \frac{p\phi(\alpha u)}{1 - q\phi(\alpha u)}. \tag{4.2}$$

Just as in the discrete case, (4.2) implies that any g.s.s. distribution on $\boldsymbol{R}_+$ is necessarily g.i.d.

Using similar techniques as in the $\boldsymbol{Z}_+$-case, one can obtain characterizations of g.i.d. and g.s.s. distributions on $\boldsymbol{R}_+$. Instead, we will use the Poisson mixtures approach of Van Harn and Steutel (1993) to extend the results of Sections 2 and 3 to $\boldsymbol{R}_+$-valued rv's.

Let $N_\lambda(\cdot)$ be a Poisson process of intensity λ and T be an $\boldsymbol{R}_+$-valued rv independent of $N_\lambda(\cdot)$. The $\boldsymbol{Z}_+$-valued rv $N_\lambda(T)$ is called a Poisson mixture. Its pgf is given by

$$P_{N_\lambda(T)}(z) = \phi_T(\lambda(1-z)), \tag{4.3}$$

where ϕ is the LST of T.

PROPOSITION 4.2. *Let X be an $\boldsymbol{R}_+$-valued rv with LST ϕ. Then*
(i) *X is g.i.d. if and only if $N_\lambda(X)$ is g.i.d. for all $\lambda > 0$.*
(ii) *X is g.s.s. if and only if $N_\lambda(X)$ is discrete g.s.s. for all $\lambda > 0$.*

PROOF. (i) Assume X is g.i.d. Then a straightforward argument shows that for any $\lambda > 0$,

$$N_\lambda(X) \stackrel{d}{=} \sum_{i=1}^{N_p} N_\lambda(X_i^{(p)}),$$

where p, $\{X_i^{(p)}\}$, and N_p, are as in (1.1). This implies that $N_\lambda(X)$ is g.i.d. Conversely, by (2.1) and (4.3), for any $\lambda > 0$ and $p \in (0,1)$ there exists a pgf $G_\lambda^{(p)}$ such that $\phi(\lambda(1-z)) = pG_\lambda^{(p)}(z)/(1 - qG_\lambda^{(p)}(z))$. Solving for $G_\lambda^{(p)}(z)$, we have $\phi(\lambda(1-z))/(p + q\phi(\lambda(1-z)))$ is a pgf for all $\lambda > 0$. It follows by Lemma A.6 of Van Harn and Steutel (1993) that for any $p \in (0,1)$, $\psi^{(p)}(u) = \phi(u)/(p + q\phi(u))$ is an LST, which can be substituted back into the LST version of (2.1), showing that X is g.i.d. To prove (ii), assume first that X is g.s.s. with LST ϕ. Then by (4.2)

$$\phi(\lambda(1-z)) = \frac{p\phi(\lambda(1-(1-\alpha+\alpha z)))}{1 - q\phi(\lambda(1-(1-\alpha+\alpha z)))} \tag{4.4}$$

for all $\lambda > 0$ and $p \in (0,1)$, and some $\alpha(= \alpha(p)) \in (0,1)$. Combining (3.3), (4.3), and (4.4), yields $N_\lambda(X)$ is discrete g.s.s. Conversely, let $0 < u < \lambda_1 < \lambda_2$. Since $N_{\lambda_1}(X)$ and $N_{\lambda_2}(X)$ are discrete g.s.s., for any $p \in (0,1)$, there exist $\alpha_1, \alpha_2 \in (0,1)$ such that $\phi(u) = p\phi(\alpha_i u)/(1 - q\phi(\alpha_i u))$, $i = 1, 2$, or, equivalently, $\phi(\alpha_1 u) = \phi(\alpha_2 u) = \phi(u)/(p + q\phi(u))$. This implies that $\alpha_1 u = \alpha_2 u$ or $\alpha_1 = \alpha_2$. Since λ_1 and λ_2 are arbitrary, we conclude that X is g.s.s. □

COROLLARY 4.3. Let X be an $\boldsymbol{R}_+$-valued rv. If X is g.i.d., then X is i.d.

PROOF. It follows from Proposition 4.2.(ii), Corollary 2.2, and Proposition A.7 in Van Harn and Steutel (1993). □

Next we establish several characterizations of the g.i.d. property. Recall that an $\boldsymbol{R}_+$-valued rv X is said to be compound-gamma if $X =^d Y(T)$ for some $\boldsymbol{R}_+$-valued Lévy process $\{Y(\cdot)\}$ and some gamma-distributed rv T, independent of $\{Y(\cdot)\}$. The LST of a compound-gamma distribution is given by $\phi(u) = \{1 - c \ln \phi_1(u)\}^{-r}$ for some $c > 0$ and $r > 0$, and some i.d. LST ϕ_1. Combining this with Theorem 1, Section XIII.7, in Feller (1971), proves that X is compound-gamma if and only if its LST is given by

$$\phi(u) = \{1 + \psi(u)\}^{-r}, \qquad u \geq 0, \tag{4.5}$$

where ψ has a completely monotone derivative with $\psi(0) = 0$, and $r > 0$. The compound-exponential distribution arises as a special case of a compound-gamma and corresponds to $r = 1$ in (4.5).

PROPOSITION 4.4. *Let X be an $\boldsymbol{R}_+$-valued rv. The following assertions are equivalent.*

(i) *X is g.i.d.;*
(ii) *$N_\lambda(X)$ is compound-geometric for all $\lambda > 0$;*
(iii) *X is compound-exponential.*

Moreover, if the distribution of X has an atom at 0, then the above assertions are equivalent to

(iv) *X satisfies the stability equation (2.3) for some $\boldsymbol{R}_+$-valued rv S.*

PROOF. We prove that (i)⇒(ii)⇒(iii)⇒(i). We then separately show (i)⇔(iv). Let ϕ be the LST of X. If X is g.i.d., then by Proposition 4.2.(i) $N_\lambda(X)$ is g.i.d. for all $\lambda > 0$ and (ii) follows from Proposition 2.1. Suppose (ii) holds. Then for any $\lambda > 0$,

$$P_{N_\lambda(X)}(z) = \phi(\lambda(1-z)) = \{1 + c_\lambda(1 - Q_\lambda(z))\}^{-1} \tag{4.6}$$

for some pgf Q_λ and some $c_\lambda > 0$. For $0 \leq u \leq \lambda$,

$$\lim_{\lambda \to \infty} c_\lambda \left(1 - Q_\lambda(1 - \lambda^{-1}u)\right) = \{\phi(u)\}^{-1} - 1,$$

and the convergence is uniform over bounded intervals. The n-th derivative of $\psi(u) = \{\phi(u)\}^{-1} - 1$ is thus given by

$$\psi^{(n)}(u) = \lim_{\lambda \to \infty} c_\lambda((-1)^{n+1}\lambda^{-n}Q_\lambda^{(n)}(1 - \lambda^{-1}u)),$$

which implies that ψ has a completely monotone derivative and hence (iii) follows. Assume X is compound-exponential with LST $\phi(u) = (1+\psi(u))^{-1}$ where $\psi(u)$ has a completely monotone derivative. Then for any $p \in (0,1)$,

$$\phi(u) = \frac{p\phi^{(p)}(u)}{1-q\phi^{(p)}(u)}, \tag{4.7}$$

where $\phi^{(p)}(u) = (1+p\psi(u))^{-1}$ is also an LST (of a compound-exponential). This implies that X is g.i.d. To show that (i)$\Leftrightarrow$(iv), firstly assume that (2.3) holds for some $\boldsymbol{R}_+$-valued rv S. Then for all $\lambda > 0$, $N_\lambda =^d B(N_\lambda(X) + N_\lambda(S))$ which, by Proposition 2.1, implies $N_\lambda(X)$ is g.i.d. for all $\lambda > 0$. (iv) then follows from Proposition 4.2 (i). Conversely, if X is g.i.d., then by the first part of the proof, ϕ satisfies (4.5) with $r=1$. Since the distribution of X has an atom at 0, then $\phi(\infty) > 0$. It follows that $0 < \psi(\infty) = \lim_{u\to\infty}\psi(u) < \infty$ which, combined with the fact that ψ' is completely monotone, implies that $\phi_1(u) = 1 - (\psi(u)/\psi(\infty))$ is a LST. Hence, $\phi(u) = \{1+\psi(\infty)(1-\phi_1(u))\}^{-1}$ which establishes (2.3) (S has LST ϕ_1). □

It is important to note that Proposition 4.4 generalizes (and provides a much simpler proof) of Theorem 1.1 of Fujita (1993) obtained under the stronger assumptions that ψ is a Bernstein function and $\lim_{u\to\infty}\psi(u) = \infty$.

Just as in the $\boldsymbol{Z}_+$-case (cf. Proposition 2.3), the next proposition shows that compound-exponential distributions, and more generally, compound-gamma distributions can arise as the marginal distribution of a Lévy process. The result is an extension of Theorem 1.2. of Fujita (1993).

PROPOSITION 4.5. *Let $X(\cdot)$ be a $\boldsymbol{R}_+$-valued Lévy process. Let $\psi(u)$ be a function on $\boldsymbol{R}_+$ with a completely monotone derivative such that $\psi(0)=0$ and let $a>0$.*

(i) *$X(t)$ has LST*

$$\phi_t(u) = (1+\psi(u))^{-t/a}, \tag{4.8}$$

for all $t>0$ if and only if $X(\cdot)$ can be represented as subordinated to a gamma process $T(\cdot)$, in the sense that $X(t)$ can be written in the form $X(t) \stackrel{d}{=} Y[T(t)]$, where for all $t>0$, $T(t)$ has a $\Gamma(t/a,1)$ distribution, and $Y(t)$ is a Lévy process with LST $\exp\{-t\psi(u)\}$.

(ii) *Moreover, if $\lim_{u\to\infty}\psi(u) < \infty$, then $X(\cdot)$ will satisfy the stability equation (2.5) and the distribution of $X(a)$, and hence of $X(t)$ for each $t>0$, will have an atom at 0.*

PROOF. The proof is the same as that of Proposition 2.3. The details are omitted. □

Next, we give a characterization of g.s.s. distributions on $\boldsymbol{R}_+$.

PROPOSITION 4.6. *An $\boldsymbol{R}_+$-valued rv X is g.s.s. if and only if its LST is given by*

$$\phi(u) = \frac{1}{1+cu^\gamma}, \tag{4.9}$$

for some $0<\gamma\le 1$ and $c>0$.

PROOF. If X is g.s.s., then by Proposition 3.2 and Proposition 4.2.(ii), we have for $0 \le u \le \lambda$, $\phi(u) = \{1+c_\lambda(u/\lambda)^{\gamma_\lambda}\}^{-1}$, for some $c_\lambda > 0$ and $0 < \gamma_\lambda \le 1$. Therefore for $0 \le u \le \lambda_1 < \lambda_2$, $c_{\lambda_1}(u/\lambda_1)^{\gamma_1} = c_{\lambda_2}(u/\lambda_2)^{\gamma_2}$. Letting $u = \lambda_1$, we have $c_{\lambda_1} = c_{\lambda_2}(\lambda_1/\lambda_2)^{\gamma_2}$, and hence, $u^{\gamma_1-\gamma_2} = \lambda_1^{\gamma_2-\gamma_1}$ for all $0 \le u \le \lambda_1$. This implies that $\gamma_1 = \gamma_2$ which in turn implies that $c_{\lambda_1}\lambda_1^{-\gamma_1} = c_{\lambda_2}\lambda_2^{-\gamma_2}$. Since λ_1 and λ_2 are arbitrary, it follows that $\gamma_\lambda (= \gamma)$ and $c_\lambda \lambda^{-\gamma} (= c)$ are independent of λ. This establishes (4.9). The converse follows from Proposition 3.2 and Proposition 4.2.(ii). □

The distribution with pgf (4.9), called the Mittag-Leffler distribution, has been studied extensively by Pillai (1990). Proposition 4.6 is a new characterization of this distribution and adds to others obtained by Pillai (1990) and Van Harn and Steutel (1993).

We conclude by sketching the construction of the continuous ($\boldsymbol{R}_+$-valued) variate counterpart of the $AR(1)$ process of (3.6). In this case we revert to the standard $AR(1)$ equation

$$X_{n+1} = \alpha X_n + \epsilon_n, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{4.10}$$

with all the variables being $\boldsymbol{R}_+$-valued. The marginal LST of interest becomes

$$P(z) = \{1 + cu^\gamma\}^{-r}, \qquad r > 0, \quad c > 0, \; 0 < \gamma \le 1. \tag{4.11}$$

Under the stationarity assumption, the innovation sequence $\{\epsilon_n\}$ is given by

$$\epsilon_n \stackrel{d}{=} \sum_{i=1}^{N} (\alpha^{U_i}) W_i, \tag{4.12}$$

where $\{U_i\}$, N, are as in (3.9), and the W_i's are iid with the common LST of (4.9). The shot-noise interpretation seen in Section 3 carries over as well (with $\odot$ replaced by the standard multplication). Such a model extends the gamma $AR(1)$ model ($GAR(1)$) of Gaver and Lewis (1980).

5. $\mathcal{N}$-infinite divisibility and stability

A more general notion of infinite divisibility based on (1.1) was studied by several authors (see Gnedenko and Korolev (1996), Section 4.6, for details and further references). The definition is as follows. Let $\mathcal{N} = \{N_p, p \in (0,1)\}$ be a family of $\boldsymbol{Z}_+$-valued rv's such that $E(N_p) = 1/p$ for any $p \in (0,1)$ and

$$H_{p_1} \circ H_{p_2}(z) = H_{p_2} \circ H_{p_1}(z), \qquad \text{for any} \quad p_1, p_2 \in (0,1), \tag{5.1}$$

where H_p is the pgf of N_p. A rv X is said to be $\mathcal{N}$-infinitely divisible if it satisfies (1.1) for any $N_p \in \mathcal{N}$. In order to characterize $\mathcal{N}$-infinite divisibility for $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$-valued rv's we need to recall that (5.1) implies (see the proof of Theorem 4.6.1 in Gnedenko and Korolev (1996)) the existence of an LST φ satisfying $\varphi(0) = -\varphi'(0) = 1$ and

$$\varphi(u) = H_p(\varphi(pu)), \qquad \text{for any} \quad u > 0 \quad \text{and } p \in (0,1). \tag{5.2}$$

By adapting the proof of Theorem 4.6.3 in Gnedenko and Korolev (1996), it can be shown that a $\boldsymbol{Z}_+$(resp. $\boldsymbol{R}_+$)-valued rv X with pgf P (resp. LST ϕ) is $\mathcal{N}$-i.d. if and only if

$$P(z) = \varphi(-\ln Q(z)) \qquad (\text{resp. } \phi(u) = \varphi(-\ln \phi_1(u))), \tag{5.3}$$

where φ is as in (5.2) and Q (resp. ϕ_1) is the pgf (resp. LST) of an i.d. distribution on $\boldsymbol{Z}_+$ (resp. $\boldsymbol{R}_+$).

Next, we define a related concept of discrete stability. A $\boldsymbol{Z}_+$-valued rv X is said to be discrete $\mathcal{N}$-stable if it satisfies (3.2) for any $N_p \in \mathcal{N}$. By an argument similar to the one used in the proof of Proposition 3.2 and by making use of the following equivalent formulation of (5.2)

$$H_p(z) = \varphi(\varphi^{-1}(z)/p), \qquad p \in (0,1), \tag{5.4}$$

it can be shown that X is discrete $\mathcal{N}$-stable if and only if its pgf P satisfies

$$P(z) = \varphi(c(1-z)^\gamma), \tag{5.5}$$

for some $c > 0$ and $0 < \gamma \leq 1$.

Similarly, $\mathcal{N}$-stability for $\boldsymbol{R}_+$-valued rv's is defined by way of (4.1) with $N_p \in \mathcal{N}$. A Poisson mixtures approach will also establish that $\mathcal{N}$-stable disributions on $\boldsymbol{R}_+$ are characterized by LST's of the form

$$\phi(u) = \varphi(cu^\gamma), \tag{5.6}$$

for some $c > 0$ and $0 < \gamma \leq 1$.

Classical (resp. geometric) infinite divisibility corresponds to the family of rv's $\mathcal{N}$ where $N_p = \frac{1}{p}$ with probability 1 (resp. N_p has distribution (1.2)).

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