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WAITING TIME PROBLEMS IN A TWO-STATE MARKOV CHAIN

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Abstract. Let F_0 be the event that l_0 0-runs of length k_0 occur and F_1 be the event that l_1 1-runs of length k_1 occur in a two-state Markov chain. In this paper using a combinatorial method and the Markov chain imbedding method, we obtained explicit formulas of the probability generating functions of the sooner and later waiting time between F_0 and F_1 by the non-overlapping, overlapping and "greater than or equal" enumeration scheme. These formulas are convenient for evaluating the distributions of the sooner and later waiting time problems.

Key words and phrases: Waiting time problems, discrete distributions of order k, Markov chain, Markov chain imbedding method, probability generating function.

1. Introduction

Since the geometric distribution of order k was introduced by Philippou *et al.* (1983), discrete distributions of order k have received a great deal of attention. One of the reasons for this, is their widespread application, for example, in psychology, quality control, non-parametric statistics, the reliability of consecutive-k-out-of-n: F systems (Chao *et al.* (1995)) and start-up demonstration test (Balakrishnan *et al.* (1995, 1997)).

The geometric distribution of order k is one of the simplest waiting time distributions. Recently, the waiting time problems associated with runs have been studied by many authors (see Philippou (1984), Hirano *et al.* (1991), Aki (1985, 1992), Ling (1990, 1992), Ling and Low (1993), Aki and Hirano (1993), Monhanty (1994), Balakrishnan *et al.* (1995, 1997), Antzoulakos and Philippou (1996, 1997), Aki *et al.* (1996), Koutras and Alexandrou (1997), Koutras (1997) and references therein).

Feller (1968) introduced the sooner waiting time problem. Ebneshahrashoob and Sobel (1990) introduced the sooner and later waiting time problems for a run of k consecutive successes or a run of r consecutive failures whichever comes sooner (later) and obtained their probability generating functions (p.g.f.s) in the case of independent and identical Bernoulli trials. Balasubramanian *et al.* (1993) obtained the p.g.f.s of the sooner and later waiting time problems for Markovian Bernoulli trials. Uchida and Aki (1995) considered a generalization of the sooner and later waiting time problems, i.e. the sooner and later waiting time between F_0 and F_1, where F_0 is the event that l_0 runs of "0" of length k_0 occur and F_1 is the event that l_1 runs of "1" of length k_1 occur in a two-state Markov chain, and obtained recurrence relations of the p.g.f.s for the waiting time problems.

In this paper, by using a combinatorial method and the Markov chain imbedding method (see Fu and Koutras (1994), Koutras and Alexandrou (1995) and Han and Aki

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(1999)), we obtain explicit formulas of p.g.f.s for the sooner and later waiting time between F_0 and F_1 in the following Markov chain.

Let $X_1, X_2, \ldots$ be a time-homogeneous $\{0,1\}$-valued Markov chain defined by $\Pr(X_1 = 1) = p_1$, $\Pr(X_1 = 0) = p_0$ and $\Pr(X_{i+1} = 1 \mid X_i = 0) = p_{01}$, $\Pr(X_{i+1} = 0 \mid X_i = 0) = p_{00}$, $\Pr(X_{i+1} = 1 \mid X_i = 1) = p_{11}$, $\Pr(X_{i+1} = 0 \mid X_i = 1) = p_{10}$, $(i = 1, 2, \ldots)$, where $p_0 + p_1 = 1$, $p_{01} + p_{00} = 1$, $p_{11} + p_{10} = 1$ and $0 \le p_1 \le 1$, $0 \le p_{01} \le 1$, $0 \le p_{11} \le 1$.

In Section 2 we obtain explicit formulas of the p.g.f.s of the sooner and later waiting time between F_0 and F_1 by the non-overlapping way of counting (Feller (1968)) in the Markov chain, by using a typical combinatorial method. In Section 3, by using the Markov chain imbedding method, we also obtain other explicit formulas of the p.g.f.s for the same problems. Using these methods, we can also obtain the p.g.f.s of the sooner and later waiting times by the overlapping and "greater than or equal" enumeration scheme.

Our results in this paper are general and new. A number of well known results can be obtained as special cases of the formulas stated here. These results are also suitable for numerical and symbolic calculations. In Section 2 we give a numerical example.

2. Combinatorial method

Let W_S (W_L) be the sooner (later) waiting time between F_0 and F_1. Let $W_1(\bullet, j_1)$ ($W_0(j_0, \bullet)$) be the waiting time for the j_1-th (j_0-th) occurrence of the 1-run (0-run) of length k_1 (k_0). Let $W_1(j_0, j_1)$ be the waiting time for the j_1-th occurrence of 1-run of length k_1 and until this time there are just j_0 0-runs of length k_0 (i.e. $W_1(j_0, j_1)$ is the waiting time of the event which j_0 0-runs of length k_0 and j_1 1-runs of length k_1 occur and the last occured run is just 1-run). Similarly, let $W_0(j_0, j_1)$ be the waiting time for the j_0-th occurrence of 0-run of length k_0 and until this time there are just j_1 1-runs of length k_1.

First, we consider the waiting time $W_1(j_0, j_1)$.

In the sequence of the event $\{W_1(j_0, j_1) = t\}$, there are j_0 0-runs and j_1 1-runs, and the last run is just 1-run. These 1-runs fall into (can be put into) three categories based on their preceding run: (1) there is not any run before the 1-run (i.e. the 1-run is the first run); (2) the preceding run is 1-run; (3) the preceding run is 0-run. Their typical sequences are

$$\text{(2.1)} \qquad \left.\begin{array}{l} (1)\ S_1\text{: nothing} \\ (2)\ S_{11}\text{: 1-run} \\ (3)\ S_{01}\text{: 0-run} \end{array}\right\} \overbrace{0\cdots0}^{0\le,\le k_0-1} \mid \underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0} \mid \overbrace{1\cdots1}^{k_1}.$$

The difference among S_1, S_{11} and S_{01} is only their initial probability. From these typical sequences, the p.g.f. of S_1 and the conditional probability generating functions (c.p.g.f.s) of S_{11} and S_{01} are

$$\text{(2.2)} \qquad \begin{aligned} f_1(z) &= (zp_1 + zp_0a_0)\left[\sum_{i=0}^{\infty}(a_1a_0)^i\right](zp_{11})^{k_1-1} \\ &= (zp_1 + zp_0a_0)\frac{1}{1 - a_1a_0}(zp_{11})^{k_1-1}, \end{aligned}$$

$$\text{(2.3)} \qquad f_{11}(z) = (zp_{11} + zp_{10}a_0)\frac{1}{1 - a_1a_0}(zp_{11})^{k_1-1},$$

$$\text{(2.4)} \qquad f_{01}(z) = (zp_{01} + zp_{00}a_0)\frac{1}{1 - a_1a_0}(zp_{11})^{k_1-1},$$

where $a_0 = [\sum_{i=0}^{k_0-2}(zp_{00})^i](zp_{01})$, $a_1 = [\sum_{i=0}^{k_1-2}(zp_{11})^i](zp_{10})$.

Similarly, we can consider 0-runs in the sequence of the event $\{W_1(j_0, j_1) = t\}$. By interchanging "0" and "1" in (2.1)-(2.4), we can introduce S_0, S_{00} and S_{10}, and compute their p.g.f.s $f_0(z)$, $f_{00}(z)$ and $f_{10}(z)$.

We can divide the sequence of the event $\{W_1(j_0, j_1) = t\}$ in $j_0 + j_1$ subsequences and these subsequences are the above S_1, S_{11}, S_{01} and S_0, S_{00}, S_{10}. Because the first subsequence must be S_1 or S_0, we have two cases: (a) the first subsequence is S_1 and (b) the first subsequence is S_0.

For case (a), supposing there are i_{11} S_{11}s, i_{01} S_{01}s, i_{00} S_{00}s and i_{10} S_{10}s in the sequence, we have $1 + i_{11} + i_{01} = j_1$, $i_{00} + i_{10} = j_0$. Because the first and the last run are both 1-runs, we have $i_{10} = i_{01}$.

The number of arrangements of the sequence of the event $\{W_1(j_0, j_1) = t\}$ in case (a) is $\binom{j_1-1}{i_{01}}\binom{j_0-1}{i_{01}-1}$. *Explanation*: There are two steps for this question. Step 1: in a row of j_1 1-runs, the first must be S_1. We select i_{01} 1-runs in the remaining $(j_1 - 1)$ 1-runs, and let these selected i_{01} 1-runs be S_{01}'s and other 1-runs be S_{11}'s. The number of selections is $\binom{j_1-1}{i_{01}}$. Step 2: We put j_0 0-runs into before every S_{01} of i_{01} S_{01}'s. Because the number of ways of putting j_0 objects into i_{01} different cells is $\binom{j_0-1}{i_{01}-1}$, when no cell is empty. Hence, by the rule of product, we have the number of arrangements is $\binom{j_1-1}{i_{01}}\binom{j_0-1}{i_{01}-1}$.

Hence, we can get the p.g.f. of $W_1(j_0, j_1)$ in the case (a)

$$f_1 \sum_{i_{01}=0}^{\min\{j_0, j_1-1\}} \binom{j_1 - 1}{i_{01}} \binom{j_0 - 1}{i_{01} - 1} f_{01}^{i_{01}} f_{11}^{j_1-1-i_{01}} f_{10}^{i_{01}} f_{00}^{j_0-i_{01}},$$

where we let $\binom{-1}{-1} = 1$ and $\binom{n}{-1} = 0$, $(n = 0, 1, \ldots)$.

Similarly, we can get the p.g.f. of $W_1(j_0, j_1)$ in the case (b)

$$f_0 \sum_{i_{10}=0}^{\min\{j_0-1, j_1-1\}} \binom{j_0 - 1}{i_{10}} \binom{j_1 - 1}{i_{10}} f_{10}^{i_{10}} f_{00}^{j_0-1-i_{10}} f_{01}^{i_{10}+1} f_{11}^{j_1-i_{10}-1}.$$

Hence, we obtain

PROPOSITION 2.1. *The p.g.f. of the waiting time $W_1(j_0, j_1)$ is*

$$\begin{aligned}\phi_{j_0,j_1}^{(1)}(z) = f_1 &\sum_{l=0}^{\min\{j_0, j_1-1\}} \binom{j_1 - 1}{l} \binom{j_0 - 1}{l - 1} f_{01}^{l} f_{11}^{j_1-1-l} f_{10}^{l} f_{00}^{j_0-l} \\ + f_0 &\sum_{l=0}^{\min\{j_0-1, j_1-1\}} \binom{j_0 - 1}{l} \binom{j_1 - 1}{l} f_{10}^{l} f_{00}^{j_0-1-l} f_{01}^{l+1} f_{11}^{j_1-1-l}.\end{aligned}$$

By interchanging "0" and "1" in Proposition 2.1, we can obtain the p.g.f. $\phi_{j_0,j_1}^{(0)}(z)$ of the waiting time $W_0(j_0, j_1)$. Because

$$\{W_S = t\} = \cup_{j_1=0}^{l_1-1}\{W_0(l_0, j_1) = t\} \cup \cup_{j_0=0}^{l_0-1}\{W_1(j_0, l_1) = t\},$$

and these events in the right-hand side are mutually exclusive events, we can obtain

THEOREM 2.1. *The p.g.f. of the sooner waiting time W_S between F_0 and F_1 is*

$$\psi_S(z) = \sum_{j_1=0}^{l_1-1} \phi^{(0)}_{l_0,j_1}(z) + \sum_{j_0=0}^{l_0-1} \phi^{(1)}_{j_0,l_1}(z)$$

where $\phi^{(0)}_{l_0,j_1}(z)$ and $\phi^{(1)}_{j_0,l_1}(z)$ are given by Propositions 2.1.

Remark 2.1. For higher-order Markov chain, when the order is less than or equal to $\min\{k_0, k_1\}$, we can obtain similar results by using the same method. But, when the order exceeds $\min\{k_0, k_1\}$, because there are many kinds of different subsequences leading to the event of interest, it is difficult to use the method.

Remark 2.2. (1) If $p_0 = p_{10} = p_{00}$ and $p_1 = p_{11} = p_{01}$ (i.i.d. trials), then we have $f_0 = f_{10} = f_{00}$ and $f_1 = f_{11} = f_{01}$. Using $\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}$ we have

$$\psi_S(z) = \sum_{j_1=0}^{l_1-1} \binom{l_0 - 1 + j_1}{l_0 - 1} f_0^{l_0} f_1^{j_1} + \sum_{j_0=0}^{l_0-1} \binom{l_1 - 1 + j_0}{l_1 - 1} f_0^{j_0} f_1^{l_1}.$$

This is Theorem 6.1 of Han and Aki (1999).

(2) If $l_0 = l_1 = 1$, we have

$$\psi_S(z) = f_0 + f_1 = \frac{(zp_0 + zp_1 a_1)(zp_{00})^{k_0-1} + (zp_1 + zp_0 a_0)(zp_{11})^{k_1-1}}{1 - a_0 a_1}.$$

This is the result of Balasubramanian *et al.* (1993) and Antzoulakos and Philippou (1997).

Let us now discuss the waiting time $W_1(\bullet, l_1)$.

This problem has been discussed by Mohanty (1994) and Koutras (1997). Aki *et al.* (1996) also discussed the problem in higher order Markov dependent trials. Actually, the waiting time $W_1(\bullet, l_1)$ is a special case of $W_1(j_0, l_1)$ for $k_0 \to \infty$ and $j_0 = 0$. So, from Proposition 2.1, we have

PROPOSITION 2.2. *The p.g.f. of the waiting time $W_1(\bullet, l_1)$ is $\phi^{(1)}_{\bullet,l_1}(z) = g_1(z)[g_{11}(z)]^{l_1-1}$, where $g_1(z) = (zp_1 + zp_0 b_0)\frac{1}{1-b_0 a_1}(zp_{11})^{k_1-1}$, $g_{11}(z) = (zp_{11} + zp_{10} b_0)$ $\frac{1}{1-b_0 a_1}(zp_{11})^{k_1-1}$ and $b_0 = [\sum_{i=0}^{\infty}(zp_{00})^i](zp_{01})$.*

Remark 2.3. Proposition 2.2 is just Theorem 4.1 of Koutras (1997).

By interchanging "0" and "1" in Proposition 2.2, we can get the p.g.f. $\phi^{(0)}_{l_0,\bullet}(z)$ of the waiting time $W_0(l_0, \bullet)$.

Because $\{W_S = t\} \cup \{W_L = t\} = \{W_0(l_0, \cdot) = t\} \cup \{W_1(\cdot, l_1) = t\}$, and the events in the right-hand (left) side are mutually exclusive events, we can obtain

THEOREM 2.2. *The p.g.f. of the later waiting time W_L is $\psi_L(z) = \phi^{(1)}_{\bullet,l_1}(z) + \phi^{(0)}_{l_0,\bullet}(z) - \psi_S(z)$.*

Remark 2.4. If $l_0 = l_1 = 1$, Theorem 2.2 reduces to Proposition 2.2 of Antzoulakos and Philippou (1997).

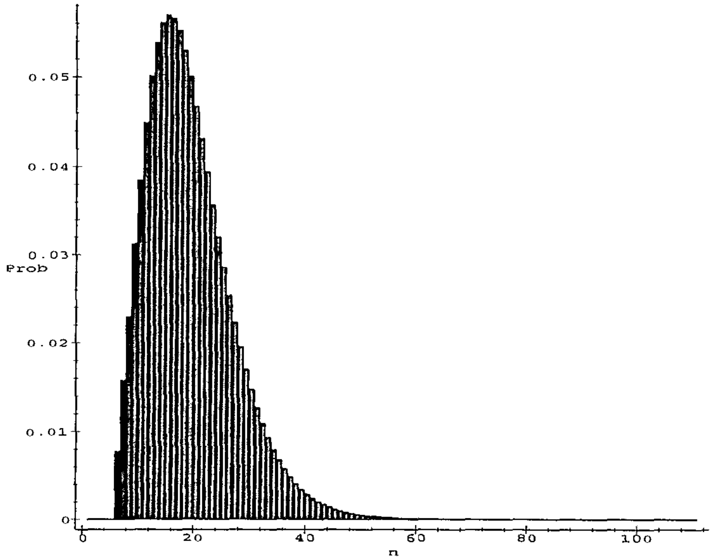

Fig. 1. The probability function of the sooner waiting time.

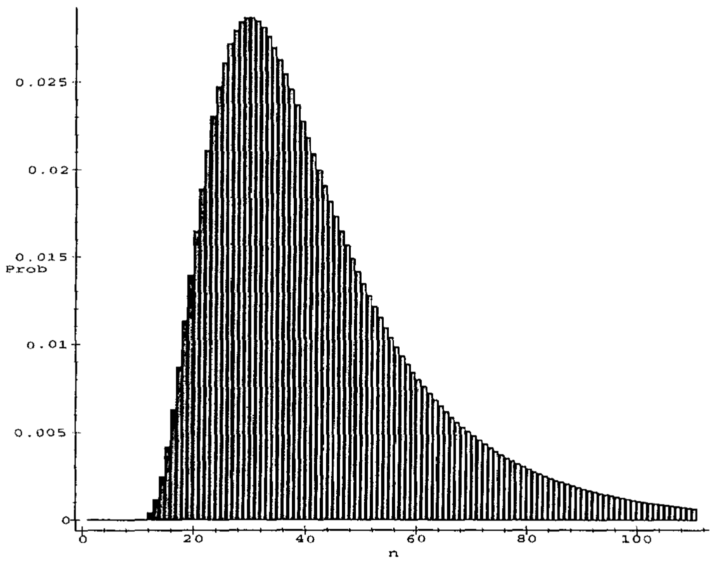

Fig. 2. The probability function of the later waiting time.

Because these results are rational functions, we can easily expand them (Stanley (1986), p. 205) and obtain the coefficient of z^t of $\psi_S(z)$, $(\psi_L(z))$ which is the probability of $\{W_S = t\}$ $(\{W_L = t\})$ $(t = 0, 1, \ldots)$. So, these results are suitable for numerical and symbolic calculations by computer algebra systems. For example, by using Maple V software, we can get the probability functions of the sooner and later waiting time. In Fig. 1 and 2, we give the bar graphs of the probability functions of the sooner and later waiting times between F_0 and F_1, where F_0 is the event that $l_0 = 3$ 0-runs of length $k_0 = 2$ and F_1 is the event $l_1 = 2$ 1-runs of length $k_1 = 3$, in a Markov chain $p_0 = 0.5$, $p_{00} = 0.35$, $p_{10} = 0.6$.

This method is applicable not only to nonoverlapping runs but to runs obtained by other counting schemes as well. Needless to say, for these schemes the typical sequences of S_1, S_{11} and S_{01} (S_0, S_{00}, S_{10}) are different.

For the overlapping way of counting (Ling (1988)), the typical sequence of S_1, S_{11} and S_{01} is, respectively

$$S_1:\ \text{nothing}\ \|\ \overbrace{0\cdots0}^{0\le,\le k_0-1}\ |\underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0}|\overbrace{1\cdots1}^{k_1},$$

$$S_{11}:\ \text{1-run}\ \|\ \begin{cases} 1 \quad \text{or} \\ \overbrace{0\cdots0}^{1\le,\le k_0-1}\ |\underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0}|\overbrace{1\cdots1}^{k_1} \end{cases}$$

$$S_{01}:\ \text{0-run}\ \|\underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0}|\overbrace{1\cdots1}^{k_1}.$$

For the "greater than or equal" way of counting (Goldstein (1990)), the typical sequence of S_1, S_{11}, S_{01} is, respectively

$$S_1:\ \text{nothing}\ \|\ \overbrace{0\cdots0}^{0\le,\le k_0-1}\ |\underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0}|\overbrace{1\cdots1}^{\ge k_1},$$

$$S_{11}:\ \text{1-run}\ \|\ \overbrace{0\cdots0}^{1\le,\le k_0-1}\ |\underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0}|\overbrace{1\cdots1}^{\ge k_1},$$

$$S_{01}:\ \text{0-run}\ \|\underbrace{\overbrace{1\cdots1}^{1\le,\le k_1-1}\ \ \overbrace{0\cdots0}^{1\le,\le k_0-1}}_{repeat\ i\ge0}|\overbrace{1\cdots1}^{\ge k_1}.$$

But, the last part of the last run is only $\overbrace{1\cdots1}^{k_1}$.

3. Markov chain imbedding method

Recently, Fu and Koutras (1994) developed a unified method for run statistics by using a Markov chain imbedding technique. Koutras and Alexandrou (1995) introduced a Markov chain imbeddable variable of Binomial type (M.V.B.) and obtained the distributions of several run statistics and scan statistics. Koutras and Alexandrou (1997) and Koutras (1997) investigated the waiting time problems by using the method. Han and Aki (1999) extended the concept of the M.V.B. to a Markov chain imbeddable vector of multinomial type and a Markov chain imbeddable vector of returnable type and obtained the joint distributions of some run statistics. Doi and Yamamoto (1998) and Koutras and Alexandrou (1998) discussed joint distributions of runs via the Markov chain imbedding method.

In this section, using the Markov chain imbedding method, we consider the sooner and later waiting time problems between F_0 and F_1 in a two-state Markov chain.

First, let us give a brief outline of the aforemetioned Markov chain imbedding technique and some of its machinery which is essential in the derivation of our results.

A $(m+1)$-dimensional random vector $\boldsymbol{X_n}$ is called a Markov chain imbeddable vector of multinomial type (M.V.M.), if

(1) there exists a Markov chain $\{Z_t, t \geq 0\}$ defined on a state space Ω,

(2) there exists a partition $\{\boldsymbol{U_x} : \boldsymbol{x} \geq \boldsymbol{0}\}$ on the state space Ω,

(3) for every $\boldsymbol{x}$, $\Pr(\boldsymbol{X}_n = \boldsymbol{x}) = \Pr(Z_t \in \boldsymbol{U_x})$

(4) $\Pr(Z_t \in \boldsymbol{U}_{\boldsymbol{x}+\boldsymbol{x}^*} \mid Z_{t-1} \in \boldsymbol{U_x}) = 0$, if $\boldsymbol{x}^* \neq \boldsymbol{0}$, or $\boldsymbol{x}^* \neq \boldsymbol{e}_k$ $(k = 0, 1, \ldots, m)$.

Without loss of generality, we assume that the sets $\boldsymbol{U_x}$ have common cardinality $s = |\boldsymbol{U_x}|$ for every $\boldsymbol{x}$, so we denote $\boldsymbol{U_x} = \{U_{\boldsymbol{x},1}, \ldots, U_{\boldsymbol{x},s}\}$. For the Markov chain $\{Z_t, t \geq 0\}$, we introduce the $s \times s$ transition probability matrices

$$A_t(\boldsymbol{x}) = (\Pr(Z_t = U_{\boldsymbol{x},j} \mid Z_{t-1} = U_{\boldsymbol{x},i}))_{s\times s},$$
$$B_t^{(k)}(\boldsymbol{x}) = (\Pr(Z_t = U_{\boldsymbol{x}+e_k,j} \mid Z_{t-1} = U_{\boldsymbol{x},i}))_{s\times s}, \qquad (k = 0, 1, \ldots, m),$$

the probability vectors of the t-th step Z_t of the Markov chain

$$\boldsymbol{f}_t(\boldsymbol{x}) = (\Pr(Z_t = U_{\boldsymbol{x},1}), \ldots, \Pr(Z_t = U_{\boldsymbol{x},s})), \boldsymbol{x} \geq \boldsymbol{0}, \qquad (t = 0, 1, \ldots),$$

and the probability generating functions

$$\varphi_t(z_0, z_1, \ldots, z_m) = \sum_{\boldsymbol{x}\geq \boldsymbol{0}} \boldsymbol{f}_t(\boldsymbol{x}) z_0^{x_0} z_1^{x_1} \cdots z_m^{x_m}.$$

If $A_t(\boldsymbol{x})$, $B_t^{(k)}(\boldsymbol{x})$ do not depend on $\boldsymbol{x}$, i.e. $A_t(\boldsymbol{x}) = A_t$, $B_t^{(k)}(\boldsymbol{x}) = B_t^{(k)}$, $(k = 0, 1, \ldots, m)$ for all $\boldsymbol{x}$, we have

$$\begin{aligned}\varphi_t(z_0, z_1, \ldots, z_m) &= \varphi_{t-1}(z_0, z_1, \ldots, z_m)\left(A_t + \sum_{k=0}^{m} z_k B_t^{(k)}\right) \\ &= \varphi_0(z_0, z_1, \ldots, z_m)\prod_{l=1}^{t}\left(A_l + \sum_{k=0}^{m} z_k B_l^{(k)}\right).\end{aligned}$$

For the homegenous case (i.e. A_t, $B_t^{(k)}$ do not depend on t), the double generating function is

$$\begin{aligned}\phi(z_0, z_1, \ldots, z_m, z) &= \sum_{t=0}^{\infty} \varphi_t(z_0, z_1, \ldots, z_m) z^t = \sum_{t=0}^{\infty}\sum_{\boldsymbol{x}\geq\boldsymbol{0}} \boldsymbol{f}_t(\boldsymbol{x}) z_0^{x_0} z_1^{x_1} \cdots z_m^{x_m} z^t \\ &= \varphi_0(z_0, z_1, \ldots, z_m)\left[I - z\left(A + \sum_{k=0}^{m} z_k B^{(k)}\right)\right]^{-1}.\end{aligned}$$

Here we are dealing with a two-state Markov chain and therefore $m = 1$.

For the nonoverlapping runs case, we establish a proper Markov chain $\{Z_t = (x_0, x_1; y_0, y_1) \mid t \geq 0\}$, where x_i $(i = 0, 1)$ is the number of "i"-run of length k_i up to the t-th trial, y_i^* is the number of trailing "i", (i.e. the number of last consecutive "i" counting backwards), and

$$y_i = \begin{cases} y_i^* - \left[\dfrac{y_i^*}{k_i}\right] k_i & \text{if } \; y_i^* \neq k_i, 2k_i, \ldots \\ -1 & \text{if } \; y_i^* = k_i, 2k_i, \ldots. \end{cases}$$

Let $\{Z_0 = (0, 0; 0, 0)\}$. The state space is

$$\Omega = \{(x_0, x_1; y_0, y_1) \mid x_i \geq 0, -1 \leq y_i \leq k_i - 1, i = 0, 1\}.$$

The partition is $\boldsymbol{U_x} = \{(\boldsymbol{x}; y_0, y_1) \mid -1 \leq y_i \leq k_i - 1, i = 0, 1\}$, and $\Omega = \cup_{\boldsymbol{x} \geq 0} \boldsymbol{U_x}$. We let $\boldsymbol{U_x} = \{(\boldsymbol{x}; 0, 0), (\boldsymbol{x}; 0, 1), \ldots, (\boldsymbol{x}; 0, k_1 - 1), (\boldsymbol{x}; 0, -1), (\boldsymbol{x}; 1, 0), \ldots, (\boldsymbol{x}; k_0 - 1, 0), (\boldsymbol{x}; -1, 0)\}$. We have $s = k_1 + k_0 + 1$,

$$\boldsymbol{f}_0(\boldsymbol{x}) = \begin{cases} (1, 0, \ldots, 0) & \text{if } \boldsymbol{x} = \boldsymbol{0} \\ (0, 0, \ldots, 0) & \text{if } \boldsymbol{x} \neq \boldsymbol{0}, \end{cases}$$

and $\varphi_0(\boldsymbol{z}) = (1, 0, \ldots, 0)$. We have

$$A + z_0 B^{(0)} + z_1 B^{(1)} =$$
$$\left(\begin{array}{ccccc|cccc}
(\boldsymbol{x};0,0) & (;0,1) & (;0,2) & \cdots & (;0,\text{-}1) & (;1,0) & (;2,0) & \cdots & (;\text{-}1,0) \\
0 & p_1 & 0 & \cdots & 0 & p_0 & 0 & \cdots & 0 \\
0 & 0 & p_{11} & \cdots & 0 & p_{10} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & p_{10} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & z_1 p_{11} & p_{10} & 0 & \cdots & 0 \\
0 & p_{11} & 0 & \cdots & 0 & p_{10} & 0 & \cdots & 0 \\
\hline
0 & p_{01} & 0 & \cdots & 0 & 0 & p_{00} & \cdots & 0 \\
0 & p_{01} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & p_{01} & 0 & \cdots & 0 & 0 & 0 & \cdots & z_0 p_{00} \\
0 & p_{01} & 0 & \cdots & 0 & p_{00} & 0 & \cdots & 0
\end{array}\right).$$

With this set up, the sooner waiting time W_S between F_0 and F_1 has

$$\{W_S = t\} = \cup_{j_1=0}^{l_1-1} \{Z_t = (l_0, j_1; -1, 0)\} \cup \cup_{j_0=0}^{l_0-1} \{Z_t = (j_0, l_1; 0, -1)\}.$$

We consider the waiting time of the state $(j_0, j_1; -1, 0)$. The state $Z_t = (j_0, j_1; -1, 0)$ denotes that a 0-run has just occurred at the t-th trial X_t, and there are j_0 0-runs and j_1 1-runs until the t-th trial X_t. So, the waiting time of the state $(j_0, j_1; -1, 0)$ is just $W_0(j_0, j_1)$ in Section 2. The probability function of $W_0(j_0, j_1)$ is

$$\begin{aligned} \Pr(W_0(j_0, j_1) = t) &= \Pr(Z_t = (j_0, j_1; -1, 0)) = \boldsymbol{f}_t((j_0, j_1)) \cdot (0, \ldots, 0, 1)' \\ &= \Pr(Z_{t-k_0} \in \boldsymbol{U}_{(j_0-1, j_1)}) \cdot [(1, 0, \ldots, 0)' p_0 \\ &\qquad + (0, \overbrace{1, \ldots, 1}^{k_1}, \overbrace{0, \ldots, 0}^{k_0})' p_{10} \\ &\qquad + (0, \ldots, 0, 1)' p_{00}] \cdot p_{00}^{k_0 - 1}, \end{aligned}$$

where $\boldsymbol{f}_t((j_0, j_1))$ and $\Pr(Z_{t-k_0} \in \boldsymbol{U}_{(j_0-1, j_1)}) = \boldsymbol{f}_{t-k_0}((j_0 - 1, j_1)) \cdot (1, \ldots, 1)'$ can be easily obtained from the joint distributions of 0-runs of length k_0 and 1-runs of length k_1 (see Han and Aki (1999) and Fu (1996)). Hence, we obtain the probability functions of the waiting times $W_0(j_0, j_1)$ and W_S, W_L. Further, we can discuss and extend the results to r-state Markov chain ($r \geq 2$) by using the same method.

In the rest of the section, we derive the explicit formulas of the p.g.f.s of W_S and W_L by using "Snake Oil" method (Wilf (1994)).

The probability generating function of $W_0(j_0, j_1)$ is

$$\phi^{(0)}_{j_0,j_1}(z) = \sum_{t=0}^{\infty} \Pr(W_0(j_0,j_1)=t) z^t = \sum_{t=0}^{\infty} \boldsymbol{f}_t((j_0,j_1)) z^t \cdot (0,\ldots,0,1)'$$
$$= \text{the coefficient of } z_0^{j_0} z_1^{j_1} \text{ of } \boldsymbol{\phi}(z_0,z_1,z)(0,\ldots,0,1)'.$$

Because $\boldsymbol{\varphi}_0(z_0,z_1) = (1,0,\ldots,0)$, by algebraic manipulations, we have

$$\boldsymbol{\phi}(z_0,z_1,z) = (1, u, (zp_{11})u, \ldots, (zp_{11})^{k_1-2}u, z_1(zp_{11})^{k_1-1}u, v, (zp_{00})v, \ldots, (zp_{00})^{k_0-2}, z_0(zp_{00})^{k_0-1}v),$$

where

$$u = \frac{zp_1(1-z_0(zp_{00})^{k_0}) + (zp_0)[a_0 + z_0(zp_{00})^{k_0-1}(zp_{01})]}{(1-z_1(zp_{11})^{k_1})(1-z_0(zp_{00})^{k_0}) - [a_0+z_0(zp_{00})^{k_0-1}(zp_{01})][a_1+z_1(zp_{11})^{k_1-1}(zp_{10})]},$$
$$v = \frac{(zp_0)}{1-z_0(zp_{00})^{k_0}} + \frac{[a_1 + z_1(zp_{11})^{k_1-1}(zp_{10})]}{1-z_0(zp_{00})^{k_0}} \cdot u.$$

Hence,

$$\phi^{(0)}_{j_0,j_1}(z) = \text{ the coefficient of } z_0^{j_0} z_1^{j_1} \text{ of } z_0(zp_{00})^{k_0-1}v.$$

Because $z_0(zp_{00})^{k_0-1}v = \frac{\alpha}{\beta}$, and $\beta = (1-a_0a_1)[1 - z_0w_0 - z_1w_1 - z_0z_1w_{01}]$, $\alpha = z_0(zp_{00})^{k_0-1}[zp_0 + zp_1a_1] + z_0z_1(zp_{00})^{k_0-1}(zp_{11})^{k_1-1}[zp_1zp_{10} - zp_0zp_{11}]$, where

$$w_0 = \frac{(zp_{00})^{k_0-1}[zp_{00} + zp_{01}a_1]}{1-a_0a_1},$$
$$w_1 = \frac{(zp_{11})^{k_1-1}[zp_{11} + zp_{10}a_0]}{1-a_0a_1},$$
$$w_{01} = \frac{(zp_{00})^{k_0-1}(zp_{11})^{k_1-1}[zp_{01}zp_{10} - zp_{00}zp_{11}]}{1-a_0a_1},$$

we have

$$z_0(zp_{00})^{k_0-1}v = \alpha \cdot \frac{1}{1-a_0a_1}\left\{\sum_{l=0}^{\infty}[z_0w_0 + z_1w_1 + z_0z_1w_{01}]^l\right\}.$$

Because the coefficient of $z_0^i z_1^j$ of $\{\sum_{l=0}^{\infty}[z_0w_0+z_1w_1+z_0z_1w_{01}]^l\}$ is $\sum_{l=\max\{i,j\}}^{i+j}\binom{l}{j}\binom{j}{i+j-l}$ $w_0^{l-j}w_1^{l-i}w_{01}^{i+j-l}$, we have

Proposition 3.1. *The p.g.f. of the waiting time* $W_0(j_0,j_1)$ *is*

$$\phi^{(0)}_{j_0,j_1}(z) = \frac{[zp_0 + zp_1a_1](zp_{00})^{k_0-1}}{1-a_0a_1}\left\{\sum_{l=\max\{j_0-1,j_1\}}^{j_0+j_1-1}\binom{l}{j_1}\binom{j_1}{j_0-1+j_1-l} \times w_0^{l-j_1}w_1^{l-j_0+1}w_{01}^{j_0+j_1-1-l}\right\}$$
$$+ \frac{(zp_{11})^{k_1-1}(zp_{00})^{k_0-1}[zp_1zp_{10} - zp_0zp_{11}]}{1-a_0a_1} \times \left\{\sum_{l=\max\{j_0-1,j_1-1\}}^{j_0+j_1-2}\binom{l}{j_1-1}\binom{j_1-1}{j_0+j_1-2-l}w_0^{l-j_1+1}w_1^{l-j_0+1}w_{01}^{j_0+j_1-2-l}\right\}.$$

We can obtain the p.g.f. $\phi^{(1)}_{j_0,j_1}(z)$ of the waiting time $W_1(j_0, j_1)$, by interchanging "0" and "1" in Proposition 3.1. Hence, we obtain

THEOREM 3.1. *The p.g.f. of the sooner waiting time* W_S *is*

$$\psi_S(z) = \sum_{j_1=0}^{l_1-1} \phi^{(0)}_{l_0,j_1}(z) + \sum_{j_0}^{l_0-1} \phi^{(1)}_{j_0,l_1}(z),$$

where $\phi^{(0)}_{l_0,j_1}(z)$ *and* $\phi^{(1)}_{j_0,l_1}(z)$ *are given by Proposition* 3.1.

Remark 3.1. Theorems 2.1 and 3.1 are different formulas of the same problem. Theorem 2.1 has a clear combinatorial meaning, but it is difficult to extend to other problems. Theorem 3.1 (the Markov chain imbedding method) has a great potential for extending to other problem, for example, the higher-order Markov chain case, the multi-state Markov chain case, the waiting time problems of pattern and etc.

Remark 3.2. (1) Let $l_0 = 1$ and $l_1 = 1$, Theorem 3.1 is similar to Theorem 1 of Aki and Hirano (1993).

(2) In i.i.d. case, we have $p_0 = p_{00} = p_{10}$, $p_1 = p_{01} = p_{11}$ and $w_{01} = 0$. Defining $0^0 = 1$, Theorem 3.1 reduces Theorem 6.1 of Han and Aki (1999).

Then, we consider the waiting time $W_0(l_0, \bullet)$ of the l_0-th 0-run of length k_0.

$$\begin{aligned}\phi^{(0)}_{l_0,\bullet}(z) &= \sum_{t=0}^{\infty} \Pr(W_0(l_0,\bullet) = t) z^t = \sum_{t=0}^{\infty}\sum_{j_1=0}^{\infty} \boldsymbol{f}_t((l_0, j_1)) z^t (0, \dots, 0, 1)' \\ &= \text{the coefficient of } z_0^{l_0} \text{ of } \boldsymbol{\phi}(z_0, z_1 = 1, z)(0, \dots, 0, 1)' \\ &= \text{the coefficient of } z_0^{l_0} \text{ of } z_0 (zp_{00})^{k_0 - 1} \cdot v\,|_{z_1=1}\,.\end{aligned}$$

Hence, we obtain

PROPOSITION 3.2. *The p.g.f. of* $W_0(l_0, \bullet)$ *is*

$$\begin{aligned}\phi^{(0)}_{l_0,\bullet}(z) = (zp_{00})^{l_0(k_0-1)} &\frac{(zp_0)[1 - (zp_{11})^{k_1}] + (zp_1)[a_1 + (zp_{11})^{k_1-1}(zp_{10})]}{(1 - a_0 a_1 - (zp_{11})^{k_1-1}[(zp_{11}) + (zp_{10})a_0])^{l_0}} \\ &\times [(zp_{00})[1 - (zp_{11})^{k_1}] + (zp_{01})[a_1 + (zp_{11})^{k_1-1}(zp_{10})]]^{l_0-1}.\end{aligned}$$

Similarly, we can obtain the p.g.f. $\phi^{(1)}_{\bullet,l_1}(z)$ of $W_1(\bullet, l_1)$. Because $b_0 = \frac{a_0 + (zp_{00})^{k_0-1}(zp_{01})}{1-(zp_{00})^{k_0}}$, it is just same as Proposition 2.2. Hence, we have

THEOREM 3.2. *The p.g.f. of the later waiting time* W_L *is* $\psi_L(z) = \phi^{(1)}_{\bullet,l_1}(z) + \phi^{(0)}_{l_0,\bullet}(z) - \psi_S(z)$.

Similarly, for overlapping run, "greater than or equal" run and "just equal" run, we can derive the p.g.f.s of W_S and W_L by establishing proper Markov chains.

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