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LINEX UNBIASEDNESS IN A PREDICTION PROBLEM

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Abstract. A statistical prediction problem under LINEX loss function is considered. Some results about LINEX-unbiased predictor are derived and the best LINEX-unbiased predictor is given. We also show that the best risk-unbiased predictor is equal to the best equivariant predictor in the location family.

Key words and phrases: Prediction problem, risk unbiasedness, LINEX loss function, adequate statistic, Rao-Blackwell theorem, location family, equivariant predictor.

1. Introduction

In some estimation problems it may be appropriate to use asymmetric loss function. Varian (1975) proposed a very useful asymmetric loss function which is called a LINEX (Linear-Exponential) loss function, though the loss function was first considered by Klebanov (1974) in the different context. Zellner (1986) showed that the sample mean is inadmissible for estimating the mean of a univariate normal distribution with respect to the LINEX loss function. Shafie and Noorbaloochi (1995) extended the inadmissibility result to the location family. Considering the concept of risk unbiasedness due to Lehmann (1951), Klebanov (1976) showed that the well-known Rao-Blackwell theorem holds under a LINEX loss function.

In this paper we discuss a prediction problem under a LINEX loss function. Introducing a concept of risk unbiasedness to the prediction problem, we consider some properties of risk-unbiased predictors, especially under a LINEX loss function. Section 2 is devoted to discuss risk-unbiased predictors. In Section 3 the Rao-blackwell theorem is shown to hold under a LINEX loss function, using an adequate statistic which plays the role of a sufficient statistic in usual estimation problems. An adequate statistic is introduced by Skibinsky (1967). See also Takeuchi and Akahira (1975) for further discussion. The best risk unbiased predictor under the LINEX loss function is also considered. In Section 4 it is shown that the unique best risk-unbiased predictor coincides with the best equivariant predictor under the location family.

2. Risk unbiased predictor

Suppose that X is observed random vector and Y a future real random variable, and the joint distribution of X and Y depends on an unknown parameter θ. After observing $X = x$, we want to predict the value of Y. A non-negative loss function $L(d, y)$ represents the loss of predicting $Y = y$ by d. Let $\delta(X)$ be a predictor of Y and

$$R(\theta, \delta) = E_\theta \{L(\delta(X), Y)\}$$

be the risk function.

DEFINITION 1. If a predictor $\delta(X)$ satisfies

$$E_\theta\{L(\delta(X), Y)\} = \min_c E_\theta\{L(\delta(X), Y + c)\}, \tag{2.1}$$

where c is a real number, then it is called a risk-unbiased predictor. If a risk-unbiased predictor minimizes the risk for all values of θ, then it is called the best risk-unbiased predictor.

Now we shall give a result which characterizes the risk-unbiased predictor (cf. Lemma 1 of Klebanov (1974)).

THEOREM 1. *Let $L(d,y)$ be a loss function which is twice continuous differentiable and convex in its second argument for each fixed value of the first. Suppose the predictor δ is such that for all θ*

$$E_\theta\left\{\sup_{|c|\le\varepsilon}\frac{\partial^2 L}{\partial y^2}(\delta(X), Y + c)\right\} < \infty, \tag{2.2}$$

where ε is some positive number. In order for δ to be risk-unbiased, it is necessary and sufficient that for all θ

$$E_\theta\left\{\frac{\partial L}{\partial y}(\delta(X), Y)\right\} = 0. \tag{2.3}$$

PROOF. Let δ be a risk-unbiased predictor of Y. Then for any c satisfying $|c| \le \varepsilon$, we have

$$\begin{aligned} &E_\theta\{L(\delta(X), Y + c)\} - E_\theta\{L(\delta(X), Y)\} \\ &\quad = cE_\theta\left\{\frac{\partial L}{\partial y}(\delta(X), Y)\right\} + \frac{1}{2}c^2 E_\theta\left\{\frac{\partial^2 L}{\partial y^2}(\delta(X), Y + v(X,Y)c)\right\}, \end{aligned}$$

where $0 \le v(X,Y) \le 1$. From the last equality and condition (2.2) it is apparent that for sufficiently small c (2.1) can hold only if (2.3) is satisfied.

Suppose that relation (2.3) holds. Then we have

$$\begin{aligned} &E_\theta\{L(\delta(X), Y + c)\} - E_\theta\{L(\delta(X), Y)\} \\ &\quad = \frac{1}{2}c^2 E_\theta\left\{\frac{\partial^2 L}{\partial y^2}(\delta(X), Y + v(X,Y)c)\right\} \ge 0, \end{aligned}$$

by virtue of the convexity of the function L in its second argument. Hence the proof is completed.

The following corollary easily follows from Theorem 1.

COROLLARY 1. *If $L(d,y) = (d-y)^2$, then $\delta(X)$ is risk-unbiased if and only if*

$$E_\theta\delta(X) = E_\theta Y. \tag{2.4}$$

Under a squared error loss the risk-unbiased predictor is called mean-unbiased, though it is the usual unbiased predictor. A LINEX loss function is defined by

$$L(d,y) = \exp[\alpha(d-y)] - \alpha(d-y) - 1, \qquad \alpha \ne 0.$$

A risk-unbiased predictor with respect to the LINEX loss is called LINEX-unbiased. The following corollary is easily obtained from Theorem 1.

COROLLARY 2. *Suppose that* $\delta(X)$ *satisfies*

$$E_\theta\left\{\exp[\alpha(\delta(X)-Y)]\right\} < \infty, \tag{2.5}$$

for all θ. *Then* $\delta(X)$ *is LINEX-unbiased if and only if*

$$E_\theta\left\{\exp[\alpha(\delta(X)-Y)]\right\} = 1 \tag{2.6}$$

for all θ.

THEOREM 2. *If* δ *is a LINEX-unbiased and satisfies* (2.5), *then it can not be mean-unbiased, unless* $P_\theta(\delta(X)=Y)=1$. *The converse is also true.*

PROOF. Suppose that δ is also mean-unbiased. Then it follows from (2.4) and (2.6) that

$$R(\theta,\delta) = E_\theta\left\{\exp[\alpha(\delta(X)-Y)] - \alpha(\delta(X)-Y) - 1\right\} = 0,$$

which implies $P_\theta(\delta(X)=Y)=1$. Hence δ can not be mean-unbiased unless $P_\theta(\delta(X)=Y)=1$. The converse is similar.

3. LINEX-unbiased predictor

In this section consideration is devoted to LINEX unbiased predictors. Now we shall give the definition of an adequate statistic which plays an important role in a prediction problem, like a sufficient statistic in an estimation problem. See Skibinsky (1967), Takeuchi and Akahira (1975), and Takada (1981).

DEFINITION 2. *A statistic* $T=T(X)$ *is said to be adequate if given* T, X *and* Y *are conditionally independent, and* T *is sufficient for the family of distributions of* X.

The following theorem, which was obtained by Sugiura and Morimoto (1969), is useful to get an adequate statistic. See also Takeuchi ((1975), p. 138).

THEOREM 3. *Let the joint density function of* X *and* Y *be* $f(x,y\mid\theta)$. *Then a necessary and sufficient condition for a statistic* $T(X)$ *to be adequate is that there exist non-negative functions* h *and* g *such that*

$$f(x,y\mid\theta) = h(x)g(T(x),y,\theta) \qquad a.e.$$

It is well known that if $\delta(X)$ is mean-unbiased and T is adequate, then

$$\delta^*(T) = E(\delta(X)\mid T), \tag{3.1}$$

is also mean-unbiased and the risk of δ^* is less than or equal to that of δ (Rao-Blackwell theorem). We show that the similar results holds under the loss funtion

$$L(d,y) = \psi(d)A(y) + B(y) + \phi(d), \tag{3.2}$$

where $A(\cdot)$ and $B(\cdot)$ are convex, twice differentiable functions, $\psi(\cdot)$ (> 0) is a strictly monotone function and $\phi(\psi^{-1}(\cdot))$ is a convex function (cf. Klevanov (1974)). The LINEX loss function is of the form. For example we can take

$$L(d,y) = \frac{y}{d} - \log\frac{y}{d} - 1,$$

for $d > 0$ and $y > 0$.

THEOREM 4. *If δ is risk-unbiased with respect to (3.2) and satisfies (2.2), then*

$$\delta^*(T) = \psi^{-1}(E[\psi(\delta(X)) \mid T]) \tag{3.3}$$

also satisfies (2.2) and is risk-unbiased. The risk of δ^* is less than or equal to that of δ.

PROOF. The condition (2.2) is equivalent to

$$E_\theta\left\{\psi(\delta(X)) \sup_{|c|\le\varepsilon} A''(Y+c)\right\} < \infty,$$

and

$$E_\theta\left\{\sup_{|c|\le\varepsilon} B''(Y+c)\right\} < \infty.$$

Since T is adequate,

$$\begin{aligned} E_\theta\left\{\psi(\delta^*(T)) \sup_{|c|\le\varepsilon} A''(Y+c)\right\} &= E_\theta\left\{E(\psi(\delta(X)) \mid T) \sup_{|c|\le\varepsilon} A''(Y+c)\right\} \\ &= E_\theta\left\{\psi(\delta(X)) \sup_{|c|\le\varepsilon} A''(Y+c)\right\}. \end{aligned}$$

Hence (2.2) is satisfied for δ^* in (3.3). So from Theorem 1 it is enough to show that δ^* satisfies (2.3).

By the risk-unbiasedness of δ and (2.3), we have

$$\begin{aligned} E_\theta\left\{\frac{\partial}{\partial y} L(\delta^*(T), Y)\right\} &= E_\theta\left\{\psi(\delta^*(T))A'(Y) + B'(Y)\right\} \\ &= E_\theta\left\{E(\psi(\delta(X)) \mid T)A'(Y) + B'(Y)\right\} \\ &= E_\theta\left\{E(\psi(\delta(X))A'(Y) \mid T) + B'(Y)\right\} \\ &= E_\theta\left\{\psi(\delta(X))A'(Y) + B'(Y)\right\} \\ &= E_\theta\left\{\frac{\partial}{\partial y} L(\delta(X), Y)\right\} \\ &= 0. \end{aligned}$$

Hence δ^* is risk-unbiased.

Next, by Jensen's inequality we have

$$\begin{aligned} \phi(\delta^*(T)) &= \phi(\psi^{-1}(E[\psi(\delta(X)) \mid T])) \\ &\le E(\phi(\delta(X)) \mid T) \qquad \text{a.e.,} \end{aligned}$$

which follows

$$E\phi(\delta^*(T)) \leq E\phi(\delta(X)).$$

Therefore

$$\begin{aligned} R(\theta, \delta^*) &= E_\theta\{\psi(\delta^*(T))A(Y) + B(Y) + \phi(\delta^*(T))\} \\ &= E_\theta\{E(\psi(\delta(X)) \mid T)A(Y) + B(Y) + \phi(\delta^*(T))\} \\ &= E_\theta\{\psi(\delta(X))A(Y) + B(Y) + \phi(\delta^*(T))\} \\ &\leq E_\theta\{\psi(\delta(X))A(Y) + B(Y) + \phi(\delta(X))\} \\ &= R(\theta, \delta), \end{aligned}$$

which completes the proof.

COROLLARY 3. *If δ is LINEX-unbiased and satisfies* (2.5), *then*

$$\delta^*(T) = \frac{1}{\alpha} \log E[\exp(\alpha\delta(X)) \mid T] \tag{3.4}$$

also satisfies (2.5) *and is* LINEX-*unbiased. The risk of δ^* is less than or equal to that of δ.*

If T is adequate and complete, then $\delta^*(T)$ in (3.1) is the unique (up to the sets of measure zero) best mean-unbiased predictor. Under some conditions a similar result holds for the risk-unbiased predictor in (3.3).

THEOREM 5. *If T is adequate and complete, and*

$$E_\theta(A'(Y) \mid T) = Q(\theta)K(T) \qquad a.e. \tag{3.5}$$

for some non-zero functions Q and K, then δ^* in (3.3) *is the unique best risk-unbiased predictor among the predictors satisfying* (2.2).

PROOF. From Theorem 4 it is enough to show that the predictor in (3.3) is the unique risk-unbiased predictor based on T. Let $\delta(T)$ be the other risk-unbiased predictor based on T and satisfy (2.2). Then from (2.3)

$$E_\theta\left\{\frac{\partial}{\partial y}L(\delta^*(T), Y)\right\} = E_\theta\left\{\frac{\partial}{\partial y}L(\delta(T), Y)\right\}.$$

So we have

$$E_\theta\left\{\psi(\delta^*(T))A'(Y)\right\} = E_\theta\left\{\psi(\delta(T))A'(Y)\right\},$$

which implies

$$E_\theta\left\{\psi(\delta^*(T))E_\theta(A'(Y) \mid T)\right\} = E_\theta\left\{\psi(\delta(T))E_\theta(A'(Y) \mid T)\right\}.$$

Hence from (3.5)

$$Q(\theta)E_\theta\left\{K(T)\psi(\delta^*(T))\right\} = Q(\theta)E_\theta\left\{K(T)\psi(\delta(T))\right\}.$$

So from the completness of T and the strict monotone of ψ we have

$$P_\theta(\delta^*(T) = \delta(T)) = 1,$$

which completes the proof.

Example 3.1. Let $X_1, \ldots, X_n$ be iid according to Poisson distribution with mean θ. Based on $X_1, \ldots, X_m$ $(m < n)$, we want to predict the total sum $Y = \sum_{i=1}^n X_i$. It follows from Theorem 3 that $\bar{X} = \frac{1}{m}\sum_{i=1}^m X_i$ is adequate. $\bar{X}$ is also complete. So it is easy to see that $n\bar{X}$ is the best mean-unbiased predictor. It is also shown that the condition (3.5) holds and hence the best LINEX-unbiased predictor is given by $c\bar{X}$ with $c = m\{1 + \frac{1}{\alpha}\log[1 + (\frac{n}{m} - 1)(1 - \exp(-\alpha))]\}$.

Example 3.2. Let $X_1, \ldots, X_n$ be iid according to $N(\mu, \sigma^2)$, and let Y be also distributed according to $N(\mu, \sigma^2)$ and the covariance between Y and X_i be $\rho\sigma^2 (i = 1, \ldots, n)$, where ρ is known and $\rho^2 < \frac{1}{n}$. Based on $X = (X_1, \ldots, X_n)$, we want to predict Y. First we assume that μ is unknown, but σ^2 is known. Then from Theorem 3 $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is adequate. Since $\bar{X}$ is complete, $\bar{X}$ is the best mean-unbiased predictor.

It is easy to see that the condition (3.5) is satisfied and hence the best LINEX-unbiased predictor is given by

$$\delta(X) = \bar{X} + \frac{1}{2}\left(2\rho - \frac{n+1}{n}\right)\alpha\sigma^2. \tag{3.6}$$

It turns out that the risk of the best mean-unbiased predictor $\bar{X}$ is larger than that of the best LINEX unbiased predictor under the LINEX loss, and the converse also holds under the squared error loss. In the following section we shall show that this result holds for the location family.

Next we assume that both μ and σ^2 are unknown. Then it is also shown that $T = (\bar{X}, S^2)$ is adequate and complete where $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2$. Hence $\bar{X}$ is the best mean-unbiased predictor. But there does not exist a LINEX-unbiased predictor, which is shown in the Appendix.

4. Risk-unbiasedness and equivariance

In this section we shall show that the unique best risk-unbiased predictor is equal to the best equivariant predictor in the location family.

Suppose that the joint density function of $X = (X_1, \ldots, X_n)$ and Y is given by

$$f(x - \theta, y - \theta),$$

and the loss function is of the form

$$L(d, y) = \rho(d - y),$$

where f and ρ are some known functions and $x - \theta = (x_1 - \theta, \ldots, x_n - \theta)$.

DEFINITION 3. A predictor $\delta(X)$ is said to be equivariant if

$$\delta(x + a) = \delta(x) + a$$

for any real number a.

It is easy to see that the risk of an equivariant predictor does not depend on θ. An equivariant predictor is said to be the best equivariant if it minimizes the risk among all equivariant predictors.

Now we shall give an expression of the best equivariant predictor. Though the expression follows from Theorem 2 of Takada (1982), we give more elementary proof for the sake of completeness.

THEOREM 6. *If $\delta^*(X)$ is the unique predictor which satisfies*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(\delta^*(X) - y) f(X-\theta, y-\theta)\,dy\,d\theta = \min_d \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(d-y) f(X-\theta, y-\theta)\,dy\,d\theta,$$

then $\delta^*(X)$ is the best equivariant.

PROOF. Let $Z = (Z_1, \ldots, Z_{n-1})$ with $Z_i = X_i - X_n$, $i = 1, \ldots, n-1$. Then it is easy to see that $\delta(X)$ is equivariant if and only if for some function w

$$\delta(X) = X_n + w(Z).$$

Since the risk of an equivariant predictor does not depend on θ,

$$R(\theta, \delta) = E_0\left\{E_0[\rho(X_n + w(Z) - Y) \mid Z]\right\}, \tag{4.1}$$

where E_0 denotes the expectation under $\theta = 0$. Since the joint conditional density function of X_n and Y given Z under $\theta = 0$ is

$$\frac{f(Z_1 + x_n, \ldots, Z_{n-1} + x_n, x_n, y)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(Z_1 + x_n, \ldots, Z_{n-1} + x_n, x_n, y)\,dx_n\,dy},$$

we have

$$E_0[\rho(X_n + w(Z) - Y) \mid Z] = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(u + w(Z) - v) f(Z_1 + u, \ldots, Z_{n-1} + u, u, v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(Z_1 + u, \ldots, Z_{n-1} + u, u, v)\,du\,dv}.$$

Making the change of variables from (u, v) to (θ, y) with $u = X_n - \theta$ and $v = y - \theta$, it follows

$$E_0[\rho(X_n + w(Z) - Y) \mid Z] = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(X_n + w(Z) - y) f(X-\theta, y-\theta)\,d\theta\,dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X-\theta, y-\theta)\,dy\,d\theta}.$$

Hence from (4.1) if $\delta(X)$ is equivariant and minimizes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(d-y) f(X-\theta, y-\theta)\,d\theta\,dy$$

with respect to d, then it is the best equivariant. Therefore it is enough to show that δ^* is equivariant, which follows from the uniqueness of δ^*.

COROLLARY 4. *Under the squared error loss, the best equivariant predictor is*

$$\delta^*(X) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(X-\theta, y-\theta)\,dy\,d\theta}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X-\theta, y-\theta)\,dy\,d\theta}.$$

COROLLARY 5. *Under the LINEX loss, the best equivariant predictor is*

$$\delta^*(X) = \frac{1}{\alpha}\log\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X-\theta, y-\theta)\,dy\,d\theta}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\alpha y) f(X-\theta, y-\theta)\,dy\,d\theta}.$$

Applying Corollaries 4 and 5 to Example 3.2, we have that $\bar{X}$ and $\delta(X)$ in (3.6) are the best equivariant predictors under the squared error loss and the LINEX loss, respectively. That is, the best risk-unbiased predictor coincides with the best equivariant predictor. We shall show that this result generally holds under the location family. To prove it, we need the following lemmas, the proofs of which are similar to those of Lehmann (1951). So the proofs are omitted.

LEMMA 1. *If δ^* is the unique (up to sets of measure zero) best risk-unbiased predictor, then it is almost equivariant.*

LEMMA 2. *If δ^* is the best equivariant, then it is risk-unbiased.*

THEOREM 7. *If there exists the unique (up to sets of measure zero) best risk-unbiased predictor, then it is equal (up to sets of measure zero) to the best equivariant predictor.*

PROOF. Suppose that δ^* is the best risk-unbiased predictor and $\tilde{\delta}$ is the best equivariant predictor. From Lemma 1 δ^* is almost equivariant. So there exists an equivariant predictor δ' such that $\delta^*(X) = \delta'(X)$, a.e.. See Berk and Bickel (1968). Hence

$$R(\theta, \delta^*) = R(\theta, \delta') \geq R(\theta, \tilde{\delta}). \tag{4.2}$$

From Lemma 2, $\tilde{\delta}$ is risk-unbiased. So

$$R(\theta, \tilde{\delta}) \geq R(\theta, \delta^*).$$

Hence from (4.2)

$$R(\theta, \delta^*) = R(\theta, \tilde{\delta}).$$

The uniqueness of δ^* implies

$$\delta^*(X) = \tilde{\delta}(X) \qquad \text{a.e.},$$

which completes the proof.

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Appendix

Suppose that there exists a LINEX-unbiased predictor δ such that

$$E_\theta \left\{\exp[\alpha(\delta(X)-Y)]\right\} = 1$$

for all $\theta = (\mu, \sigma^2)$. Since the conditional distribution of Y given X is $N(\mu + n\rho(\bar{X} - \mu), \sigma^2(1-n\rho^2))$, we have

$$E_\theta\{\exp[\alpha(\delta(X) - \mu - n\rho(\bar{X}-\mu))]\} = \exp\left(-\frac{\alpha^2\sigma^2}{2}(1-n\rho^2)\right).$$

In the sequel, we assume $\mu = 0$. Then

$$E_\theta\{\exp[\alpha(\delta(X) - n\rho\bar{X})]\} = \exp\left(-\frac{\alpha^2\sigma^2}{2}(1-n\rho^2)\right). \tag{A.1}$$

Let $V = \sum_{i=1}^n X_i^2$. Then V is complete and sufficient, and the density of V is $\frac{1}{\sigma^2} g(\frac{v}{\sigma^2})$, where $g(z)$ is the density function of a chi-squared random variable with n degrees of freedom. Let

$$\delta^*(V) = E\left\{\exp[\alpha(\delta(X) - n\rho\bar{X})] \mid V\right\}.$$

Then from (A.1)

$$E_\theta\delta^*(V) = \exp\left(-\frac{\alpha^2\sigma^2}{2}(1-n\rho^2)\right). \tag{A.2}$$

Multiplying both sides of (A.2) by $(\sigma^2)^r$ $(r > 0)$ and integrating with respect to σ^2 from zero to infinity, we get the following equation

$$\begin{aligned}\int_0^\infty (\sigma^2)^r \left\{\int_0^\infty \delta^*(v)\sigma^{-2} g\left(\frac{v}{\sigma^2}\right) dv\right\} d\sigma^2 \\ = \int_0^\infty (\sigma^2)^r \exp\left(-\frac{\alpha^2\sigma^2}{2}(1-n\rho^2)\right) d\sigma^2.\end{aligned} \tag{A.3}$$

The right hand side of (A.3) is finite if $r > 1$. By Fubini's theorem, the left hand side of (A.3) is equal to

$$\begin{aligned}&\int_0^\infty \delta^*(v)\left\{\int_0^\infty (\sigma^2)^{r-1} g\left(\frac{v}{\sigma^2}\right) d\sigma^2\right\} dv \\ &= \frac{1}{c}\int_0^\infty \delta^*(v)v^{n/2-1}\left\{\int_0^\infty (\sigma^2)^{r-n/2}\exp\left(-\frac{v}{2\sigma^2}\right) d\sigma^2\right\} dv \\ &= \frac{1}{c}\int_0^\infty \delta^*(v)v^{n/2-1}\left\{\int_0^\infty \tau^{n/2-r-2}\exp\left(-\frac{\tau v}{2}\right) d\tau\right\} dv,\end{aligned}$$

where $c = 2^{n/2}\Gamma(\frac{n}{2})$. Since δ^* is positive, this integral is infinite if $r > \frac{n}{2} - 1$, which contradicts to the fact that the right hand side of (A.3) is finite if $r > 1$. Hence there does not exist a LINEX-unbiased predictor when μ and σ^2 are unknown.

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