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HYPOTHESES TESTING FOR ERROR-IN-VARIABLES MODELS

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Abstract. In this paper, hypotheses testing based on a corrected score function are considered. Five different testing statistics are proposed and their asymptotic distributions are investigated. It is shown that the statistics are asymptotically distributed according to the chisquare distribution or can be written as a linear combination of chisquare random variables with one degree of freedom. A small scale numerical Monte Carlo study is presented in order to compare the empirical size and power of the proposed tests. A comparative calibration example is used to illustrate the results obtained.

Key words and phrases: Asymptotic tests, comparative calibration, consistent estimator, measurement error, naive test.

1. Introduction

Most studies in life sciences, biology, engineering, demography and economics involve covariates that can not be recorded exactly. Errors arise, most notably as measurement errors. Examples include a follow-up study of A-bomb survivors where the variable radiation received is measured with error (Okajima *et al.* (1985), Pierce *et al.* (1992)), amount of nitrogen in the soil in a study related to the yield of a certain grain (Fuller (1987)), biologic covariates, such as systolic blood pressure, daily intake of saturated fat in the famous Framingham Heart prospective study dealing with cardiovascular disease (Gordon and Kannel (1968)). Frequently, interests are on assessing the statistical relationship between the unobserved covariates and the response.

The present paper is primarily concerned with testing for association between the true covariates and the response variable. A simple approach considers the naive test obtained from substituting the unobserved covariates with the observed ones. Tosteson and Tsiatis (1988) have compared the local power, assuming a general measurement error structure, of the naive score test and the optimal score test obtained by a flexible procedure in generalized linear models. Lagakos (1988) has also computed the efficiency loss for naive tests in univariate linear, Cox and logistic regression models. Stefanski and Carroll (1990) have considered Wald tests. They have compared the naive Wald and a corrected Wald test (Stefanski (1985)) assuming an additive measurement error structure.

Nakamura (1990) introduced an approach which allows the derivation of consistent and asymptotically normal estimators for the parameters of a linear or nonlinear measurement errors-in-variables model. Additional results on corrected score functions are established by Gimenez and Bolfarine (1997). We recall that most of the approaches considered for estimation in such models produces only approximate unbiased estimates, with no formal theoretical justification, such as the regression calibration (Carroll and Stefanski (1990)) or James-Stein (Whittemore (1989)) type estimators. These less biased estimators are used to avoid the attenuation problem typically associated with the naive or ordinary regression estimators. Resampling techniques are then required for obtaining the estimated standard errors associated with such estimates, making it difficult to obtain general valid asymptotic results to be used in conjunction with such estimators. Nakamura's approach allows its use in more general situations without making assumptions concerning the true covariates, having associated general expressions for the asymptotic covariance matrix. The main object of the paper is to derive asymptotic valid tests (Carroll *et al.* (1995), p. 207) for some measurement error models, which are validated by the asymptotic distributions associated with the procedures. A review of the approach is considered in Section 2. In Section 3 the asymptotic tests are formally obtained by using the asymptotic properties of the estimators. Wald, score and likelihood type statistics are proposed. A small scale numerical study is presented in Section 4 for comparing the asymptotic tests. The applicability of these results is illustrated in a comparative calibration model in Section 5.

2. Corrected score estimator approach

Let $\boldsymbol{Z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}'_n)'$ denote the matrix of independent variables (covariates), $\boldsymbol{Y} = (y_1, \ldots, y_n)'$ the vector of dependent variables and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ the p-dimensional vector of unknown parameters, lying in a parameter space $\boldsymbol{\Theta}$. The notation considered above is used for simplicity. However, more general situations where $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are matrices, leading to multivariate models, for example, can be handled similarly. Moreover, let $l(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y})$ be the log-likelihood function corresponding to the sample and

$$\boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y}) = \frac{\partial l(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y})}{\partial \boldsymbol{\theta}} \quad \text{and} \quad \boldsymbol{I}(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y}) = -\frac{\partial \boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y})}{\partial \boldsymbol{\theta}}, \quad \boldsymbol{\theta} \in \mathcal{F},$$

the score and information matrix, respectively, where $\mathcal{F}$ is an open convex subspace of $\boldsymbol{\Theta}$. Let $\boldsymbol{\theta}_{\boldsymbol{z}}$ be the maximum likelihood estimator of $\boldsymbol{\theta}$, that is, the value of $\boldsymbol{\theta}$ such that $\boldsymbol{U}(\boldsymbol{\theta}_{\boldsymbol{z}}; \boldsymbol{Z}, \boldsymbol{Y}) = 0$ and $\boldsymbol{\theta}_0 \in \mathcal{F}$, be the true parameter value. Let $E^{+}(\cdot)$ denote the expectation with respect to the vector $\boldsymbol{Y}$ given $\boldsymbol{Z}$. Under some regularity conditions the maximum likelihood estimator (MLE) is consistent and asymptotically normal. These important properties of the MLE are based strongly on the fact that under the true parameter value $E^{+}\{\boldsymbol{U}(\boldsymbol{\theta}_0; \boldsymbol{Z}, \boldsymbol{Y})\} = \boldsymbol{0}$.

We are concerned with the situation that $\boldsymbol{Z}$ can not be recorded directly, but instead we observe a surrogate $\boldsymbol{X} = (\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_n)'$ having measurement error (Carroll *et al.* (1995)). In this paper an additive error model

$$\boldsymbol{x}_i = \boldsymbol{z}_i + \boldsymbol{u}_i, \qquad i = 1, \ldots, n$$

is considered, where the random errors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, are mutually independent and also are independent of $\boldsymbol{Z}$ and $\boldsymbol{Y}$, each having normal distribution with zero mean and covariance matrix $\boldsymbol{\Sigma}_u$. This covariance matrix may be assumed known or estimated from validation studies (Fuller (1987)). Thus, calling $\boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y})$ the naive score function, we have that, in general, $E\{\boldsymbol{U}(\boldsymbol{\theta}_0; \boldsymbol{X}, \boldsymbol{Y})\} \neq \boldsymbol{0}$, implying that $\boldsymbol{\theta}_{\boldsymbol{x}}$ which solves $\boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{0}$ is not necessarily a consistent estimator of $\boldsymbol{\theta}$.

Nakamura (1990) considers a correction for score functions. The corrected score method depends on the existence of a corrected score function $\boldsymbol{U}^{*}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y})$ such that

$$E\{\boldsymbol{U}^{*}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y}) \mid \boldsymbol{Y}, \boldsymbol{Z}\} = \boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y}), \tag{2.1}$$

for all $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{\theta}$. Moreover, assuming $\boldsymbol{U}^*$ differentiable in $\mathcal{F}$, we have a corrected observed information

$$\boldsymbol{I}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y}) = -\frac{\partial \boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})}{\partial \boldsymbol{\theta}}.$$

It follows under the true model that $E\{\boldsymbol{U}^*(\boldsymbol{\theta}_0;\boldsymbol{X},\boldsymbol{Y})\} = 0$. Since $\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$ is unbiased, regularity conditions ensures the existence of a consistent sequence of estimators $\hat{\boldsymbol{\theta}}$ satisfying $\boldsymbol{U}^*(\hat{\boldsymbol{\theta}};\boldsymbol{X},\boldsymbol{Y}) = 0$. The following proposition showing the asymptotic properties of $\hat{\boldsymbol{\theta}}$ under various regularity conditions have been established in Gimenez and Bolfarine (1997).

PROPOSITION 2.1. *Let y's be mutually independent and $\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$ a function satisfying property* (2.1). *Under some regularity conditions, there exists a solution $\hat{\boldsymbol{\theta}}$ of the system of equations $\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{0}$ which is consistent and asymptotically normal with mean $\boldsymbol{\theta}_0$ and covariance matrix $n^{-1}\boldsymbol{\Omega}_n(\boldsymbol{\theta}_0)$, where*

$$\boldsymbol{\Omega}_n(\boldsymbol{\theta}_0) = \boldsymbol{\Lambda}_n^{-1}(\boldsymbol{\theta}_0)\boldsymbol{\Gamma}_n(\boldsymbol{\theta}_0)\{\boldsymbol{\Lambda}_n^{-1}(\boldsymbol{\theta}_0)\}', \tag{2.2}$$

with

$$\boldsymbol{\Lambda}_n(\boldsymbol{\theta}) = \frac{1}{n}E\{\boldsymbol{I}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})\} = \frac{1}{n}\sum_{i=1}^{n} E\{\boldsymbol{I}^*(\boldsymbol{\theta};\boldsymbol{x}_i,y_i)\}$$

and

$$\boldsymbol{\Gamma}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} E\{\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{x}_i,y_i)\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{x}_i,y_i)'\}.$$

In cases where the corrected score function is obtained as $\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y}) = \partial l^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})/\partial\boldsymbol{\theta}$, where $l^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$ is the corrected log-likelihood function such that $E\{l^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y}) \mid \boldsymbol{Z},\boldsymbol{Y}\} = l(\boldsymbol{\theta};\boldsymbol{Z},\boldsymbol{Y})$, for all $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{\theta}$, the asymptotic properties of $\hat{\boldsymbol{\theta}}$ have been described by Nakamura (1990). On the other hand, it may be possible to find a corrected score function $\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$ that can not be obtained as $\partial l^*/\partial\boldsymbol{\theta}$ and then care must be taken when defining $\hat{\boldsymbol{\theta}}$. The numerical application in Section 5 illustrates this last situation and presents a model for which the corrected score method can be employed without assuming $\boldsymbol{\Sigma}_u$ known or previously estimated.

The corrected score method depends critically on the distributional assumption of the measurement error. Corrected score functions satisfying (2.1) may not necessarily exist and finding them may not be always an easy task. These issues are studied in details in Stefanski (1989).

Regularity conditions for proving consistency and asymptotic normality of $\hat{\boldsymbol{\theta}}$ can be found in Gimenez and Bolfarine (1997). These regularity conditions taking into consideration the dependency of the matrices $\boldsymbol{\Lambda}_n(\boldsymbol{\theta})$ and $\boldsymbol{\Gamma}_n(\boldsymbol{\theta})$ on the incidental parameter vector $\boldsymbol{Z}$. It is not assumed that these matrices converge to any limit.

In order to simplify notation, $\boldsymbol{U}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$, $\boldsymbol{I}^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$ and $l^*(\boldsymbol{\theta};\boldsymbol{X},\boldsymbol{Y})$ are denoted by just $\boldsymbol{U}^*(\boldsymbol{\theta})$, $\boldsymbol{I}^*(\boldsymbol{\theta})$ and $l^*(\boldsymbol{\theta})$ respectively, in the remaining of the paper.

3. Tests based on a corrected score function

In this section, we consider five different test statistics for composite hypotheses based on the corrected score approach. The asymptotic properties of the proposed statistics are based on the results established in the previous section. Let's consider the partition $\boldsymbol{\theta} = (\boldsymbol{\psi}', \boldsymbol{\lambda}')' \in \mathcal{R}^p$, where $\boldsymbol{\psi}$ is the $s \times 1$ parameter of interest, representing the object of the research and $\boldsymbol{\lambda}$ is the $(p-s) \times 1$ vector of nuisance parameters. Thus, given the n observations $(y_1, \boldsymbol{x}_1), \ldots, (y_n, \boldsymbol{x}_n)$, the main object is testing the null hypothesis $H_0 : \boldsymbol{\psi} = \boldsymbol{\psi}_0$, in the presence of the nuisance parameter vector $\boldsymbol{\lambda}$. It is assumed that there exist positive definite matrices $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ and $\boldsymbol{\Gamma}(\boldsymbol{\theta})$ to which the matrices $\boldsymbol{\Lambda}_n(\boldsymbol{\theta})$ and $\boldsymbol{\Gamma}_n(\boldsymbol{\theta})$ weakly converge, respectively. Using notation typically associated with partitioned matrices, we write

$$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{\Lambda}_{\psi\psi}(\boldsymbol{\theta}) & \boldsymbol{\Lambda}_{\psi\lambda}(\boldsymbol{\theta}) \\ \boldsymbol{\Lambda}_{\lambda\psi}(\boldsymbol{\theta}) & \boldsymbol{\Lambda}_{\lambda\lambda}(\boldsymbol{\theta}) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{\Gamma}_{\psi\psi}(\boldsymbol{\theta}) & \boldsymbol{\Gamma}_{\psi\lambda}(\boldsymbol{\theta}) \\ \boldsymbol{\Gamma}_{\lambda\psi}(\boldsymbol{\theta}) & \boldsymbol{\Gamma}_{\lambda\lambda}(\boldsymbol{\theta}) \end{pmatrix},$$

with the partitioning dimensions following the dimensions of $\boldsymbol{\psi}$ and $\boldsymbol{\lambda}$, respectively, that is, $\boldsymbol{\Lambda}_{\psi\psi}(\boldsymbol{\theta})$ is of dimension $s \times s$ and so on. In a similar fashion, we write

$$\boldsymbol{U}^*(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{U}^*_{\psi}(\boldsymbol{\theta}) \\ \boldsymbol{U}^*_{\lambda}(\boldsymbol{\theta}) \end{pmatrix}.$$

Moreover, let $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\psi}}', \hat{\boldsymbol{\lambda}}')'$ and $\hat{\boldsymbol{\theta}}_0 = (\boldsymbol{\psi}_0', \hat{\boldsymbol{\lambda}}_0')'$, consistent estimators of $\boldsymbol{\theta}$ satisfying

$$\boldsymbol{U}^*(\hat{\boldsymbol{\theta}}) = \mathbf{0} \tag{3.1}$$

and

$$\boldsymbol{U}^*_{\lambda}(\hat{\boldsymbol{\theta}}_0) = \mathbf{0}. \tag{3.2}$$

The following testing statistics can be defined, with the suffix n abbreviated where understood,

$$W_1 = n(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0)'\hat{\boldsymbol{\Lambda}}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0)(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0), \tag{3.3}$$

$$Q_1 = n^{-1}\boldsymbol{U}^*_{\psi}(\hat{\boldsymbol{\theta}}_0)'\hat{\boldsymbol{\Lambda}}^{-1}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0)\boldsymbol{U}^*_{\psi}(\hat{\boldsymbol{\theta}}_0), \tag{3.4}$$

where

$$\hat{\boldsymbol{\Lambda}}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0) = \hat{\boldsymbol{\Lambda}}_{\psi\psi}(\hat{\boldsymbol{\theta}}_0) - \hat{\boldsymbol{\Lambda}}_{\psi\lambda}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Lambda}}^{-1}_{\lambda\lambda}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Lambda}}_{\lambda\psi}(\hat{\boldsymbol{\theta}}_0), \tag{3.5}$$

$$W_2 = n(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0)'\hat{\boldsymbol{\Omega}}^{-1}_{\psi\psi}(\hat{\boldsymbol{\theta}}_0)(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0) \tag{3.6}$$

and

$$Q_2 = n^{-1}\boldsymbol{U}^*_{\psi}(\hat{\boldsymbol{\theta}}_0)'\hat{\boldsymbol{\Lambda}}^{-1}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Omega}}^{-1}_{\psi\psi}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Lambda}}^{-1}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0)\boldsymbol{U}^*_{\psi}(\hat{\boldsymbol{\theta}}_0), \tag{3.7}$$

where

$$\hat{\boldsymbol{\Omega}}(\hat{\boldsymbol{\theta}}_0) = \hat{\boldsymbol{\Lambda}}^{-1}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Gamma}}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Lambda}}^{-1}(\hat{\boldsymbol{\theta}}_0),$$

with

$$\hat{\boldsymbol{\Lambda}}(\boldsymbol{\theta}) = \hat{\boldsymbol{\Lambda}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{I}^*(\boldsymbol{\theta}; \boldsymbol{x}_i, y_i)$$

and

$$\hat{\boldsymbol{\Gamma}}(\boldsymbol{\theta}) = \hat{\boldsymbol{\Gamma}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{U}^*(\boldsymbol{\theta}; \boldsymbol{x}_i, y_i)\boldsymbol{U}^*(\boldsymbol{\theta}; \boldsymbol{x}_i, y_i)'.$$

If there exists a function $W(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y})$ such that

$$E\{W(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y}) \mid \boldsymbol{Z}, \boldsymbol{Y}\} = \boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y})\boldsymbol{U}(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y})',$$

Nakamura (1990) proposes a better variance estimator $\hat{\boldsymbol{\Omega}}(\hat{\boldsymbol{\theta}}_0)$, based on a variance decomposition.

In the special case where the corrected score is the gradient of the corrected likelihood $l^*(\boldsymbol{\theta})$, that is, $\boldsymbol{U}^*(\boldsymbol{\theta}) = \partial l^*(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$, it is also possible to define a likelihood ratio type statistic, which we write as

$$L = 2\{l^*(\hat{\boldsymbol{\theta}}) - l^*(\hat{\boldsymbol{\theta}}_0)\}, \tag{3.8}$$

where $\hat{\boldsymbol{\theta}}$ is a global maximum of $l^*(\boldsymbol{\theta})$, and $\hat{\boldsymbol{\theta}}_0$ maximizes $l^*(\boldsymbol{\theta})$ under H_0. As shown next, W_1, Q_1 and L are asymptotically equivalent. W_2 and Q_2 are also shown to be asymptotically equivalent.

THEOREM 3.1. *Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}_0$ consistent roots of equations* (3.1) *and* (3.2) *respectively. Then, under some regularity conditions and H_0, it follows*

a) $W_1 \rightarrow^D \sum_{i=1}^{s} \mu_i V_i$, *where $V_1, \ldots, V_s$ are iid chisquare random variables with one degree of freedom and $\mu_1, \ldots, \mu_s$ are the eigenvalues of the matrix*

$$\boldsymbol{\Lambda}_{\psi\psi.\lambda}(\boldsymbol{\theta}_0)\boldsymbol{\Omega}_{\psi\psi}(\boldsymbol{\theta}_0), \tag{3.9}$$

where "$\rightarrow^D$" denotes convergence in distribution (Rao (1973)). *Moreover, W_1 is asymptotically equivalent to Q_1 and L.*

b) $W_2 \rightarrow^D \chi_s^2$, *where χ_s^2 denotes a random variable distributed according to the chisquare distribution with s degrees of freedom. Moreover, W_2 is asymptotically equivalent to Q_2.*

The proof can be found in the Appendix. The following remarks are direct consequences of the above results.

Remark 1. If the matrix in (3.9) is the identity matrix then $W_1 \rightarrow^D \chi_s^2$ as in the classical case. Thus, if the product $\boldsymbol{\Lambda}(\boldsymbol{\theta}_0)\boldsymbol{\Omega}(\boldsymbol{\theta}_0)$ is close to the identity matrix, then the asymptotic distribution of L, W_1 and Q_1 can be approximated by a χ_s^2 distribution.

Remark 2. If $\boldsymbol{\Lambda}(\boldsymbol{\theta}_0)$ is a block diagonal matrix, that is, $\boldsymbol{\Lambda}_{\psi\lambda}(\boldsymbol{\theta}_0) = \boldsymbol{0}$ then (3.9) reduces to $\boldsymbol{\Gamma}_{\psi\psi}(\boldsymbol{\theta}_0)\boldsymbol{\Lambda}_{\psi\psi}^{-1}(\boldsymbol{\theta}_0)$. This is the case when $p - s = 0$, that is, no nuisance parameters are in the model.

Remark 3. If $\boldsymbol{\Lambda}_{\psi\lambda}(\boldsymbol{\theta}_0) = \boldsymbol{0}$ and $s = 1$, then (3.9) equals the ratio $\boldsymbol{\Gamma}_{\psi\psi}(\boldsymbol{\theta}_0)/\boldsymbol{\Lambda}_{\psi\psi}(\boldsymbol{\theta}_0)$ which can be seen as a correction factor of the usual χ_1^2 distribution.

Remark 4. In the estimation of matrices $\boldsymbol{\Gamma}(\boldsymbol{\theta}_0)$ and $\boldsymbol{\Lambda}(\boldsymbol{\theta}_0)$ any consistent estimator of $\boldsymbol{\theta}_0$ can be used. Notice that, if the statistic Q_1 is used then the obvious choice would be $\hat{\boldsymbol{\theta}}_0$, since in this case, the parametric model has only to be adjusted under H_0.

Table 1. Empirical size for a 5% nominal level.

σ_u^2	L	W_1	Q_1
0.0	5.6	12.0	6.6
0.1	5.7	12.2	6.9
0.3	5.6	12.6	6.8
0.5	4.4	10.4	6.4

Remark 5. We need to compute quantiles of the distribution of $\sum_{i=1}^{s}\hat{\mu}_i V_i$, where $\hat{\mu}_i$ are the eigenvalues of the consistent estimator of the matrix $\mathbf{\Lambda}_{\psi\psi.\lambda}(\boldsymbol{\theta}_0)\mathbf{\Omega}_{\psi\psi}(\boldsymbol{\theta}_0)$. In the case where $p > 1$, some algorithms in Marazzi (1980) and Griffiths and Hill (1985) can be used. Another approach would be to simulate from the distribution of $\sum_{i=1}^{s}\hat{\mu}_i V_i$.

Remark 6. In the special case that $s = 1$, writing $\hat{\mu}_1 = \hat{\mathbf{\Lambda}}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0)\hat{\mathbf{\Omega}}_{\psi\psi}(\hat{\boldsymbol{\theta}}_0)$, we have

$$\frac{W_1}{\hat{\mu}_1} = n\frac{(\hat{\psi}-\psi_0)^2}{\hat{\Omega}_{\psi\psi}(\hat{\boldsymbol{\theta}}_0)} = W_2 \quad \text{and} \quad \frac{Q_1}{\hat{\mu}_1} = \frac{n^{-1}\boldsymbol{U}^*_{\psi}(\hat{\boldsymbol{\theta}}_0)^2}{\hat{\mathbf{\Lambda}}_{\psi\psi.\lambda}(\hat{\boldsymbol{\theta}}_0)\hat{\Omega}_{\psi\psi}(\hat{\boldsymbol{\theta}}_0)} = Q_2.$$

Thus, $W_1/\hat{\mu}_1 = W_2$ and $Q_1/\hat{\mu}_1 = Q_2$ are asymptotically distributed according to the χ_1^2 distribution.

Remark 7. Regularity conditions in Theorem 3.1 are those assumed in Proposition 2.1 (Gimenez and Bolfarine (1997)), and the convergence of matrices $\mathbf{\Lambda}_n$ and $\mathbf{\Gamma}_n$.

4. Simulation study

In this section we perform a Monte Carlo simulation study for comparing the empirical power and size of the test statistics L, W_1 and Q_1 presented in Section 3. The simulation study is based on an exponential regression model for lifetime data.

A set of independent random variables $\boldsymbol{T}' = (T_1, \ldots, T_{20})$ is generated for each repetition. $\boldsymbol{T}$ is a vector of realizations of an exponential distribution with parameter $\exp(\alpha+\beta z)$ and the null hypothesis of interest is $\beta = 0$. The true covariate is generated as a standard normal and the error variable as a normal distribution with mean 0 and variance σ_u^2. The parameter α in this study is set equal to zero and 1000 replications are run for each simulation. Note that according to the Remark 6 in Section 3, for this model, $W_2 = W_1/\hat{\mu}_1$ and $Q_2 = Q_1/\hat{\mu}_1$. When $s > 1$, W_1 is not equivalent to W_2 and Q_1 is not equivalent to Q_2. Moreover, is not difficult to prove that for the exponencial model, with $s \geq 1$, Q_2 coincides with the so-called naive score test of Tosteson and Tsiatis.

The simulations are performed for several values of the error variance ($\sigma_u^2 = 0, 0.1, 0.3, 0.5$). Table 1 displays the empirical levels of the tests for a 5% nominal level. Even for a small sample of size 20 the empirical sizes of the likelihood ratio and score tests are very close to the nominal levels. The same does not happen with the Wald test. There are some evidences from simulation studies in the literature that in general, distributions of the likelihood ratio and score statistics approach their limiting distributions considerably more rapidly than the distribution of the Wald statistics (Lawless (1982)). This fact and a small sample size used in the simulation study might explain the poor empirical size of the Wald test. Moreover, the Wald test is known to behave aberrantly in some models (Hauck and Donner (1977)).

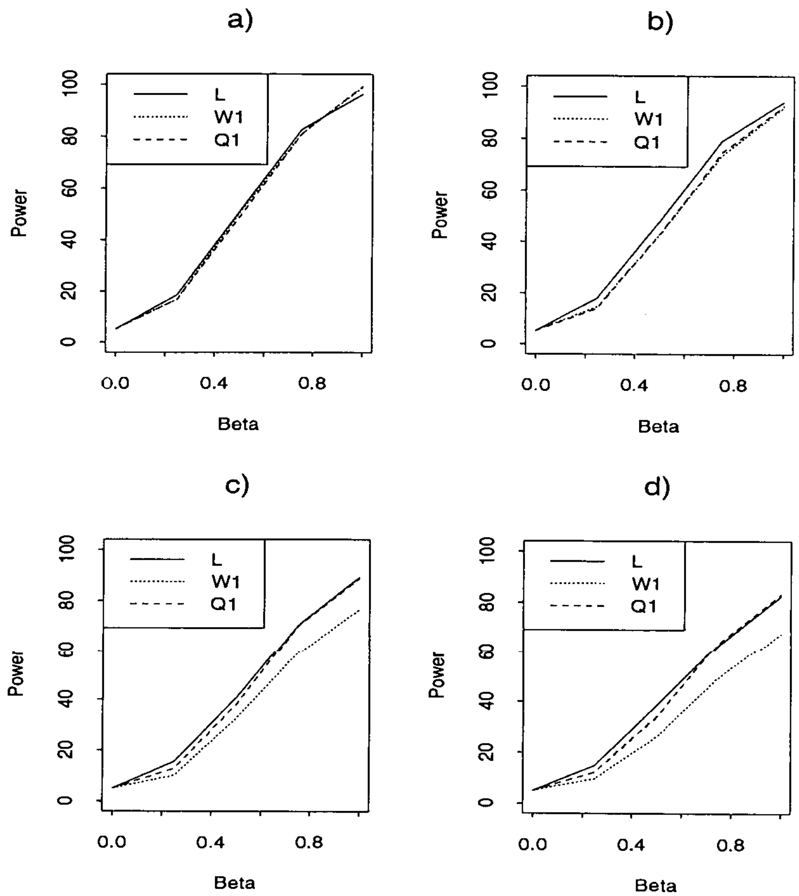

Fig. 1. Power curves for the tests L, W_1 and Q_1. a) $\sigma_u^2 = 0.0$, b) $\sigma_u^2 = 0.1$, c) $\sigma_u^2 = 0.3$, d) $\sigma_u^2 = 0.5$.

An adjustment was performed using the empirical distribution of the statistics such that the empirical size is corrected with 5%. That is, a critical value is determined for each test statistic such that the empirical size of the three statistics coincides at 5%. Then, the empirical power of each test is the proportion of samples in which the test statistic exceeds its critical value, under the different values of parameter β. Figure 1 displays the empirical power of these three tests for the values $\sigma_u^2 = 0, 0.1, 0.3, 0.5$, respectively. It can be observed that Wald test is losing more power than the other two as the error variance increases, then we can not recommend the Wald type test. Moreover, it seems that likelihood ratio test presents a better performance than the score test. The critical value is obtained as $3.84\hat{\mu}_1$, as considereded in Remark 6 of Section 3.

5. A comparative calibration model application

The results obtained in the previous sections are now illustrated by considering the comparative calibration problem which is presented in Jaech (1985). See also Kimura (1992). Such experiment aims at comparing different ways of measuring the same unknown quantity z in a group of n subjects. In Section 5.1, we derive Wald and score statistics for testing hypotheses for the comparative calibration model. In Section 5.2 we report an application of the proposed statistics to a data set considered in Jaech (1985).

5.1 *The linear calibration model*

A model usually considered in the literature for p different methods (Barnett (1969), Kimura (1992)) is

$$\boldsymbol{y}_i = \boldsymbol{\alpha} + \boldsymbol{\beta} z_i + \boldsymbol{\varepsilon}_i, \tag{5.1}$$

$$x_i = z_i + u_i, \tag{5.2}$$

for $i = 1, \ldots, n$, where $\boldsymbol{y}_i = (y_{i1}, \ldots, y_{ip})'$, $\boldsymbol{\epsilon}_i = (\varepsilon_{i1}, \ldots, \varepsilon_{ip})'$, are p-random vectors with $\boldsymbol{\varepsilon}_i \sim^{\text{iid}} N(0, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_p^2)$, $u_i \sim^{\text{iid}} (0, \sigma_u^2)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)'$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$. Expression (5.1) considers that the additive and multiplicative bias corresponding to method k are α_k and β_k, respectively, $k = 1, \ldots, p$. Expression (5.2), assumed by Barnett (1969) and Kimura (1992), indicates that one of the methods measures the unknown quantity z_i without bias. In order to make the model identifiable, we consider the situation where $\lambda_k = \sigma_k^2/\sigma_u^2$, $k = 1, \ldots, p$, are known and taken without loss of generality equals to one, $i = 1, \ldots, n$, that is, $\sigma_k^2 = \sigma_u^2 = \phi$, $k = 1, \ldots, p$.

Interest centers on testing the hypothesis that the methods are measuring the quantity z without bias, which means that

$$H_0 : \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_p \\ \mathbf{1}_p \end{pmatrix},$$

where $\mathbf{1}_p = (1, \ldots, 1)'$ and we can use the Wald or the score statistics based on the corrected score function defined in (3.6) and (3.7) respectively. In order to evaluate the expressions of these statistics we have initially to obtain the corrected score estimator and its corresponding asymptotic covariance matrix. Maximum likelihood estimation is discussed in Kimura (1992) and Bolfarine and Galeas-Rojas (1995).

The unobserved log-likelihood for the model (5.1)–(5.2) can be written as

$$l(\boldsymbol{\theta}; \boldsymbol{Z}, \boldsymbol{Y}) = \sum_{i=1}^{n} l(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i)$$

where $l(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i) = -\frac{p}{2}\log(2\pi) - \frac{p}{2}\log\phi - \frac{1}{2\phi}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} z_i)'(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} z_i)$, with $\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}', \phi)'$ of dimension $2p + 1$, leading to

$$\boldsymbol{U}(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i) = (\boldsymbol{U}_\alpha(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i)', \boldsymbol{U}_\beta(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i)', U_\phi(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i))'$$

with

$$\boldsymbol{U}_\alpha(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i) = \frac{1}{\phi}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} z_i),$$

$$\boldsymbol{U}_\beta(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i) = \frac{1}{\phi}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} z_i) z_i,$$

$$U_\phi(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i) = -\frac{p}{2\phi} + \frac{1}{2\phi^2}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} z_i)'(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} z_i),$$

for $i = 1, \ldots, n$. Thus, considering

$$\boldsymbol{U}^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i) = (\boldsymbol{U}_\alpha^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i)', \boldsymbol{U}_\beta^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i)', U_\phi^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i))'$$

with

$$U^*_\alpha(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i) = \frac{1}{\phi}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} x_i), \tag{5.3}$$

$$U^*_\beta(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i) = \frac{1}{\phi}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} x_i)x_i + \boldsymbol{\beta}, \tag{5.4}$$

$$U^*_\phi(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i) = \frac{1}{2\phi^2}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} x_i)'(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} x_i) - \frac{1}{2\phi}(\boldsymbol{\beta}'\boldsymbol{\beta} + p), \tag{5.5}$$

for $i = 1, \ldots, n$. It follows that

$$E[\boldsymbol{U}^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i) \mid z_i, \boldsymbol{y}_i] = \boldsymbol{U}(\boldsymbol{\theta}; z_i, \boldsymbol{y}_i), \qquad i = 1, \ldots, n,$$

which implies that $\boldsymbol{U}^*(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^n \boldsymbol{U}^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i)$ is a corrected score function. It can be observed that,

$$\frac{\partial l^*(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i)}{\partial \phi} = -\frac{p}{2\phi} + \frac{1}{2\phi^2}(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} x_i)'(\boldsymbol{y}_i - \boldsymbol{\alpha} - \boldsymbol{\beta} x_i) \neq U^*_\phi(\boldsymbol{\theta}; x_i, \boldsymbol{y}_i),$$

implying that the corrected score function can not be obtained through differentiating the corrected log-likelihood.

Solving $\boldsymbol{U}^*(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{0}$, the corrected score estimator $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}', \hat{\phi})'$ is obtained as the solution to the following system of equations:

$$\boldsymbol{\alpha} = \bar{\boldsymbol{y}} - \boldsymbol{\beta}\bar{x}, \tag{5.6}$$

$$\boldsymbol{S}_{xy} - \boldsymbol{\beta} S_{xx} + \boldsymbol{\beta}\phi = \boldsymbol{0}, \tag{5.7}$$

$$\phi = \frac{1}{(\boldsymbol{\beta}'\boldsymbol{\beta} + p)} \sum_{k=1}^{p} (S_{y_k y_k} + \beta_k^2 S_{xx} - 2\beta_k S_{x y_k}), \tag{5.8}$$

where

$$\bar{\boldsymbol{y}} = \frac{1}{n}\sum_{i=1}^n \boldsymbol{y}_i, \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i,$$

$$\boldsymbol{S}_{xy} = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(\boldsymbol{y}_i - \bar{\boldsymbol{y}}), \qquad S_{xx} = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2,$$

$$\boldsymbol{S}_{yy} = \frac{1}{n}\sum_{i=1}^n (\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})'.$$

The solution of the system of equations (5.6)–(5.8) can be obtained easily by using an iterative method. Let $\hat{\boldsymbol{\beta}}^{(m)} = (\hat{\beta}_1^{(m)}, \ldots, \hat{\beta}_p^{(m)})'$ be the solution of (5.7) at step m, in the step $m+1$ the algorithm proceeds as follows:

1: Find $\hat{\phi}^{(m+1)} = \phi(\hat{\boldsymbol{\beta}}^{(m)})$, according to (5.8).

2: Find $\hat{\beta}_k^{(m+1)}$, $k = 1, \ldots, p$, as the solution to (5.7), that is,

$$\hat{\beta}_k^{(m+1)} = \frac{S_{x y_k}}{S_{xx} - \hat{\phi}^{(m+1)}}, \qquad k = 1, \ldots, p.$$

An estimate of $\boldsymbol{\alpha}$ follows directly from (5.6).

The system of equations (5.6)–(5.8) has multiple roots. Therefore a careful choice must be made in order to pick a consistent root. Taking the naive estimate as an initial guess, an appropriate root is usually obtained after two or three steps in the numerical process (Stefanski (1989)).

It follows from the assumptions

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i \xrightarrow{p} \mu < \infty \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})^2 \xrightarrow{p} \nu^2 < \infty, \tag{5.9}$$

that

$$\hat{\mathbf{\Lambda}}_n(\boldsymbol{\theta}) \xrightarrow{p} \mathbf{\Lambda}(\boldsymbol{\theta}) \quad \text{and} \quad \hat{\mathbf{\Gamma}}_n(\boldsymbol{\theta}) \xrightarrow{p} \mathbf{\Gamma}(\boldsymbol{\theta}),$$

where

$$\mathbf{\Lambda}(\boldsymbol{\theta}) = \frac{1}{\phi}\begin{pmatrix} \boldsymbol{A}\otimes \boldsymbol{I}_p & (0,-1)'\otimes\boldsymbol{\beta} \\ \mathbf{0} & \dfrac{\boldsymbol{\beta}'\boldsymbol{\beta}+p}{2\phi} \end{pmatrix}, \tag{5.10}$$

$$\mathbf{\Gamma}(\boldsymbol{\theta}) = \frac{1}{\phi}\begin{pmatrix} (\boldsymbol{A}+2\boldsymbol{B})\boldsymbol{\beta}\boldsymbol{\beta}' + (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{I}_p & (\boldsymbol{C}\otimes\boldsymbol{\beta}) \\ (\boldsymbol{C}'\otimes\boldsymbol{\beta}') & \dfrac{1}{2}(\boldsymbol{\beta}'\boldsymbol{\beta})^2 + \boldsymbol{\beta}'\boldsymbol{\beta} + \dfrac{p}{2} \end{pmatrix},$$

with

$$\boldsymbol{A} = \begin{pmatrix} 1 & \mu \\ \mu & \nu^2+\mu^2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & \phi \end{pmatrix}, \quad \boldsymbol{C} = -\begin{pmatrix} 0 \\ \boldsymbol{\beta}'\boldsymbol{\beta}+1 \end{pmatrix},$$

and $\boldsymbol{I}_p$ the p-dimensional identity matrix. It follows from Proposition 2.1 that $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}', \hat{\phi})'$, solution of (5.6)–(5.8), is consistent and asymptotically normal with a mean vector $\boldsymbol{\theta}_0 = (\boldsymbol{\alpha}_0', \boldsymbol{\beta}_0', \phi_0)'$ and covariance matrix $n^{-1}\mathbf{\Omega}_n = n^{-1}\mathbf{\Lambda}_n^{-1}\mathbf{\Gamma}_n\mathbf{\Lambda}_n^{-1}$.

According to the notation of Section 3, $\boldsymbol{\psi} = (\boldsymbol{\alpha}', \boldsymbol{\beta}')'$ is the parameter of interest and $\lambda = \phi$ is the nuisance parameter. $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}', \hat{\phi})'$ is the unrestricted corrected score estimator, that is, is the solution of the system of equations (5.6)–(5.8) and $\hat{\boldsymbol{\theta}}_0 = (\mathbf{0}', \mathbf{1}_p', \hat{\phi}_0)'$ is the restricted estimator under H_0, such that, from (5.8)

$$\hat{\phi}_0 = \frac{1}{2p}\sum_{k=1}^{p}(S_{y_k y_k} + S_{xx} - 2S_{xy_k}).$$

The Wald statistic W_2 described in (3.6) can be written as

$$W_2 = n[\hat{\boldsymbol{\alpha}}', (\hat{\boldsymbol{\beta}} - \mathbf{1}_p)']\hat{\mathbf{\Omega}}_{\psi\psi}^{-1}(\hat{\boldsymbol{\theta}}_0)[\hat{\boldsymbol{\alpha}}', (\hat{\boldsymbol{\beta}} - \mathbf{1}_p)']',$$

where

$$\mathbf{\Omega}_{\psi\psi}^{-1}(\boldsymbol{\theta}) = \frac{1}{\phi}\begin{pmatrix} (\boldsymbol{I}_p+\boldsymbol{\beta}\boldsymbol{\beta}')^{-1} & \mu(\boldsymbol{I}_p+\boldsymbol{\beta}\boldsymbol{\beta}')^{-1} \\ \mu(\boldsymbol{I}_p+\boldsymbol{\beta}\boldsymbol{\beta}')^{-1} & \mu^2(\boldsymbol{I}_p+\boldsymbol{\beta}\boldsymbol{\beta}')^{-1} + \dfrac{\nu^4}{\nu^2+\phi}(\boldsymbol{I}_p + r\boldsymbol{\beta}\boldsymbol{\beta}')^{-1} \end{pmatrix}, \tag{5.11}$$

with

$$r = \frac{1}{\nu^2+\phi}\left[\nu^2 + \frac{2p(p-1)\phi}{(\boldsymbol{\beta}'\boldsymbol{\beta}+p)^2}\right].$$

By using (5.2) and (5.9)

$$\bar{x} \xrightarrow{p} \mu \quad \text{and} \quad S_{xx} - \phi \xrightarrow{p} \nu^2, \quad \text{as } n \to \infty, \tag{5.12}$$

so that, $\hat{\Omega}_{\psi\psi}^{-1}$ is obtained by assigning $\bar{x}$ and $S_{xx}-\phi$ to μ and ν^2, respectively, in the right hand of (5.11). Thus, we have

$$(5.13)\quad W_2 = \frac{n}{\hat{\phi}_0}\{\hat{\boldsymbol{\alpha}}'(\boldsymbol{I}_p+\mathbf{1}_p\mathbf{1}_p')^{-1}\hat{\boldsymbol{\alpha}} + 2\bar{x}\hat{\boldsymbol{\alpha}}'(\boldsymbol{I}_p+\mathbf{1}_p\mathbf{1}_p')^{-1}(\hat{\boldsymbol{\beta}}-\mathbf{1}_p)$$
$$+ (\hat{\boldsymbol{\beta}}-\mathbf{1}_p)'[\bar{x}^2(\boldsymbol{I}_p+\mathbf{1}_p\mathbf{1}_p')^{-1} + \frac{(S_{xx}-\hat{\phi}_0)^2}{S_{xx}}(\boldsymbol{I}_p+\hat{r}\mathbf{1}_p\mathbf{1}_p')^{-1}](\hat{\boldsymbol{\beta}}-\mathbf{1}_p)\}$$

where $\hat{r} = 1 - \frac{(p+1)-\hat{\phi}_0}{2pS_{xx}}$.

The score statistic Q_2 is given by

$$Q_2 = n^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0)'\hat{\boldsymbol{\Lambda}}_{\psi\psi.\phi}^{-1}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Omega}}_{\psi\psi}^{-1}(\hat{\boldsymbol{\theta}}_0)\hat{\boldsymbol{\Lambda}}_{\psi\psi.\phi}^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0),$$

where from (5.3) and (5.4),

$$(5.14)\qquad \boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0) = \frac{n}{\hat{\phi}_0}\begin{pmatrix} \bar{\boldsymbol{y}}-\mathbf{1}_p\bar{x} \\ \boldsymbol{S}_{xy}+\bar{x}\bar{\boldsymbol{y}}-\mathbf{1}_p(S_{xx}-\hat{\phi}_0+\bar{x}^2)\end{pmatrix},$$

and from (5.10),

$$(5.15)\qquad \boldsymbol{\Lambda}_{\psi\psi.\phi}^{-1} = \phi(\boldsymbol{A}^{-1}\otimes\boldsymbol{I}_p),$$

where $\boldsymbol{A}^{-1} = \frac{1}{\nu^2}\begin{pmatrix}\nu^2+\mu^2 & -\mu \\ -\mu & 1\end{pmatrix}$.

Q_2 can be rewritten, according to (5.11), (5.12), (5.14) and (5.15), as

$$(5.16)\quad Q_2 = \frac{n}{\hat{\phi}_0}\Big\{(\bar{\boldsymbol{y}}-\mathbf{1}_p\bar{x})'\left[(\boldsymbol{I}_p+\mathbf{1}_p\mathbf{1}_p')^{-1}+\frac{\bar{x}^2}{S_{xx}}(\boldsymbol{I}_p+\hat{r}\mathbf{1}_p\mathbf{1}_p')^{-1}\right](\bar{\boldsymbol{y}}-\mathbf{1}_p\bar{x})$$
$$-\frac{2\bar{x}}{S_{xx}}[\boldsymbol{S}_{xy}+\bar{x}\bar{\boldsymbol{y}}-\mathbf{1}_p(S_{xx}-\hat{\phi}_0+\bar{x}^2)]'(\boldsymbol{I}_p+\hat{r}\mathbf{1}_p\mathbf{1}_p')^{-1}(\bar{\boldsymbol{y}}-\mathbf{1}_p\bar{x})$$
$$+\frac{1}{S_{xx}}[\boldsymbol{S}_{xy}+\bar{x}\bar{\boldsymbol{y}}-\mathbf{1}_p(S_{xx}-\hat{\phi}_0+\bar{x}^2)]'(\boldsymbol{I}_p+\hat{r}\mathbf{1}_p\mathbf{1}_p')^{-1}[\boldsymbol{S}_{xy}+\bar{x}\bar{\boldsymbol{y}}$$
$$-\mathbf{1}_p(S_{xx}-\hat{\phi}_0+\bar{x}^2)]\Big\}.$$

5.2 Numerical results

The data set considered next is taken from Jaech (1985). The densities of 43 cylindrical nuclear reactor fuel pellets of sintered uranium were measured by different methods. Full details of the experiment can be found in the reference just mentioned. We use the data corresponding to three methods: Method 1, a geometric method based on weighting the pellet and finding its volume by measuring the pellet diameter and length and other two methods named Method 2 and Method 3. The Method 1 measures the true quantity without bias. In particular, we want to test the hypothesis that the Methods 2 and 3 are measuring with no bias the quantity z.

Returning to our comparative calibration problem with $n = 43$ and $p = 2$, we have the following statistics

$$\bar{x} = 4.397,\qquad \bar{\boldsymbol{y}} = (4.370, 4.436)'$$
$$S_{xx} = 0.04391,\qquad \boldsymbol{S}_{xy} = (0.03449, 0.03933)'\quad\text{and}$$
$$\boldsymbol{S}_{yy} = \begin{pmatrix}0.04013 & 0.03483 \\ & 0.06442\end{pmatrix}.$$

Solving the system of equations (5.6)–(5.8), we obtain corrected estimates:

$$\hat{\boldsymbol{\alpha}} = (-0.21963, -0.79769)',$$
$$\hat{\boldsymbol{\beta}} = (1.04381, 1.19029)' \quad \text{and}$$
$$\hat{\phi} = 0.01397.$$

The values of the Wald and score statistics given by (5.13) and (5.16), are $W_2 = 10.46$ and $Q_2 = 10.62$, respectively. Comparing with a chi-squared distribution with four degrees of freedom, we can conclude at a 5% level that Methods 2 and 3 are measuring with bias the density of cylindrical nuclear reactor fuel pellets of sintered uranium.

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Appendix. Proof of Theorem 3.1

Replacing $\hat{\boldsymbol{\Lambda}}(\hat{\boldsymbol{\theta}}_0)$ and $\hat{\boldsymbol{\Gamma}}(\hat{\boldsymbol{\theta}}_0)$ by $\boldsymbol{\Lambda}(\boldsymbol{\theta}_0)$ and $\boldsymbol{\Gamma}(\boldsymbol{\theta}_0)$ (denoted just by $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ to simplify notation), in the expressions for W_1, Q_1, W_2 and Q_2, asymptotically equivalent statistics are obtained. We denote these statistics by W_1^*, Q_1^*, W_2^* and Q_2^*, respectively, and find their asymptotic distributions.
The proof of a) is considered first. Taylor series expansion of $\boldsymbol{U}^*(\boldsymbol{\theta})$ about $\hat{\boldsymbol{\theta}}$ yields

$$\text{(A.1)} \qquad \frac{1}{\sqrt{n}}\boldsymbol{U}^*(\boldsymbol{\theta}_0) = \frac{1}{\sqrt{n}}\boldsymbol{U}^*(\hat{\boldsymbol{\theta}}) + \bar{\boldsymbol{I}}_n^*(\boldsymbol{\theta}^*)\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0),$$

where $\bar{\boldsymbol{I}}_n^*(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^n \boldsymbol{I}^*(\boldsymbol{\theta}; \boldsymbol{x}_i, y_i)$ and $\|\boldsymbol{\theta}^* - \boldsymbol{\theta}_0\| < \|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\|$. Since

$$\text{(A.2)} \qquad \bar{\boldsymbol{I}}_n^*(\boldsymbol{\theta}^*) \xrightarrow{p} \boldsymbol{\Lambda}.$$

and

$$\text{(A.3)} \qquad \sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = O_p(1)$$

(see Proposition 2.1), it follows from (3.1), (A.1), (A.2) and (A.3) that

$$\text{(A.4)} \qquad \frac{1}{\sqrt{n}}\boldsymbol{U}^*(\boldsymbol{\theta}_0) = \boldsymbol{\Lambda}\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + o_p(1).$$

From (A.4) and a similar expansion about $\hat{\boldsymbol{\theta}}_0$, it can be verified that

$$\text{(A.5)} \qquad \hat{\boldsymbol{\lambda}}_0 = \hat{\boldsymbol{\lambda}} + \boldsymbol{\Lambda}_{\lambda\lambda}^{-1}\boldsymbol{\Lambda}_{\lambda\psi}(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0).$$

Thus, it follows that

$$\text{(A.6)} \qquad \sqrt{n}(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0) = \frac{1}{\sqrt{n}}\boldsymbol{\Lambda}_{\psi\psi.\lambda}^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0) + o_p(1),$$

with $\boldsymbol{\Lambda}_{\psi\psi.\lambda}$ as defined in (3.5). Moreover, replacing (A.6) in the expression for W_1^*, it follows that

$$W_1^* = n^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0)'\boldsymbol{\Lambda}_{\psi\psi.\lambda}^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0) + o_p(1),$$

that is, $W_1^* = Q_1^* + o_p(1)$, so that W_1^* and Q_1^* are asymptotically equivalent.

Now, if $\boldsymbol{U}^*(\boldsymbol{\theta}) = \partial l^*(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$ then Taylor series expansion of $l^*(\boldsymbol{\theta}_0)$ about $\hat{\boldsymbol{\theta}}$ yields

$$2\{l^*(\boldsymbol{\theta}_0) - l^*(\hat{\boldsymbol{\theta}})\} = -n(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'\bar{\boldsymbol{I}}_n^*(\boldsymbol{\theta}^*)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0),$$

where $\boldsymbol{\theta}^*$ is such that $\|\boldsymbol{\theta}^* - \boldsymbol{\theta}_0\| < \|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\|$. Thus, (A.2) and (A.3) implies that

$$2\{l^*(\boldsymbol{\theta}_0) - l^*(\hat{\boldsymbol{\theta}})\} = -n(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'\boldsymbol{\Lambda}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + o_p(1). \tag{A.7}$$

From (A.7) and the analogous expansion about $\hat{\boldsymbol{\theta}}_0$, it follows that

$$\begin{aligned} L = 2\{l^*(\hat{\boldsymbol{\theta}}) - l^*(\hat{\boldsymbol{\theta}}_0)\} &= n(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'\boldsymbol{\Lambda}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \\ &\quad - n(\hat{\boldsymbol{\lambda}}_0 - \boldsymbol{\lambda}_0)'\boldsymbol{\Lambda}_{\lambda\lambda}(\hat{\boldsymbol{\lambda}}_0 - \boldsymbol{\lambda}_0) + o_p(1). \end{aligned} \tag{A.8}$$

Replacing (A.5) in (A.8) and considering that $\boldsymbol{\Lambda}_{\lambda\psi} = \boldsymbol{\Lambda}'_{\psi\lambda}$, we can write

$$L = n(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0)'\boldsymbol{\Lambda}_{\psi\psi.\lambda}(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0) + o_p(1).$$

Thus, $L = W_1^* + o_p(1)$, that is, L and W_1^* are asymptotically equivalent. In order to derive the distribution of these statistics it is easier to deal with W_1^*. From Proposition 2.1, it follows that

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} N_p(\boldsymbol{0}, \boldsymbol{\Omega}),$$

so that, in particular

$$\sqrt{n}(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0) \xrightarrow{D} N_s(\boldsymbol{0}, \boldsymbol{\Omega}_{\psi\psi}). \tag{A.9}$$

Then, a) follows from (A.9) and some results for the quadratic form of normal variables (Rao (1973)).

To prove b), it follows by using (A.9) that

$$W_2^* = n(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0)'\boldsymbol{\Omega}_{\psi\psi}^{-1}(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0) \xrightarrow{D} \chi_s^2.$$

Moreover, it can be shown that the statistic

$$Q_2^* = n^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0)'\boldsymbol{\Lambda}_{\psi\psi.\lambda}^{-1}\boldsymbol{\Omega}_{\psi\psi}^{-1}\boldsymbol{\Lambda}_{\psi\psi.\lambda}^{-1}\boldsymbol{U}_\psi^*(\hat{\boldsymbol{\theta}}_0)$$

is asymptotically equivalent to W_2^*.

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