Ann. Inst. Statist. Math.
Vol. 52, No. 3, 488–506 (2000)

BIVARIATE SIGN TESTS BASED ON THE SUP, L_1 AND L_2 NORMS*

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(Received June 23, 1997; revised March 19, 1999)

Abstract. The bivariate location problem is considered. The sup, L_1 and L_2 norms are used to construct bivariate sign tests from the univariate sign statistics computed on the projected observations on all lines passing through the origin. The tests so obtained are affine-invariant and distribution-free under the null hypothesis. The sup-norm gives rise to Hodges' test. A class of tests derived from the L_2-norm, with Blumen's test as a member, is seen to be related to a class proposed by Oja and Nyblom (1989, *J. Amer. Statist. Assoc.*, **84**, 249–259). The L_1-norm gives rise to a new test. Its asymptotic null distribution is seen to be the same as that of the L_1-norm of a certain normal process related to the standard Wiener process. An explicit expression of its cumulative distribution function is given. A simulation study will examine the merits of the three approaches.

Key words and phrases: Location problem, distribution-free, affine-invariance, normal process, Wiener process, L_1-norm, L_2-norm, Hodges' test, Blumen's test.

1. Introduction

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and identically distributed (iid) random vectors. It is assumed that, for any line passing through the origin, the conditional univariate distribution of $(X_1 - \delta_1, Y_1 - \delta_2)$ given that it is on that line has median 0. This is true for example if (X_1, Y_1) is distributed symetrically around (δ_1, δ_2), that is, $(X_1 - \delta_1, Y_1 - \delta_2)$ and $(-X_1 + \delta_1, -Y_1 + \delta_2)$ are identically distributed. However the former is a weaker assumption including many skewed distribution models. We wish to confront the hypotheses

$$(1.1) \qquad H_0 : (\delta_1, \delta_2) = (0, 0) \quad \text{and} \quad H_1 : (\delta_1, \delta_2) \neq (0, 0).$$

Obviously, if we want to test $H_0 : (\delta_1, \delta_2) = (\delta_{10}, \delta_{20})$ with $(\delta_{10}, \delta_{20})$ not necessarily equals to $(0, 0)$, then the simple transformation $(X_i - \delta_{10}, Y_i - \delta_{20})$, $i = 1, \ldots, n$, brings the problem back to the form (1.1). Various generalizations of the univariate sign test have been proposed to treat this problem. The oldest two suggestions were made by Hodges

* Dedicated to the memory of Serge Tardif. This research was supported by the Natural Sciences and Engineering Research Council of Canada, by the Fonds pour la formation de chercheurs et l'aide à la recherche of the Province of Québec and by la Faculté des études supérieures de l'Université de Montréal.

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*** Serge Tardif died on September 26, 1998 after an earlier version of this paper had been prepared. The remaining authors take full responsability for the present version.

(1955) and Blumen (1958). These two statistics are distribution-free (under H_0) and affine-invariant, i.e., they remain unchanged after a nonsingular linear transformation of the observations. Section 2 examines the relation between these tests and the present work.

Asymptotically distribution-free tests based on a quadratic form involving coordinate-wise sign statistics have been proposed by Bennett (1962) for the multivariate location problem. The large sample test procedure of Chatterjee (1966) is identical to the bivariate procedure of Bennett (1962) but, in addition, Chatterjee gave a small-sample version of the test which is conditionally distribution-free. These tests are scale-invariant but not rotation-invariant and thus not affine-invariant. Later, Dietz (1982) proposed a modification of these tests that yields affine-invariance.

Oja and Nyblom (1989) introduced a class of bivariate affine-invariant distribution-free sign tests that includes both Blumen's and Hodges' test. In particular, they showed that Blumen's test is optimal among their class for elliptical alternatives. Brown and Hettmansperger (1989) proposed a conditionally (and asymptotically) distribution-free, affine-invariant sign test. This test is based on the gradient of Oja's measure of scatter. It is shown in Brown *et al.* (1992) that this test and Blumen's test are asymptotically equivalent for elliptical alternatives and the connection with the class of Oja and Nyblom (1989) is explored.

Möttönen and Oja (1995) introduced a multivariate rotation invariant (not affine-invariant) sign test (and signed-rank test) with a spatial analogue of the sign concept. For bivariate spherical distributions, their spatial sign test is asymptotically equivalent to Blumen's test, and thus to the sign test of Brown and Hettmansperger (1989). Chakraborty *et al.* (1998) proposed to apply the spatial sign test to transformed observations (by a certain type of data dependent transformation). Their method produces a version of the spatial sign test that is affine-invariant.

Chapter 6 of Hettmansperger and McKean (1998) gives an account of the procedures described above.

In this paper, a different perspective is employed to look at the notion of bivariate sign statistics. We begin with the set of univariate sign statistics computed from the projections of the observations on all lines passing through the origin. The sup, L_1 and L_2 norms are then used to construct bivariate sign statistics from these univariate sign statistics. All the tests studied are affine-invariant and distribution-free under the null hypothesis. We present the method in Section 2 and examine the cases of the sup and L_2 norms. For the L_2-norm, a relation with the class of Oja and Nyblom (1989) is found. In Section 3, we see that the L_1-norm gives rise to a new distribution-free bivariate sign test. Small sample critical points of the tests are given and its null asymptotic distribution is obtained. The results from a small simulation study are described in Section 4 along with an example using a real data set. Concluding remarks appear in Section 5. Some of the proofs are sketched in the Appendix.

2. A class of tests based on the L_2-norm

Let θ be in the interval $[-\pi/2, \pi/2]$. For each j, $1 \leq j \leq n$, define $P_j(\theta) = X_j \cos(\theta) + Y_j \sin(\theta)$ as the projection of (X_j, Y_j) on the directed line passing through the origin with angle θ. Let $s(u)$ be the sign function, i.e., $s(u) = 1$ or -1 as $u \geq 0$ or $u < 0$. Define

$$S(\theta) = \sum_{j=1}^{n} s(P_j(\theta)) \tag{2.1}$$

as the sign statistic for the projected observations on the axis with angle θ. Under the hypothesis H_0, $s(P_1(\theta)),\ldots,s(P_n(\theta))$ are iid random variables (r.v.) taking the values 1 or -1, each with probability 1/2. A large value of $|S(\theta)|$ is thus a criterion to reject H_0. The approach of this paper is to take an average of the values in the set $P=\{|S(\theta)|:-\pi/2\leq\theta\leq\pi/2\}$ as a test statistic. More precisely, we will focus on the sup and on weighted L_1 and L_2 norms of these values.

Let us look at the sup-norm first. In that case, the test for H_0 is simply based on the statistic

$$\sup_{-\pi/2\leq\theta\leq\pi/2}|S(\theta)|. \tag{2.2}$$

But this is to Hodges' test statistic; see Hodges (1955), Joffe and Klotz (1962), Klotz (1964) and Killeen and Hettmansperger (1972). Moreover, Killeen and Hettmansperger (1972) introduced the test based on the sup-norm of the Wilcoxon signed-rank statistic computed on the projected observations.

We will now examine a class of statistics derived from a weighted L_2-norm of the values in the set P. For each i, $1\leq i\leq n$, let

$$\theta_i=-\arctan\left(\frac{X_i}{Y_i}\right)+\frac{\pi}{2} \tag{2.3}$$

be the angle ($\in[0,\pi)$) between the vector (X_i,Y_i) and the X-axis. Also, define $\psi_i=\theta_i-\pi/2$, $1\leq i\leq n$, and let F_ψ be the cumulative distribution function (c.d.f.) of ψ_1. If F_ψ was known, an intuitively appealing statistic for testing the hypothesis H_0 would be

$$\frac{1}{n}\int_{-\pi/2}^{\pi/2}\left(\sum_{j=1}^{n}s(P_j(\theta))\right)^2 dF_\psi(\theta). \tag{2.4}$$

A little algebra, using the fact that $s(P_j(\theta))=s(Y_j)s(\theta-\psi_j)$, shows that (2.4) is equal to

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}s(Y_i)s(Y_j)[1-2(F_\psi(\max\{\psi_i,\psi_j\})-F_\psi(\min\{\psi_i,\psi_j\}))]. \tag{2.5}$$

Since the function F_ψ is unknown, we replace it by the empirical c.d.f. of $\psi_1,\ldots,\psi_n$. Then (2.5) becomes

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}s(Y_i)s(Y_j)\left[1-\frac{2}{n}|R_i-R_j|\right] \tag{2.6}$$

where R_i is the rank of θ_i among $\theta_1,\ldots,\theta_n$ (and also the rank of ψ_i among $\psi_1,\ldots,\psi_n$). Let $\phi:[0,1]\to\Re$ be a score function satisfying $\phi(u)=-\phi(1-u)$ for all $u\in[0,1]$. Examination of (2.6) gives rise to the class of statistics

$$W_\phi=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}s(Y_i)s(Y_j)\phi\left(\frac{|R_i-R_j|}{n}\right). \tag{2.7}$$

The test consists in rejecting H_0 for large values of W_ϕ. Obviously, the choice $\phi(u)=1-2u$ corresponds to the statistic (2.6).

The strong relationship between this class and the class of tests studied in Oja and Nyblom (1989) will be given shortly, but first, let us examine W_ϕ more closely. Note that we can write

$$W_\phi = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} s(Y_i^*)s(Y_j^*)\phi\left(\frac{|i-j|}{n}\right) \tag{2.8}$$

where, for each i, $1 \le i \le n$, Y_i^* is the value of Y_j for which $R_j = i$. Since the two vectors $(s(Y_1),\ldots,s(Y_n))$ and $(\theta_1,\ldots,\theta_n)$ are independent under H_0, we have that $(s(Y_1^*),\ldots,s(Y_n^*))$ and $(s(Y_1),\ldots,s(Y_n))$ are identically distributed, i.e., $s(Y_1^*),\ldots,$ $s(Y_n^*)$ are iid and each can take the values 1 and -1 with probability 1/2 respectively. Combining this and the representation (2.8), we see that W_ϕ is distribution-free under H_0 and consequently, exact critical points can be obtained. But the representation (2.8) can also be used to obtain the asymptotic null distribution of W_ϕ which is given in the next theorem whose proof is sketched in the Appendix. The only assumption we make for now and for the rest of this article is that

$$\frac{X_1}{Y_1} \text{ has a continuous distribution and } P(X_1 = 0, Y_1 = 0) = 0. \tag{2.9}$$

THEOREM 2.1. *Suppose that ϕ possesses a bounded continuous derivative on* (0,1). *Under the assumption* (2.9) *and under* H_0,

$$W_\phi \xrightarrow{D} \frac{1}{2}\sum_{k=1}^{\infty} a_k C_k \quad \text{as} \quad n \to \infty$$

where, for $k = 1, 2, \ldots,$

$$a_k = 2\int_0^1 \phi(x)\cos(k\pi x)dx,$$

are the coefficients in the cosine Fourier series developpement of ϕ and where $C_1, C_2, \ldots$ are iid χ_2^2 random variables.

When applying the test, p-values can be computed either from the exact null distribution or from the asymptotic distribution given above using formula (5.10) of Oja and Nyblom (1989) for instance.

Oja and Nyblom (1989) have made a study of bivariate sign tests. In particular, they considered the following class of statistics:

$$S_h = \frac{1}{n^2}\sum_{k=0}^{n-1}\left[\sum_{i=1}^{n}(z_{k+i} - 1/2)h(i/n)\right]^2 \tag{2.10}$$

where h is a general score function and where the z's are defined in their paper. However, it is seen that, for $i = 1, \ldots, n$,

$$2(z_i - 1/2) = s(Y_i^*) \quad \text{and} \quad 2(z_{n+i} - 1/2) = -s(Y_i^*). \tag{2.11}$$

Oja and Nyblom (1989) showed that the statistic S_h is affine-invariant and it is not difficult to see that the same is true for the statistic W_ϕ by verifying that it is scale-invariant, rotation-invariant and invariant for a reflection about the X-axis since any nonsingular linear transformation can be decomposed into a sequence of those types of transformations.

After substituting (2.11) in (2.10) and expanding the square, a little algebra shows that

$$S_h = \frac{1}{4n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} s(Y_i^*) s(Y_j^*) \phi_{nh}(|i-j|) \tag{2.12}$$

where $\phi_{nh} : \{0, 1, \ldots, n-1\} \to \Re$ is defined by

$$\phi_{nh}(y) = \sum_{k=1}^{n-y} h\left(\frac{k}{n}\right) h\left(\frac{k+y}{n}\right) - \sum_{k=1}^{y} h\left(\frac{k}{n}\right) h\left(1 + \frac{k-y}{n}\right).$$

The similarity between (2.12) and (2.8) is evident. The only difference is that ϕ_{nh} varies with n while ϕ is a fixed function. Nevertheless, the next theorem tells us that, for every choice of the function h in the class of Oja and Nyblom (1989), there is a function ϕ in the class (2.8) such that the two statistics are asymptotically equivalent under H_0. For any square integrable function h, introduce the function

$$\phi_h(y) = \int_0^1 [h(1-x)h(1-x+y)I(x \geq y) - h(x)h(x+1-y)I(x \leq y)]dx \tag{2.13}$$

where I denotes the indicator function. Since $\phi_h(y) = -\phi_h(1-y)$ for all $y \in [0,1]$, ϕ_h is a proper score function in the class (2.8) and this choice gives the asymptotic equivalence with (2.12).

THEOREM 2.2. *Let S_h be defined by* (2.10) *where h possesses a bounded continuous derivative on* (0,1) *and let W_{ϕ_h} be the statistic* (2.7) *with the score function ϕ_h defined by* (2.13). *Under the assumption* (2.9) *and under H_0,*

$$|4S_h - W_{\phi_h}| \xrightarrow{P} 0 \quad \textit{as} \quad n \to \infty.$$

PROOF. We will show that

$$E(4S_h - W_{\phi_h})^2 = 16E(S_h^2) + E(W_{\phi_h}^2) - 8E(S_h W_{\phi_h}) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.14}$$

Note that

$$W_{\phi_h} = \frac{2}{n} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s(Y_i^*) s(Y_j^*) \phi_h\left(\frac{|i-j|}{n}\right) + \phi_h(0).$$

Also,

$$S_h = \frac{1}{2n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s(Y_i^*) s(Y_j^*) \phi_{nh}(|i-j|) + \frac{1}{4n} \phi_{nh}(0).$$

Since, for $i < j$ and $k < l$, $E[s(Y_i^*)s(Y_j^*)s(Y_k^*)s(Y_l^*)] = 0$ unless $i = k$ and $j = l$ and since $E[s(Y_i^*)s(Y_j^*)] = 0$ if $i \neq j$, we have

(2.15) $E(S_h W_{\phi_h})$

$$= \frac{1}{n^3}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n} E[s(Y_i^*)s(Y_j^*)s(Y_k^*)s(Y_l^*)]\phi_h\left(\frac{|i-j|}{n}\right)\phi_{nh}(|i-j|)$$

$$+ \frac{1}{2n^2}\phi_{nh}(0)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E[s(Y_i^*)s(Y_j^*)]\phi_h\left(\frac{|i-j|}{n}\right)$$

$$+ \frac{1}{2n^2}\phi_h(0)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E[s(Y_i^*)s(Y_j^*)]\phi_{nh}(|i-j|)$$

$$+ \frac{\phi_h(0)}{4}\frac{\phi_{nh}(0)}{n}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\phi_h\left(\frac{|i-j|}{n}\right)\phi_{nh}(|i-j|) + \frac{\phi_h(0)}{4}\frac{\phi_{nh}(0)}{n}.$$

The first term in (2.15) is equal to

$$\frac{1}{n^2}\sum_{i=1}^{n-1}(1-i/n)\phi_h(i/n)\phi_{nh}(i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n-1}(1-i/n)\phi_h(i/n)\left[\sum_{k=1}^{n-i} h\left(\frac{k}{n}\right)h\left(\frac{k+i}{n}\right) - \sum_{k=1}^{i} h\left(\frac{k}{n}\right)h\left(1+\frac{k-i}{n}\right)\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{k=1}^{n}(1-i/n)\phi_h(i/n)\left[h\left(1-\frac{k}{n}+\frac{1}{n}\right)h\left(1+\frac{i-k}{n}+\frac{1}{n}\right)I(k>i)\right.$$
$$\left. - h\left(\frac{k}{n}\right)h\left(1+\frac{k-i}{n}\right)I(k\le i)\right].$$

This last term converges, as $n \to \infty$, to

$$\int_0^1 (1-x)\phi_h(x)\left[\int_0^1 [h(1-y)h(1+x-y)I(y>x)\right.$$
$$\left. - h(y)h(1+y-x)I(y<x)]dy\right]dx$$

$$= \int_0^1 (1-x)\phi_h^2(x)dx.$$

On the other hand, the second term in (2.15) equals

$$\int_0^1 h^2(x)dx\frac{1}{4n}\sum_{k=1}^{n} h^2(k/n) \to \frac{1}{4}\left(\int_0^1 h^2(x)dx\right)^2 \quad \text{as} \quad n \to \infty.$$

Consequently, as $n \to \infty$,

$$E(S_h W_{\phi_h}) \to \int_0^1 (1-x)\phi_h^2(x)dx + \frac{1}{4}\left(\int_0^1 h^2(x)dx\right)^2.$$

Similarly, as $n \to \infty$,

$$E(S_h^2) \to \frac{1}{4}\int_0^1 (1-x)\phi_h^2(x)dx + \frac{1}{16}\left(\int_0^1 h^2(x)dx\right)^2$$

and

$$E(W_{\phi_h}^2) \to 4\int_0^1 (1-x)\phi_h^2(x)dx + \left(\int_0^1 h^2(x)dx\right)^2.$$

Combining the above results with (2.14) concludes the proof.

Theorem 2.2 also indicates that the two statistics are asymptotically equivalent under a contiguous sequence of alternatives and thus that they are equivalent in the Pitman efficiency sense. Here are three examples of score functions h with their corresponding score functions ϕ_h. If we select $h(t) = \sin(\pi t)$ in Oja-Nyblom's class, then the resulting statistic is identical to the statistic (2.8) with the choice $\phi(u) = \cos(\pi u)$ and this yields Blumen's test. Also, the choice $h(t) \equiv 1$ corresponds exactly to the statistic (2.6), i.e., the choice $\phi(u) = 1 - 2u$. Finally, if we select $h(t) = 1/2 - t$, then

$$4S_h = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} s(Y_i^*)s(Y_j^*)\left[\frac{n^3 - 6n|i-j|^2 + 4|i-j|^3}{12n^3} + \frac{2n - 4|i-j|}{12n^3}\right]$$

which is asymptotically equivalent to the statistic (2.8) with $\phi(u) = u^3/3 - u^2/2 + 1/12$. These three choices of the function h correspond to the B_n, U_n and T_n tests of Oja and Nyblom (1989) respectively.

3. An L_1-norm bivariate sign test

In the foregoing section, we studied bivariate sign tests based on the sup and L_2 norms. In this section, we will use a weighted L_1-norm of the sign statistics computed from the projected observations to obtain our bivariate sign test. In order to give a motivation for the statistic to come, consider the random variable

$$\int_{-\pi/2}^{\pi/2}\left|\sum_{i=1}^{n} s(P_i(\theta))\right| dF_\psi(\theta). \tag{3.1}$$

As in the last section, a little algebra using the fact that $s(P_i(\theta)) = s(Y_i)s(\theta - \psi_i)$ shows that (3.1) is equal to

$$\sum_{k=0}^{n}\left|\sum_{i=1}^{n} s(Y_i^*)s(k-i)\right| \left(F_\psi(\psi_{(k+1)}) - F_\psi(\psi_{(k)})\right) \tag{3.2}$$

where $\psi_{(1)}, \ldots, \psi_{(n)}$ are the order statistics of $\psi_1, \ldots, \psi_n$ and where we define $\psi_{(0)} = -\pi/2$ and $\psi_{(n+1)} = \pi/2$. Again, since F_ψ is unknown, we replace it by the empirical c.d.f. of $\psi_1, \ldots, \psi_n$ and (3.2) becomes

$$L_n = \frac{1}{n}\sum_{k=0}^{n-1}\left|\sum_{i=1}^{n} s(Y_i^*)s(k-i)\right|. \tag{3.3}$$

The test consists in rejecting H_0 for large values of L_n. Since L_n is based only on the r.v.'s $s(Y_1^*), \ldots, s(Y_n^*)$, it is distribution-free under H_0. Furthermore, it is not difficult to show that L_n is affine-invariant. Table 4 in the Appendix gives a set of exact null critical points of nL_n for sample sizes 12 to 20.

For larger sample sizes however, having asymptotic null critical points would be useful. The rest of this section will be concerned with the asymptotic null distribution of $n^{-1/2}L_n$. Let $\{W(t) : 0 \leq t \leq 1\}$ be the standard Wiener process and define the stochastic process $\{G(t) : 0 \leq t \leq 1\}$ given by

$$G(t) = 2W(t) - W(1).$$

It is straightforward to verify that $G(t)$ is a normal process with mean 0 and covariance function given by

$$\mathrm{Cov}(G(t_1), G(t_2)) = 1 - 2|t_1 - t_2|, \qquad 0 \leq t_1, t_2 \leq 1.$$

The following result, whose proof is given in the Appendix, covers the case of the statistic L_n and will be helpful in obtaining an expression for its asymptotic null c.d.f..

THEOREM 3.1. *Let V_1, $V_2, \ldots$ be iid random variables with $E(V_1) = 0$ and $\mathrm{Var}(V_1) = 1$. Then,*

$$\frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \left| \sum_{i=1}^{n} V_i s(k-i) \right| \xrightarrow{D} \int_0^1 |G(t)| dt$$

where $G(t)$ is the process defined above.

COROLLARY 3.1. *Under the assumption (2.9) and under H_0,*

$$\frac{1}{\sqrt{n}} L_n \xrightarrow{D} \int_0^1 |G(t)| dt.$$

In what follows, we will find an explicit formula of the c.d.f. of the random variable

$$L = \int_0^1 |G(t)| dt. \tag{3.4}$$

The study of the L_1-norm of stochastic processes is much more complicated than that of the L_2-norm where general results are well-known, see Shorack and Wellner (1986) for example. An explicit expression of the c.d.f. of the L_1-norm of the Brownian bridge, based on previous work by Shepp (1982) and Rice (1982), is given in Johnson and Killeen (1983). From earlier work by Kac (1946), Aki and Kashiwagi (1989) obtained an expression for the c.d.f. of the L_1-norm of the Wiener process.

From now on, we will follow closely the treatment of Kac (1946). In view of Theorem 3.1, we can choose any distribution for the iid r.v.'s V_1, $V_2, \ldots$, as long as they have 0 expectation and unit variance. Hence, we will suppose that $V_1, V_2, \ldots$ possess the *standardized double-exponential distribution with density function*

$$\frac{\exp(-\sqrt{2}|x|)}{\sqrt{2}}, \qquad -\infty < x < \infty.$$

The first step is to obtain the limiting, as $n \to \infty$, Laplace transform of

$$D_n = \frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \left| \sum_{i=1}^{n} V_i s(k-i) \right|, \tag{3.5}$$

which is the Laplace transform of L. Then, we will invert it to get an explicit expression for the c.d.f. of L.

Let α_j and α_j' denote the j-th zero of the Airy function and of the derivative of the Airy function respectively, see page 446 of Abramowitz and Stegun (1964).

THEOREM 3.2. *For $z > 0$, the Laplace transform of L is*

$$E(e^{-zL}) = 2 \sum_{j=1}^{\infty} (e^{\alpha_j'(2z^2)^{1/3}} - e^{\alpha_j (2z^2)^{1/3}}).$$

PROOF. For $z > 0$, the Laplace transform of D_n $(n \geq 3)$ is

$$E(e^{-zD_n}) = \frac{1}{2^{n/2}} \int_{\Re^n} \exp\left(\frac{-z}{n^{3/2}} \sum_{k=1}^{n} \left| \sum_{i=1}^{n} x_i s(k-i) \right| \right) \exp\left(-\sqrt{2} \sum_{i=1}^{n} |x_i| \right) dx_1 \cdots dx_n.$$

By making the change of variables

$$y_k = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} x_i s(k-i), \qquad k = 1, \ldots, n,$$

whose Jacobian is $1/(\sqrt{2})^{n-2}$, and by letting

$$K(s,t) = \frac{e^{-\beta|s|} e^{-|s-t|} e^{-\beta|t|}}{2} \qquad \text{and} \qquad \beta = \frac{z}{\sqrt{2} n^{3/2}},$$

we obtain

$$E(e^{-zD_n}) = 2 \int_{\Re^n} K(-y_n, y_1) K(y_1, y_2) \cdots K(y_{n-1}, y_n) dy_1 \cdots dy_n. \tag{3.6}$$

Now, letting $\lambda_1, \lambda_2, \ldots$ be the eigenvalues and $f_1(t), f_2(t), \ldots$ be the normalized eigenfunctions of the integral equation

$$\int_{-\infty}^{\infty} K(s,t) f(t) dt = \lambda f(s), \tag{3.7}$$

(3.6) becomes

$$2 \sum_{j=1}^{\infty} \lambda_j^n \int_{-\infty}^{\infty} f_j(t) f_j(-t) dt. \tag{3.8}$$

Note that the integral equation (3.7) is solved in Kac (1946). Let J_ν denote the Bessel function of the first kind of order ν. The eigenvalues are seen to be

$$\lambda_{2i-1} = \frac{1}{(r_i \beta)^2}, \qquad \lambda_{2i} = \frac{1}{(t_i \beta)^2}, \qquad i = 1, 2, \ldots,$$

where r_i is the i-th positive root of $J_{1/\beta}(x) = 0$ and t_i is the i-th positive root of $J'_{1/\beta}(x) = 0$. The corresponding eigenfunctions are

$$f_{2i-1}(t) = \frac{s(t)J_{1/\beta}(r_1 e^{-\beta|t|})e^{-\beta|t|}}{M_{2i-1}} \quad \text{and} \quad f_{2i}(t) = \frac{J_{1/\beta}(t_1 e^{-\beta|t|})e^{-\beta|t|}}{M_{2i}}, \quad i = 1, 2, \ldots,$$

where $M_1, M_2, \ldots$ are normalizing constants. Noticing that

$$\int_{-\infty}^{\infty} f_{2i-1}(t) f_{2i-1}(-t)dt = -1 \quad \text{and} \quad \int_{-\infty}^{\infty} f_{2i}(t) f_{2i}(-t) = 1,$$

we can write, in view of (3.8),

$$E(e^{-zD_n}) = 2\sum_{j=1}^{\infty}\left(\frac{1}{(t_j\beta)^{2n}} - \frac{1}{(r_j\beta)^{2n}}\right). \tag{3.9}$$

The final step in order to obtain the Laplace transform of L is to take the limit, as $n \to \infty$, of (3.9). Let

$$P(y) = \frac{\sqrt{2y}}{3}\left(J_{-1/3}\left(\frac{(2y)^{3/2}}{3}\right) + J_{1/3}\left(\frac{(2y)^{3/2}}{3}\right)\right).$$

Kac (1946) also showed that

$$(\beta t_j)^{2n} \to e^{\delta_j (2z)^{2/3}} \quad \text{and} \quad (\beta r_j)^{2n} \to e^{\gamma_j (2z)^{2/3}} \quad \text{as } n \to \infty,$$

where γ_j and δ_j are the j-th positive zeros of $P(y)$ and $P'(y)$ respectively. Therefore,

$$\lim_{n\to\infty} E(e^{-zD_n}) = E(e^{-zL}) = 2\sum_{j=1}^{\infty}(e^{\delta_j(2z)^{2/3}} - e^{\gamma_j(2z)^{2/3}}),$$

where the passage to the limit is justified since, for $n \geq 3$ and $j \geq 2$,

$$\frac{1}{(t_j\beta)^2} \leq \frac{c}{(j-1)^2} \quad \text{and} \quad \frac{1}{(r_j\beta)^2} \leq \frac{c}{(j-1)^2}$$

where c is a fixed constant. The first inequality is taken directly from Kac (1946) and the second follows since the t_j's and r_j's alternate, i.e., $t_1 < r_1 < t_2 < r_2 \cdots$.

Finally, the fact that $\gamma_j = -\alpha_j/2^{1/3}$ and $\delta_j = -\alpha'_j/2^{1/3}$, $j = 1, 2, \ldots$, completes the proof.

The next step consists in inverting the Laplace transform of L. Aki and Kashiwagi (1989) noted that $\exp(-z^{2/3})$ is the Laplace transform of the positive stable distribution of order $2/3$ whose c.d.f. may be represented as

$$F(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \exp\left(-\frac{1}{x^2}h(u)\right)du$$

where

$$h(u) = \left(\frac{\sin((2u+\pi)/3)}{\cos(u)}\right)^2 \frac{\cos((\pi-u)/2)}{\cos(u)}.$$

THEOREM 3.3. *The cumulative distribution function of L is*

$$F_L(x) = 2\sum_{j=1}^{\infty}\left(F\left(\frac{x}{\sqrt{2}(-\alpha_j')^{3/2}}\right) - F\left(\frac{x}{\sqrt{2}(-\alpha_j)^{3/2}}\right)\right), \qquad x > 0.$$

PROOF. The expression of $F_L(x)$ comes from the termwise inversion of the Laplace transform of L. Since $\alpha_j' > \alpha_j \ \forall j$, the terms in the series

$$\sum_{j=1}^{\infty}\left(F\left(\frac{x}{\sqrt{2}(-\alpha_j')^{3/2}}\right) - F\left(\frac{x}{\sqrt{2}(-\alpha_j)^{3/2}}\right)\right) \tag{3.10}$$

are nonnegative. Hence, to justify the termwise inversion, it suffices to show that (3.10) converges. Define, for $j = 1, 2, \ldots$,

$$x_{1j} = \frac{x}{\sqrt{2}(-\alpha_j')^{3/2}} \quad \text{and} \quad x_{2j} = \frac{x}{\sqrt{2}(-\alpha_j)^{3/2}}.$$

Note that $h(u)$ is increasing on $[-\pi/2, \pi/2)$ with $h(-\pi/2) = 4/27$. Furthermore, from Johnson and Killeen (1983), we have that

$$(-\alpha_j') = 2^{1/3}\delta_j \geq \frac{1}{2^{2/3}}\left(3\pi\left(j - 2 + \frac{7}{12}\right)\right)^{2/3} \qquad \forall j \geq 3.$$

Hence,

$$\begin{aligned}\frac{1}{\pi}(F(x_{1j}) - F(x_{2j})) &\leq \frac{1}{\pi}F(x_{1j}) \\ &\leq \exp\left(\frac{-4}{27x_{1j}^2}\right) \\ &= \exp\left(\frac{-8(-\alpha_j')^3}{27x^2}\right) \\ &\leq \exp\left(\frac{-2\pi^2}{3x^2}(j - 2 + 7/12)^2\right)\end{aligned}$$

and the series (3.10) converges.

The first few terms in the series expansion of F_L can be computed by numerical integration using the values of α_j and α_j' given on page 478 of Abramowitz and Stegun (1964) for $j = 1, \ldots, 10$. The convergence is very rapid and only the first three or four terms are needed. Table 1 gives $F_L(x)$ for selected values of x.

Recall that F_L is the asymptotic null c.d.f. of the test statistic $L_n/\sqrt{n}$. We recommend that the exact critical points given in Table 4 should be used for sample sizes less or equal than 20 and that asymptotic critical points could be used for larger sample sizes. To give an idea of the approximation, if one used the .05 level critical point from the asymptotic distribution, the actual exact levels obtained for sample sizes 12 to 20 would be (12,.023), (13,.038), (14,.032), (15,.040), (16,.044), (17,.045), (18,.044), (19,.042) and (20,.047).

Table 1. Cumulative distribution function of L.

x	$F_L(x)$	x	$F_L(x)$	x	$F_L(x)$	x	$F_L(x)$
.3617	.05	.6553	.40	.9976	.75	1.3937	.94
.4162	.10	.6957	.45	1.0689	.80	1.4366	.95
.4606	.15	.7378	.50	1.1547	.85	1.4874	.96
.5009	.20	.7821	.55	1.2660	.90	1.5504	.97
.5396	.25	.8293	.60	1.2933	.91	1.6352	.98
.5778	.30	.8800	.65	1.3232	.92	1.7708	.99
.6161	.35	.9356	.70	1.3564	.93	2.1612	.999

4. Simulation study and example

In this section, we will present the results from a small simulation study and conclude with a real data example. The four tests considered are the test based on L_n, Blumen's test, Hodges' test and for comparison purposes, Hotelling's test. The test based on L_n is a L_1 norm sign test, Blumen's test is a L_2 norm sign test (the one with the choice $\phi(u) = \cos(\pi u)$ in the class (2.7)) and Hodges' test is a sup norm sign test. Four distributions were used to generate pseudo-observations (X, Y). The first is the standard bivariate normal distribution with independent marginals. The second one is a bivariate Cauchy distribution where X and Y are distributed independently as a standard Cauchy variate. The third distribution considered is the distribution for which X and Y are independent, X being standard normal and Y being standard Cauchy. The fourth is a skewed distributions generated via $(X, Y)(1 + X/\sqrt{(X^2 + Y^2)})$ with (X, Y) being a bivariate normal vector with independent marginals. This distribution is thus skewed in the direction of the first component X. The null hypothesis (amount of shift (0,0)) and between 2 and 10 shift alternatives are considered for each distributions. Namely, we considered 2 alternatives for the bivariate normal distribution, 4 for the bivariate Cauchy distribution (2 in the X-direction and 2 on the main diagonal), 6 for the normal-Cauchy distribution (2 in the X-direction, 2 in the Y-direction and 2 on the main diagonal) and 10 for the skewed-normal distribution (2 in the X-direction, 2 in the (−X)-direction, 2 in the Y-direction, 2 on the main diagonal and 2 on the axis perpendicular to the main diagonal). Each power was estimated with 10000 replications.

Tables 2 and 3 give the observed probability of rejecting H_0 for sample sizes of 16 and 49 respectively. All tests were performed at or near (as described below) the 5% level. For Hotelling's test, the first number is the result when using the critical points from an F-distribution while the second one (between parentheses) is the result when using the χ^2 distribution. For n=16, the critical points for the three sign tests were obtained from their exact null distributions. Thus, the exact levels of the test based on L_n, Blumen's and Hodges' tests are .0444, .0439 and .0440 respectively. For n=49, the asymptotic critical points were used for Blumen's test and the test based on L_n. An exact critical point was still used for Hodges' test giving an exact level of .05038. Those sample sizes were chosen in regards to the small number of available natural levels for Hodges' test.

The results show that Hotelling's test is, obviously, the more powerful for the bivariate normal distribution. In that case and among the three sign tests, Blumen's and L_n tests are the better one when n=16 but the former seems to get a little edge when n=49. This is in accordance with the fact, demonstrated in Oja and Nyblom (1989), that Blumen's test is the locally most powerful invariant sign test for elliptical distribu-

Table 2. Observed probability of rejecting H_0 for $n = 16$.

Population	Amount of shift	L_n	Blumen	Hodges	Hotelling's T^2
Bivariate	(0,0)	.0440	.0446	.0461	.0528 (.0958)
normal	(.45,0)	.2267	.2274	.2174	.2895 (.4085)
	(.73,0)	.5223	.5221	.5042	.6462 (7617)
Bivariate	(0,0)	.0453	.0445	.0430	.0169 (.0432)
Cauchy	(.74,0)	.2104	.2125	.2163	.0793 (.1420)
	(1.5,0)	.4964	.5046	.5465	.2514 (.3314)
	(.47,.47)	.1986	.1971	.1885	.0682 (.1276)
	(.9,.9)	.5417	.5354	.4929	.2123 (.3103)
Normal-	(0,0)	.0481	.0464	.0449	.0360 (.0736)
Cauchy	(.47,0)	.2219	.2214	.2145	.3027 (.4262)
	(.77,0)	.5123	.5165	.5114	.6954 (.8068)
	(0,.7)	.1995	.2003	.2138	.0935 (.1569)
	(0,1.53)	.5255	.5346	.5867	.2762 (.3625)
	(.38,.38)	.2217	.2211	.2082	.2197 (.3337)
	(.64,.64)	.5310	.5281	.4934	.5630 (.6973)
Skewed-	(0,0)	.0440	.0446	.0461	.2550 (.4503)
normal	(.05,0)	.2281	.2152	.2100	.3338 (.5502)
	(.105,0)	.5248	.5020	.4623	.4374 (.6661)
	(.048,.048)	.2121	.2124	.2141	.3331 (.5508)
	(.115,.115)	.5122	.5131	.5113	.4836 (.7098)
	(0,.14)	.2120	.2113	.2324	.2856 (.4894)
	(0,.4)	.5101	.5147	.5442	.5093 (.7026)
	(−.26,−.26)	.2090	.2061	.2012	.1427 (.2546)
	(−.6,−.6)	.5006	.4951	.4762	.4234 (.5467)
	(−.56,0)	.2104	.2067	.1965	.0827 (.1264)
	(−1.1,0)	.5114	.5069	.4676	.3935 (.4766)

tions. For the bivariate Cauchy distribution (not an elliptical distribution), Hotelling's test has no power and can not keep its level. Among the three sign tests, we see that when the shift is in only one direction (the X-direction here), then Hodges's test is the better one followed by Blumen's test and the test based on L_n comes in third. The situation is reversed when the shift is on the main diagonal as the test based on L_n and Blumen's test become the more powerful (L_n having a slight edge) while Hodges' test comes in third place. For the normal-Cauchy distribution (again, not an elliptical distribution), we see that the performance of Hotelling's test depends strongly on the direction of the shift. Among the three sign tests, when the shift is on the X (normal) direction, the peformance of the three sign tests is similar with the exception that Blumen's test seems to take the edge in the high power region for n=49. When the shift is on the Y (Cauchy) direction, Hodges's test is the better one. When the shift is on the main diagonal, L_n and Blumen's test are the better ones. Finally, for the skewed-normal distribution, Hotelling's test is clearly not an appropriate test since the expectation of this distribution under H_0 (amount of shift (0,0)) is not even (0,0). Its levels are thus completely off but we still include it for the sake of completeness. Among the three sign tests, when the shift is on the X-direction (the direction of the skew), L_n test is by far superior to the other two. When the shift is on the main diagonal, the three tests are

Table 3. Observed probability of rejecting H_0 for $n = 49$.

Population	Amount of shift	Statistics L_n	Blumen	Hodges	Hotelling's T^2
Bivariate	(0,0)	.0502	.0529	.0507	.0549 (.0665)
normal	(.22,0)	.1987	.2015	.1910	.2605 (.2969)
	(.38,0)	.5180	.5239	.4907	.6359 (.6752)
Bivariate	(0,0)	.0451	.0465	.0472	.0144 (.0231)
Cauchy	(.34,0)	.1889	.1919	.1994	.0307 (.0412)
	(.61,0)	.4769	.4990	.5332	.0644 (.0820)
	(.24,.24)	.1975	.1977	.1831	.0284 (.0407)
	(.41,.41)	.5114	.4998	.4597	.0600 (.0769)
Normal-	(0,0)	.0419	.0430	.0456	.0340 (.0453)
Cauchy	(.23,0)	.1975	.1978	.1944	.2652 (.3047)
	(.38,0)	.4695	.4803	.4584	.6400 (.6817)
	(0,.35)	.2042	.2127	.2197	.0508 (.0637)
	(0,.61)	.4902	.5159	.5564	.0865 (.1037)
	(.185,.185)	.1992	.1973	.1846	.1839 (.2146)
	(.33,.33)	.5385	.5338	.4901	.5251 (.5694)
Skewed-	(0,0)	.0502	.0529	.0507	.9788 (.9858)
normal	(.016,0)	.2314	.2136	.2318	.9860 (.9911)
	(.0387,0)	.5836	.5283	.5354	.9925 (.9958)
	(.013,.013)	.2127	.2111	.2440	.9848 (.9903)
	(.034,.034)	.4987	.5021	.5337	.9919 (.9947)
	(0,.03)	.1967	.2062	.2605	.9798 (.9862)
	(0,.097)	.4823	.5013	.5966	.9829 (.9882)
	(−.075,−.075)	.2091	.2114	.2686	.9056 (.9291)
	(−.21,−.21)	.5086	.5093	.5368	.7148 (.7611)
	(−.24,0)	.2123	.2078	.2028	.4714 (.5268)
	(−.46,0)	.4980	.4786	.4375	.0801 (.0993)

similar when n=16 but Hodges' test takes the edge for n=49. Hodges's test is still the better one, and by far, when the shift is on the Y-direction. For shifts in the direction perpendicular to the main diagonal, the Blumen and L_n tests are the best when n=16 while Hodges's test becomes the better one when n=49. The last case is when the shift is on the $(-X)$-direction, there, L_n test is superior to the other two.

The results of this simulation show that there is no clear winner among the three bivariate sign tests considered here. Their relative performance depends on the distribution and on the direction of the shift and each of them can be in turn the best one or the worst. In accordance with the fact that they are distribution-free under H_0, they all maintained very well their prescribed levels.

To conclude this section, the same four test procedures were carried out on a subset of a data set from Ryan *et al.* (1976). The data consists in systolic and diastolic blood pressures measurements on 15 male Peruvian Indians over 21 years old who were born at a high altitude and whose parents were also born at a a high altitude. The data, also analyzed in Hettmansperger (1984), are (170,76), (125,75), (148,120), (140,78), (106,72), (108,62), (124,70), (134,64), (116,76), (114,74), (118,68), (138,78), (134,86), (124,64) and (114,66). As Hettmansperger (1984) and within the framework of hypothesis (1.1), we wish to test $H_0 : (\delta_1, \delta_2) = (120, 80)$ which are the standard blood pressures values for healthy males over 21 years old in the United States. For the transformed data set $(X_i - 120, Y_i - 80)$, $i, = 1, \ldots, 15$, the values of Hotelling's statistic, nL_n, $2W_\phi$ (with $\phi(u) = \cos(\pi u)$) and Hodges's statistic are 4.12, 97, 9.32 and 13 respectively. With a p-value of .0411, Hotelling's test is quite inconclusive. On the other hand, the exact p-values (from their exact null distributions) of the test based on L_n, Blumen's test and Hodges' test are .0110, .0037 and .0119 respectively all indicating more evidence against H_0 *than Hotelling's test.*

5. Concluding remarks

In this paper, we studied sign tests in the bivariate location problem. The starting point was the set of univariate sign statistics computed from the projections of the observations on all lines passing through the origin. Affine-invariant test statistics and distribution-free bivariate sign tests were constructed by taking the sup, L_1 and L_2 norms of these statistics. The use of the sup-norm gives rise to Hodges' test which has been studied extensively. Also, it was seen that the class of tests derived from the L_2-norm was closely related to a class proposed by Oja and Nyblom (1989). The L_1-norm induced a new test statistic. Small sample, as well as asymptotic critical points were obtained. The asymptotic null distribution of this statistic is the same as that of the L_1-norm of a certain normal process which is related to the standard Wiener process. The result giving an explicit expression of its c.d.f. is interesting in itself. The results from a simulation study show that each of the three approach have its own merits and that there is no clear winner among them.

Acknowledgements

The authors wish to thank an anonymous referee who pointed out an error in an earlier version of this paper. His comments and suggestions were also helpful in preparing the final version.

Appendix

PROOF OF THEOREM 2.1. Since the proof is a direct application of Theorem 2 of de Wet and Venter (1973), we will use their notation. Write

$$W_\phi = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijn} s(Y_i^*) s(Y_j^*)$$

where $c_{ijn} = \phi(|i-j|/n)/n$. The cosine Fourier series of ϕ, which converges uniformly since ϕ has a bounded continuous derivative on (0,1), is

$$\phi(x) = \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

For each $m = 1, 2, \ldots$, let

$$\gamma_{2m-1} = \gamma_{2m} = \frac{a_m}{2},$$

$$b_{i,2m-1,n} = \frac{\sqrt{2}}{\sqrt{n}} \cos\left(\frac{im\pi}{n+1}\right) \quad \text{and} \quad b_{i,2m,n} = \frac{\sqrt{2}}{\sqrt{n}} \sin\left(\frac{im\pi}{n+1}\right).$$

Let us verify the conditions of their theorem.

Condition (B4):

$$\begin{aligned}\sum_{i=1}^{n} b_{i,2m-1,n} b_{i,2k-1,n} &= \frac{2}{n}\sum_{i=1}^{n} \cos\left(\frac{im\pi}{n+1}\right)\cos\left(\frac{ik\pi}{n+1}\right)\\ &\to 2\int_0^1 \cos(xm\pi)\cos(xk\pi)dx \quad \text{as} \quad n\to\infty\\ &= \delta_{mk}.\end{aligned}$$

The other cases follow similarly. The verification of condition (B5) is immediate.

Condition (C):

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijn}^2 &= \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\phi^2\left(\frac{k}{n}\right) + \frac{4\phi^2(0)}{n}\\ &\to 2\int_0^1 (1-x)\phi^2(x)dx \quad \text{as} \quad n\to\infty\\ &= \int_0^1 \phi^2(x)dx \quad \text{since} \quad \phi(x) = -\phi(1-x)\\ &= \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \quad \text{by Parseval's Theorem}\\ &= \sum_{n=1}^{\infty}\gamma_n^2.\end{aligned}$$

Condition (C1):

$$\sum_{n=1}^{\infty}|\gamma_n| = \sum_{n=1}^{\infty}|a_n| < \infty$$

since ϕ possesses a bounded continuous derivative.

Condition (BC1): as $n\to\infty$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijn} b_{i,2m-1,n} b_{j,2m-1,n}$$

$$\to 2\int_0^1\int_0^1 \cos(m\pi x)\cos(m\pi y)\phi(|x-y|)dxdy. \tag{A.1}$$

But, since

$$\phi(|x-y|) = \sum_{n=1}^{\infty} a_n(\cos(n\pi x)\cos(n\pi y) - \sin(n\pi x)\sin(n\pi y)),$$

we find easily that (A.1) equals $a_m/2 = \gamma_{2m-1}$ which establishes this condition. Finally, since

$$\sum_{i=1}^{n} c_{iin} = \phi(0) = \sum_{n=1}^{\infty} a_n\cos(n\pi 0) = \sum_{n=1}^{\infty}\gamma_n,$$

Theorem 2 of de Wet and Venter (1973), along with their following remark, entails that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijn} s(Y_i^*) s(Y_j^*) \xrightarrow{D} \sum_{m=1}^{\infty} \gamma_m Z_m^2 \quad \text{as} \quad n \to \infty,$$

where $Z_1, Z_2, \ldots$ are iid standard normal variates and this in turn concludes the proof.

PROOF OF THEOREM 3.1. We will use Proposition 1.10.3, page 87 of Prakasa Rao (1987). Let, for each $k = 0, 1, \ldots, n$,

$$Z_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i s(k-i).$$

Consider the process $G_n(t)$ defined by

$$G_n(t) = Z_{[tn]}, \qquad 0 \le t \le 1,$$

where $[u]$ is the integer part of u. In other words, if $k/n \le t < (k+1)/n$, then $G_n(t) = Z_k$. Let us invoke the Cramér-Wold device in order to verify condition (i) of Prakasa Rao. Let $t_1, \ldots, t_k$ be in $[0,1]$. We will show that

$$\sum_{r=1}^{k} \lambda_r G_n(t_r) \xrightarrow{D} \sum_{r=1}^{k} \lambda_r G(t_r) \quad \text{as} \quad n \to \infty,$$

where $\lambda_1, \ldots, \lambda_k$ are arbitrary constants. Write

$$\sum_{r=1}^{k} \lambda_r G_n(t_r) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_{in} V_i$$

where $a_{in} = \sum_{r=1}^{k} \lambda_r s([t_r n] - i)$. Now, we use Theorem V.1.2 of Hájek and Šidák (1967) which also holds for a triangular array of constants a_{in} as in our case. Since, as $n \to \infty$,

$$\frac{1}{n} \sum_{i=1}^{n} a_{in}^2 \to \sum_{r=1}^{k}\sum_{u=1}^{k} \lambda_r \lambda_u (1 - 2|t_r - t_u|) < \infty$$

and

$$\frac{1}{\sqrt{n}} \max_{1 \le i \le n} |a_{in}| < \frac{1}{\sqrt{n}} \sum_{r=1}^{k} |\lambda_r| \to 0,$$

the conclusion of Theorem V.1.2 of Hájek and Šidák (1967) is that

$$\sum_{r=1}^{k} \lambda_r G_n(t_r) \xrightarrow{D} N\left(0, \sum_{r=1}^{k}\sum_{u=1}^{k} \lambda_r \lambda_u (1 - 2|t_r - t_u|)\right).$$

Thus condition (i) of Prakasa Rao is satisfied. As for condition (ii), let $0 \le t_1 < t_2 < t_3 \le 1$ be such that $G_n(t_i) = Z_{k_i}$, $i = 1, 2, 3$, with $0 \le k_1 < k_2 < k_3 \le n$. Note that

$$G_n(t_2) - G_n(t_1) = \frac{2}{\sqrt{n}} \sum_{i=k_1+1}^{k_2} V_i$$

and

Table 4. Critical points of nL_n. The table gives the smallest value of x for which $P(nL_n \geq x) \leq c$.

	c												
n	.9	.8	.7	.6	.5	.4	.3	.2	.1	.05	.01	.005	.001
12	20	24	28	32	36	40	44	48	56	60	72	-	-
13	23	27	31	33	37	41	47	51	59	69	81	85	-
14	26	30	34	38	42	46	54	58	70	78	90	94	-
15	29	33	37	41	45	51	57	63	75	83	101	105	113
16	32	36	40	44	52	56	64	72	84	92	108	116	124
17	33	39	45	49	55	61	67	77	91	99	121	125	141
18	34	42	46	54	62	66	74	86	98	110	130	138	150
19	39	45	51	57	65	71	79	91	107	119	139	147	165
20	40	48	56	64	68	76	88	100	116	128	152	164	180

$$G_n(t_3) - G_n(t_2) = \frac{2}{\sqrt{n}} \sum_{i=k_2+1}^{k_3} V_i.$$

Since these last two r.v.'s are independent, we have

$$\begin{aligned} E[(G_n(t_2) - G_n(t_1))^2 (G_n(t_3) - G_n(t_2))^2] &= \mathrm{Var}\left(\frac{2}{\sqrt{n}} \sum_{i=k_1+1}^{k_2} V_i\right) \mathrm{Var}\left(\frac{2}{\sqrt{n}} \sum_{i=k_2+1}^{k_3} V_i\right) \\ &= 16 \frac{(k_2 - k_1)}{n} \frac{(k_3 - k_2)}{n} \\ &\leq 16(t_3 - t_1)^2. \end{aligned}$$

This verifies condition (ii) with $\alpha_1 = \alpha_2 = 2$, $c = 16$ and $\beta = 1$. Finally, the fact that

$$\int_0^1 |G_n(t)| dt = \frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \left| \sum_{i=1}^{n} V_i s(k - i) \right|$$

concludes the proof.

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