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RANK TESTS BASED ON EXCEEDING OBSERVATIONS

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Abstract. Rank tests based on the maximum number of exceeding observations for several standard nonparametric hypotheses are proposed. An approach to constructing nonparametric rank tests via metrics on the permutation group is used. The test statistics are based on a metric induced by Chebyshev's norm.

Key words and phrases: Induced rank test statistic, location problem, multi-sample problems, partial rankings, metric on permutations, invariance, Chebyshev's metric, subgroups of permutations.

1. Distances involved in rank tests

Distances between two sets of permutations are involved in many rank statistics. Critchlow (1992) proposed a construction based on distances which produces many familiar rank test statistics. The method allows the creation of families of statistics for standard nonparametric hypotheses, based on the same distance. The proposed test statistics are minimum interpoint distance between appropriate sets of permutations. We enlarge the class of test statistics for five standard nonparametric hypothesis testing situations with new statistics based on Chebyshev's metric. We also derive some combinatorial and group theoretic properties of Chebyshev's metric that play a key role in the computation of the corresponding test statistics. The basic notation follows.

Let $X = (X_1, \dots, X_n)$ be a random variable with values in a measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. The space $\mathcal{X}$ is assumed to be a symmetric Borel subset of $\boldsymbol{R}^n$ and $\mathcal{B}(\mathcal{X})$ is the Borel field inherited from $\boldsymbol{R}^n$. For $x = (x_1, \dots, x_n)$ where no two coordinates coincide, let $\alpha_x(i)$ be the number of coordinates not greater than x_i. The corresponding statistic $\alpha_X(i)$ is called the rank of X_i. The vector $\alpha_X = (\alpha_X(1), \dots, \alpha_X(n))$ is a permutation of $1, \dots, n$. We use the notation S_n for the space of all permutations of n integers.

To test any nonparametric hypothesis H versus an alternative A let $\alpha \in S_n$ be the rank vector corresponding to the observed sample. Identify two suitable sets of permutations. The equivalence class $[\alpha]$ consists of all permutations in S_n which are equivalent (for the particular testing problem) to the observed permutation α. The set E of extremal permutations consists of all permutations in S_n which are least in agreement with H and most in agreement with A. (See Critchlow ((1992), Subsection 7.1) for a more detailed formulation and motivation of these sets.) Then the proposed test statistic is the minimum interpoint distance between the sets $[\alpha]$ and E:

$$d([\alpha], E) = \min_{\substack{\pi \in [\alpha] \\ \sigma \in E}} d(\pi, \sigma),$$

where d is an arbitrary metric on S_n. The test rejects the null hypothesis for small values of $d([\alpha], E)$.

Critchlow (1992) has investigated this method for generating rank test statistics via metrics on the permutation group. He has constructed rank test statistics for two-sample and multi-sample problems using different metrics on the permutation group. The method gives some well-known tests induced by Kendall's tau, Spearman's rho and Spearman's footrule and it also gives some new tests by using of the Hamming distance and the Ulam distance.

Although the construction gives rise to many familiar rank tests statistics, it also induces many new statistics, whose behaviour is not known. The distribution, mean, variance, efficiency, etc. therefore have to be obtained on a case by case basis for many test statistics. Fueda (1993) studies the asymptotic distribution and the efficiency of a test statistic suggested by Critchlow (1992) for the two-sample problem. Further, Fueda (1996) introduces a convex sum distance and proves the limiting normality of a class of test statistics for the two-sample problem derived by Critchlow's method.

The goal of this paper is to study test statistics $d([\alpha], E)$ induced by Chebyshev's metric.

For $\alpha, \beta \in S_n$, Chebyshev's metric

$$M(\alpha, \beta) = \max_{1 \le i \le n} |\alpha(i) - \beta(i)|$$

is the maximum of the absolute values of the differences between the ranks. It is easily checked that the function M is a right-invariant metric on S_n, in the sense that $M(\alpha \circ \nu, \beta \circ \nu) = M(\alpha, \beta)$ for all $\alpha, \beta, \nu \in S_n$. Moreover, M possesses the transposition property.

Let τ_{ij} denote the element of S_n which interchanges i and j, leaving all other elements of $\{1, \ldots, n\}$ fixed.

DEFINITION 1. (Transposition property) The metric d on S_n satisfies the transposition property if

$$d(\alpha, \beta) \le d(\alpha, \beta') \tag{1.1}$$

for all i and j, where α, β, and β' are permutations satisfying $\alpha(i) < \alpha(j)$, $\beta(i) < \beta(j)$, and $\beta' = \beta \circ \tau_{ij}$.

PROPOSITION 1. *Chebyshev's metric satisfies the transposition property.*

The Proof of Proposition 1 is in the Appendix.

Metrics possessing the transposition property are used for defining monotone rank statistics for some hypothesis testing problems. Such statistics have a monotone power function for stochastically ordered alternatives and produce an unbiased test.

For comprehensive accounts of statistical measures on permutations one is referred to Diaconis (1988) and Critchlow (1985).

2. Nonparametric hypotheses

As Hajek and Sidak (1967) treat the rank statistics, a density $p \in H_0$ if and only if

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i), \tag{2.1}$$

where $f(x)$ is an arbitrary one-dimensional density. Under H_0 the observations X_i are assumed to be independent and identically distributed according some density f.

The hypothesis H_1 denote the family of n-dimensional densities p such that

$$p = \prod_{i=1}^{n} f(x_i), \tag{2.2}$$

where $f(x)$ is an arbitrary symmetric one-dimensional density $f(x) = f(-x)$. Note that $H_1 \subset H_0$.

Under H_0 a randomized test is called a rank test if its critical function Φ is a function of the rank vector α. Under H_0 the random vector $\alpha_X = (\alpha_X(1), \dots, \alpha_X(n))$ is distributed uniformly. The Neyman-Pearson lemma gives the most powerful rank test of H_0 against some simple alternative. (See for instance Hajek and Sidak (1967).) The exact evaluation of the critical function is rarely possible because the distribution under the alternative is difficult to compute. One of the few exceptions is the translation of the uniform distribution (Theorem 2.3).

2.1 *A two-sample rank test for one-sided alternative*

Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be random samples with distribution functions F and G respectively. We wish to test the hypothesis H_0 defined by (2.1) that F and G are identical. The alternative A is that $F(x) \geq G(x)$, with strict inequality for some x.

Denote the rank of X_i among $X_1, \dots, X_m, Y_1, \dots, Y_n$ by $\alpha(i)$ and the rank of Y_i by $\alpha(m+i)$. Thus $\alpha \in S_{m+n}$. Let $S_m \times S_n$ be the subgroup of S_{m+n} given by:

$$S_m \times S_n = \{\gamma \in S_{m+n} : \gamma(i) \leq \gamma(j), i = 1, \dots, m, j = m+1, \dots, m+n, \}.$$

Thus $S_m \times S_n$ consists of all rankings which permute the first m items among the first m ranks, and which permute the remaining n items among the remaining n ranks. The equivalence class $[\alpha]$, that assigns the same set of ranks to the first population as α, is the left coset $\alpha(S_m \times S_n)$.

The extremal set E is the subgroup $S_m \times S_n$. Thus the test statistic for H_0 versus A is given by

$$M([\alpha], E) = \min_{\substack{\pi \in \alpha(S_m \times S_n) \\ \sigma \in S_m \times S_n}} M(\pi, \sigma). \tag{2.3}$$

Let $a_1 < \cdots < a_m$ be the ranks assigned by α to the first population, and $a_{m+1} < \cdots < a_{m+n}$ be the ranks assigned by α to the second population.

THEOREM 2.1. *The test statistic for the two-sample problem with a one-sided alternative induced by Chebyshev's metric is equal to*:

$$M(\alpha(S_m \times S_n), S_m \times S_n) = \max\{a_m - m, m + 1 - a_{m+1}\}. \tag{2.4}$$

PROOF. Since Chebyshev's metric is right-invariant the test statistic (2.3) is equal to

$$\min_{\pi \in \alpha(S_m \times S_n)} M(\pi, e),$$

where e is the identity permutation $e(i) \equiv i$ for all i.

Define $\alpha_0 \in S_n$ by $\alpha_0(i) = a_i$ for $i = 1, \dots, m+n$. Clearly, $\alpha_0 \in S_m \times S_n$ since α_0 assigns the same set of ranks to the first population as α. The transposition property implies that $M(\alpha_0, e) = \min_{\pi \in \alpha(S_m \times S_n)} M(\pi, e)$.

The next theorem states the exact distribution of the M-test statistic.

Theorem 2.2. *Let $M = \max\{a_m - m, m+1-a_{m+1}\}$ be the M-test statistic. Then under H_0*

$$(2.5)\quad \begin{aligned} P(M=0) &= \binom{m+n}{m}^{-1}; \\ P(M=k) &= \binom{m+n}{m}^{-1}\binom{2k-2}{k-1}\frac{3k-2}{k}, \quad \text{for} \quad 1 \le k \le \min(m,n); \\ P(M=k) &= \binom{m+n}{m}^{-1}\binom{l+k-1}{l-1}, \quad \text{for} \quad \min(m,n) < k \le \max(m,n), \end{aligned}$$

where $l = \min(m,n)$.

Proof. Formulas (2.5) follow by applying well-known combinatorial rules. The event $\{M = k\}$ is decomposed into events $\{a_m = m+k, a_{m+1} = j\}$ with $j = m+1-k, \ldots, m$, and $\{a_m = j, a_{m+1} = m+1-k\}$ with $j = m+1, \ldots, m+n$. Lemma 1 below gives the total number of ways these events can occur.

Table 1. The number of points in the lower tail $P\{M \le k\} \le 0.01$ of the distribution of M under H_0.

$n\backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
3	*																						
4	*	*																					
5	*	0	1																				
6	*	1	1	2																			
7	0	1	2	2	3																		
8	0	1	2	3	3	4																	
9	1	2	3	3	4	4	5																
10	1	2	3	4	4	5	5	6															
11	1	2	3	4	5	5	6	7	7														
12	1	2	3	4	5	6	6	7	8	8													
13	1	3	4	5	5	6	7	7	8	9	9												
14	2	3	4	5	6	6	7	8	8	9	10	10											
15	2	3	4	5	6	7	8	8	9	9	10	11	11										
16	2	3	4	5	6	7	8	9	9	10	10	11	12	12									
17	2	3	5	6	7	7	8	9	10	10	11	11	12	13	13								
18	2	4	5	6	7	8	8	9	10	11	11	12	12	13	14	14							
19	2	4	5	7	8	8	9	10	10	11	12	12	13	14	14	15	15						
20	2	4	6	7	8	9	10	10	11	11	12	13	13	14	15	15	16	16					
21	3	5	6	7	9	9	10	11	11	12	12	13	14	14	15	16	16	17	17				
22	3	5	7	8	9	10	11	11	12	13	13	13	14	15	15	16	17	17	18	18			
23	3	5	7	8	10	11	11	12	13	13	14	14	14	15	16	16	17	18	18	19	19		
24	3	5	7	9	10	11	12	13	13	14	14	15	15	16	16	17	17	18	19	19	20	20	
25	3	6	8	9	11	12	13	13	14	15	15	16	16	16	17	17	18	18	19	20	20	21	21

Table 2. The number of points in the lower tail $P\{M \leq k\} \leq 0.05$ of the distribution of M under H_0.

$n\backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
3	0																						
4	0	1																					
5	1	2	2																				
6	1	2	3	3																			
7	1	2	3	4	4																		
8	2	3	3	4	5	5																	
9	2	3	4	4	5	6	6																
10	2	3	4	5	6	6	7	7															
11	2	3	4	5	6	7	7	8	8														
12	3	4	5	6	6	7	8	8	9	9													
13	3	4	5	6	7	7	8	9	9	10	10												
14	3	5	6	7	7	8	8	9	10	10	11	11											
15	4	5	6	7	8	9	9	9	10	11	11	12	12										
16	4	6	7	8	9	9	10	10	10	11	12	12	13	13									
17	5	6	8	9	9	10	10	11	11	11	12	13	13	14	14								
18	5	7	8	9	10	11	11	12	12	12	12	13	14	14	15	15							
19	5	7	9	10	11	11	12	12	13	13	13	14	14	15	15	16	16						
20	6	8	9	10	11	12	13	13	13	14	14	14	15	15	16	16	17	17					
21	6	8	10	11	12	13	13	14	14	15	15	15	15	16	16	17	17	18	18				
22	6	9	10	12	13	13	14	14	15	15	16	16	16	16	17	17	18	18	19	19			
23	7	9	11	12	13	14	15	15	16	16	16	17	17	17	17	18	18	19	19	20	20		
24	7	10	11	13	14	15	15	16	16	17	17	18	18	18	18	19	19	19	20	20	21	21	
25	7	10	12	13	14	15	16	17	17	18	18	18	19	19	19	19	20	20	20	21	21	22	22

Lemma 1. *The number of permutations $\alpha \in S_{m+n}$ for which $\mathrm{a}_m = k$ and $\mathrm{a}_{m+1} = l$ is*

$$
\#\{\mathrm{a}_m = k, \mathrm{a}_{m+1} = l\} = \begin{cases} 0 & \text{if} \quad k < m \ \text{or} \ l > m+1 \\ m!n! & \text{if} \quad k = m \ \text{or} \ l = m+1 \\ m!n! \dbinom{k-l-1}{m-l} & \text{if} \quad m+n \geq k \geq m \ \text{or} \ m \geq l \geq 1. \end{cases} \tag{2.6}
$$

The proof of the lemma is in the Appendix.

Tables of the number of points in the lower tail $P\{M \leq k\} \leq \alpha$ of the distribution of M under H_0 are given for $n = 1, \ldots, 25$; $m = 1, \ldots, n$; $\alpha = 0.01, 0.05$.

Translation of the uniform distribution. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be random samples with uniform distributions. We test H_0 against

$$
q_\Delta(x_1, \ldots, x_{m+n}) = \begin{cases} 1, & \text{if} \quad 0 < x_1, \ldots, x_m < 1, \Delta < x_{m+1}, \ldots, x_{m+n} < 1 + \Delta, \\ 0, & \text{otherwise}, \quad 0 < \Delta < 1. \end{cases} \tag{2.7}
$$

(2.7) means that the second sample is shifted to the right with respect to the first sample.

THEOREM 2.3. *The most powerful rank test of* H_0 *against the alternative* q_Δ *depends on* α *through* $a_m = \max\{\alpha(1), \ldots, \alpha(m)\}$ *and* $a_{m+1} = \min\{\alpha(m+1), \ldots, \alpha(m+n)\}$ *only and is given by*

$$Q_\Delta(\alpha_X = \alpha) = \sum_{i=0}^{m+n-a_m} \sum_{j=0}^{a_{m+1}-1} \Delta^{i+j}(1-\Delta)^{m+n-i-j} \frac{1}{i!j!(m+n-i-j)!}.$$

PROOF. (2.7) entails

$$\begin{aligned}
Q_\Delta(\alpha_X = \alpha) &= \int\cdots\int_{\substack{0<z_1<\ldots<z_{m+n}<1+\Delta \\ \Delta<z_{a_{m+1}}, z_{a_m}<1}} dz_1, \ldots, dz_{m+n} \\
&= \frac{1}{(a_{m+1}-1)!(m+n-a_m)!} \\
&\quad \int\cdots\int_{\Delta<z_{a_{m+1}}<\cdots<z_{a_m}<1} z_{a_{m+1}}^{a_{m+1}-1}(1+\Delta-z_{a_m})^{m+n-a_m} dz_{a_{m+1}}, \ldots, dz_{a_m} \\
&= \frac{1}{(a_{m+1}-1)!(m+n-a_m)!} \sum_{i=0}^{m+n-a_m} \binom{m+n-a_m}{i} \Delta^i \\
&\quad \int\cdots\int_{\Delta<z_{a_{m+1}}<\cdots<z_{a_m}<1} z_{a_{m+1}}^{a_{m+1}-1}(1-z_{a_m})^{m+n-a_m-i} dz_{a_{m+1}}, \ldots, dz_{a_m} \\
&= \sum_{i=0}^{m+n-a_m} \frac{\Delta^i}{i!(a_{m+1}-1)!(m+n-a_{m+1}-i)!} \\
&\quad \int_\Delta^1 z_{a_{m+1}}^{a_{m+1}-1}(1-z_{a_{m+1}})^{m+n-a_{m+1}-i} dz_{a_{m+1}} \\
&= \sum_{i=0}^{m+n-a_m} \frac{\Delta^i}{i!(m+n-i)!} \sum_{j=0}^{a_{m+1}-1} \binom{m+n-i}{j} \Delta^i (1-\Delta)^{m+n-i-j} \\
&= \sum_{i=0}^{m+n-a_m} \sum_{j=0}^{a_{m+1}-1} \Delta^{i+j}(1-\Delta)^{m+n-i-j} \frac{1}{i!j!(m+n-i-j)!}.
\end{aligned}$$

It follows that $Q_\Delta(\alpha_X = \alpha)$ depends on α through (a_m, a_{m+1}) only, and is a decreasing of a_m and an increasing function of a_{m+1}. Thus, if (a_m, a_{m+1}) corresponds to α and (a'_m, a'_{m+1}) to α' such that $(a'_m \geq a_m, a'_{m+1} \leq a_{m+1})$, then, for $0 < \Delta < 1$,

$$Q_\Delta(\alpha_X = \alpha) \geq Q_\Delta(\alpha_X = \alpha'),$$

where equality is reached if and only if $a'_m = a_m, a'_{m+1} = a_{m+1}$.

The statistic $M = \max\{a_m - m, m+1-a_{m+1}\}$ generates the locally most powerful rank test for H_0 against a shift Δ of the uniform distribution over $(0,1)$ for Δ close to 1, that is for $1-\varepsilon < \Delta < 1$.

3. Extension to other nonparametric hypotheses

The test statistics induced by Chebyshev's metric for other hypothesis testing situations are now considered briefly. For each testing situation, the sets $[\alpha]$ and E are as given in Critchlow (1992), Section 6). For the *two-sample rank test with a two-sided alternative* we test H_0 against $A : \{F(x) \geq G(x)\} \cup \{G(x) \geq F(x)\}$ with strict inequality for some x.

The equivalence class $[\alpha]$ is the same as in the one-sided alternative case. The extremal set E consists of all permutations which rank all of the X_i before all of the Y_j, as well all permutations which rank all of the Y_j before all of the X_i. In other words, $E = S_2 \odot (S_m \times S_n)$, where $S_2 \odot (S_m \times S_n)$ is defined by $\{\gamma \in S_{m+n} : \exists \beta \in S_2 : \gamma(i) < \gamma(j) \forall i \in N_{\beta(1)}, j \in N_{\beta(2)}\}$, and $N_1 = \{1, \dots, m\}$, $N_2 = \{m+1, \dots, m+n\}$.

THEOREM 3.1. *The test statistic for the two-sample problem with a two-sided alternative induced by Chebyshev's metric is given by:*

$$\begin{aligned} &M(\alpha(S_m \times S_n), S_2 \odot (S_m \times S_n)) \\ &\quad = \min\{\max\{\mathrm{a}_m - m, m + 1 - \mathrm{a}_{m+1}\}, \max\{\mathrm{a}_{m+n} - n, n + 1 - \mathrm{a}_1\}\}, \end{aligned}$$

where $\mathrm{a}_m = \max\{\alpha(1), \dots, \alpha(m)\}$, $\mathrm{a}_{m+1} = \min\{\alpha(m+1), \dots, \alpha(m+n)\}$, $\mathrm{a}_{m+n} = \max\{\alpha(m+1), \dots, \alpha(m+n)\}$, *and* $\mathrm{a}_1 = \min\{\alpha(1), \dots, \alpha(m)\}$.

The proof of Theorem 3.1 is a special case of Theorem 3.3.

A multi-sample rank test for ordered alternatives. Let $X_1, \dots, X_{n_1}$, $X_{n_1+1}, \dots$, $X_{n_1+n_2}, \dots, X_{n_1+\dots+n_{r-1}+1}, \dots, X_n$ be $r > 2$ random samples with samples sizes n_1, $n_2, \dots, n_r$, $(\sum n_i = n)$ and distribution functions $F_1, \dots, F_r$, respectively. The null hypothesis H_0 is: $F_1(x) \equiv \dots \equiv F_r(x)$, and the alternative A is: $F_1(x) \geq \dots \geq F_r(x)$, where each inequality is strict for some x.

Denote the rank of X_i among $X_1, \dots, X_n$ by $\alpha(i)$ $(i = 1, \dots, n)$. Thus $\alpha \in S_n$.

Let $N_1, \dots, N_r$ be a partition of $\{1, \dots, n\}$ such that N_1 contains the first n_1 integers, N_2 contains next n_2 integers, and so on.

Let $S_{n_1}, \dots, S_{n_r}$ be the subgroups of S_n given by

$$\begin{aligned} S_{n_1} &= \{\pi \in S_n : \pi(i) = i, \forall i \notin N_1\} \\ S_{n_2} &= \{\pi \in S_n : \pi(i) = i, \forall i \notin N_2\} \\ &\dots \\ S_{n_r} &= \{\pi \in S_n : \pi(i) = i, \forall i \notin N_r\}. \end{aligned} \tag{3.1}$$

Then the subgroup $S = S_{n_1} \times \dots \times S_{n_r}$ consists of all permutations in S_n which permute the first n_1 ranks among the first n_1 integers, the next n_2 ranks among the next n_2 integers, and so on. The equivalence class $[\alpha]$, that assigns the same set of ranks to each population as α, is the left coset αS.

The extremal set E consists of all permutations from the subgroup S.

The test statistic for H_0 versus A is given by

$$d([\alpha], E) = \min_{\substack{\pi \in \alpha S \\ \sigma \in S}} d(\pi, \sigma).$$

Notation. Let $a_1 < \cdots < a_{n_1}$ be the ranks assigned by α to the first population; $a_{n_1+1} < \cdots < a_{n_1+n_2}$ be the ranks assigned by α to the second population, etc. So $a_{n_1+\cdots+n_{r-1}+1} < \cdots < a_n$ are the ranks assigned by α to the last population.

Theorem 3.2 *The test statistic for the multi-sample problem with ordered alternatives induced by Chebyshev's metric is given by:*

$$M(\alpha S, S) = \max_{1\leq j\leq r}[\max\{k_j + 1 - a_{k_j+1}, a_{k_{j+1}} - k_{j+1}\}],$$

where $k_j = \sum_{i=1}^{j-1} n_i, j = 1, \ldots, r+1$.

The proof of Theorem 3.2 relies on the same arguments as Theorem 2.1 and is omitted.

For *the multi-sample problem with unordered alternatives* the null hypothesis is the same as for the ordered alternatives case, but the alternative is $A = \cup_{\beta\in S_r} A_{1,\beta}$, where $A_{1,\beta}: \; F_{\beta(1)}(x) \geq \cdots \geq F_{\beta(r)}(x)$ for $\beta \in S_r$.

The extremal set E is $E = S_r \odot S$, where $S = S_{n_1} \times \cdots \times S_{n_r}$. The dot product $S_r \odot S$ is defined by

$$\{\gamma \in S_n : \exists \beta \in S_r : \gamma(i_1) < \cdots < \gamma(i_r) \forall (i_1, \ldots, i_r) \in N_{\beta(1)} \times \cdots \times N_{\beta(r)}\},$$

where the sets $N_1, \ldots, N_r$ are defined as in ordered alternatives case.

Notation. For $j = 1, \ldots, r$ let $a_{1j} < \cdots < a_{n_j j}$ be the ranks assigned by α to population j.

THEOREM 3.3. *The test statistic for the multi-sample problem with unordered alternatives induced by Chebyshev's metric is given by:*

$$M(\alpha S, S_r \odot S) = \min_{\beta\in S_r} \max_{1\leq j\leq r} \left[\max\left\{ \left| a_{1\beta(j)} - \left(1 + \sum_{k=1}^{j-1} n_{\beta(k)}\right)\right|, \right.\right.$$
$$\left.\left. \left| a_{n_{\beta(j)}\beta(j)} - \left(n_{\beta(j)} + \sum_{k=1}^{j-1} n_{\beta(k)}\right)\right| \right\}\right].$$

The Proof of Theorem 3.3 is in the Appendix.

Test for symmetry. Let $X_1, \ldots, X_m$ be m independent random variables where X_i has density f_i. The null hypothesis H_1 given by (2.2) is that all the variables have the same density f symmetric about 0. The alternative A is that $f_i = f(x + \Delta)$ for all i, f symmetric about 0.

Create a new population of $n = 2m+1$ observations: $X_1, \ldots, X_m, 0, -X_1, \ldots, -X_m$. The ordering of these n items defines a permutation $\alpha \in S_n$.

Let $N_1 = \{1, \ldots, m\}$, $N_2 = \{m+1\}$, $N_3 = \{m+2, \ldots, n\}$. The equivalence class $[\alpha]$ consists of all permutations which assign to $X_1, \ldots, X_m$ the same set of ranks as α, and the same set of ranks to $-X_1, \ldots, -X_m$ as α. $[\alpha]$ is a left coset $\alpha(S_m \times S_1 \times S_m)$ of the subgroup $S_m \times S_1 \times S_m = \{\gamma \in S_n : \gamma(N_j) = N_j \forall j = 1, 2, 3\}$.

The extremal set E is the subgroup $S_m \times S_1 \times S_m$. Thus E consists of all permutations which assign the first m ranks to $X_1, \ldots, X_m$, the middle rank to 0, and the last m ranks to $-X_1, \ldots, -X_m$.

The test statistic is $M(\alpha(S_m \times S_1 \times S_m), S_m \times S_1 \times S_m)$. It is algebraically equivalent to a particular case of the statistic for the multi-sample problem with a one-sided alternative, although the hypotheses tested are different.

Notation. Let $\mathrm{a}_1 < \cdots < \mathrm{a}_m$ be an enumeration of the set $\alpha_{N_1} = \alpha\{1, \ldots, m\}$, let $\mathrm{a}_{m+1} = m + 1 = \alpha(m + 1)$, and let $\mathrm{a}_{m+2} < \cdots < \mathrm{a}_n$ be an enumeration of the set $\alpha_{N_3} = \alpha\{m + 2, \ldots, n\}$.

THEOREM 3.4. *The test statistic for the one-sample location problem with a one-sided alternative induced by Chebyshev's metric is given by*:

$$M(\alpha(S_m \times S_1 \times S_m), S_m \times S_1 \times S_m) = \mathrm{a}_m - m.$$

The Proof of Theorem 3.4 is in the Appendix.

4. Two-sample tests based on exceeding observations

In this section, we return to the two-sample problem, and the M-test is considered briefly in relation to other tests based on exceeding observations. The notation is adapted from Hajek and Sidak (1967). Given two samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ with densities f_1 and f_2, respectively, we test the hypothesis H_0 against the alternative of shift in location $f_1(x) = f(x - \Delta)$, $f_2(x) = f(x)$, where $\Delta > 0$, or $\Delta < 0$ (one-sided alternatives), or $\Delta \neq 0$ (two-sided alternative).

Let A and B' denote the number of observations among $X_1, \ldots, X_m$ larger than $\max_{1 \le j \le n} Y_j$ or smaller than $\min_{1 \le j \le n} Y_j$, and let A' and B denote the number of observations among $Y_1, \ldots, Y_n$ larger than $\max_{1 \le i \le m} X_i$ or smaller than $\min_{1 \le i \le m} X_i$.

According this notation the M-test defined by Theorem 2.1,

$$M = \max\{\mathrm{a}_{m+n} - m + n, n + 1 - \mathrm{a}_1\},$$

with a_{m+n} the maximum rank among $Y_1, \ldots, Y_n$ and a_1 the minimum rank among $X_1, \ldots, X_m$, is equal to

$$M = \max\{m - A, n - B\}.$$

The *E-test* introduced by Hajek and Sidak (1967) is based on the statistic

$$E = \min(A, B) - \min(A', B').$$

It can be used against both two-sided and one-sided alternatives. Sidak (1977) gives tables of the one-sided significance level $P\{E \ge k\}$ for $2 \le k \le 6$, for $3 \le m \le n \le 25$. The simpler statistic $\min(A, B)$ can be used against the one-sided alternative $\Delta > 0$. This statistic generates the locally most powerful rank test for testing H_0 against a shift Δ of the uniform distribution over $(0, 1)$ for Δ close to 0. For equal sample sizes the statistic $\min(A, B)$ is equivalent to the M-test.

The *Haga test* (Haga (1960)) is based on the statistic $T = A + B - A' - B'$. It also can be used against both two-sided and one–sided alternatives. Against the one-sided alternative the simpler statistic $A + B$ shares with the M-test the property that it generates the locally most powerful test for H_0 against a shift Δ of the uniform distribution over $(0, 1)$ for Δ close to 1.

The simplest statistic based on A has been suggested by Rosenbaum (1957). Sidak and Vondracek (1957) proposed the statistic $A + B'$.

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Appendix

PROOF OF PROPOSITION 1. (Transposition property) Let α, β and $\beta' \in S_n$ be permutations such that $\alpha(i) < \alpha(j)$, $\beta(i) < \beta(j)$ and $\beta' = \beta \circ \tau_{ij}$.

$M(\alpha, \beta')$ and $M(\alpha, \beta)$ can be represented in the form:

$$M(\alpha,\beta) = \max\left\{|\alpha(i)-\beta(i)|, |\alpha(j)-\beta(j)|, \max_{k\neq i,j}|\alpha(k)-\beta(k)|\right\},$$
$$M(\alpha,\beta') = \max\left\{|\alpha(i)-\beta'(i)|, |\alpha(j)-\beta'(j)|, \max_{k\neq i,j}|\alpha(k)-\beta'(k)|\right\}.$$

First of all, suppose $i \in \{1,\ldots,n\}$ maximizes $|\alpha(\cdot)-\beta(\cdot)|$, i.e. $M(\alpha,\beta) = |\alpha(i)-\beta(i)|$.

If $\alpha(i) \le \beta(i)$, then $|\alpha(i)-\beta(i)| \le |\alpha(i)-\beta(j)| = |\alpha(i)-\beta'(i)| \le M(\alpha,\beta')$.

Similarly, if $\beta(i) \le \alpha(i)$, then $|\alpha(i)-\beta(i)| \le |\alpha(j)-\beta(i)| = |\alpha(j)-\beta'(j)| \le M(\alpha,\beta')$.

Next, suppose $j \in \{1,\ldots,n\}$ maximizes $|\alpha(\cdot)-\beta(\cdot)|$, i.e. $M(\alpha,\beta) = |\alpha(j)-\beta(j)|$.

If $\beta(j) \ge \alpha(j)$, then $|\alpha(j)-\beta(j)| \le |\alpha(i)-\beta(j)| = |\alpha(i)-\beta'(i)| \le M(\alpha,\beta')$, and similarly, if $\beta(j) \le \alpha(j)$, then$|\alpha(j)-\beta(j)| \le |\alpha(j)-\beta(i)| = |\alpha(i)-\beta'(i)| \le M(\alpha,\beta')$.

Finally, if $k \in \{1,\ldots,n\}$, $k \neq i,j$ maximizes $|\alpha(\cdot)-\beta(\cdot)|$, it follows directly that $M(\alpha,\beta) \le M(\alpha,\beta')$.

PROOF OF LEMMA 1. Formulas (2.6) follow by analyzing the possibilities in two-sample box model. Let k and l $(1 \le k, l \le m+n)$ be fixed numbers.

1. Obviously $\#\{a_m = k, a_{m+1} = l\} = 0$ for $k < m$ or $l > m+1$.

2. Let $k = m$ and $l = m+1$. This case corresponds to number of permutations which permute the first m objects among the first m ranks, and the remaining n objects among the remaining n ranks. The number of these permutations is $m!n!$.

3. Let $k \ge m+1$ and $l \le m$ and fix $a_m = k$ and $a_{m+1} = l$. The event $\{a_m = k, a_{m+1} = l\}$ occurs if and only if integers $1,\ldots,l-1$ are assigned ranks $a_1,\ldots,a_{m-1}$; integers $k+2,\ldots,m+n$ are assigned ranks $a_{m+1},\ldots,a_{m+n}$; the remaining $k-l-1$ integers are assigned to any of the remaining $k-l-1$ ranks arbitrarily; which is possible in $\frac{(m-1)!}{(m-l)!}\frac{(n-1)!}{(k-m-1)!}(k-l-1)!$ ways.

Leaving k and l to be in any position $a_1,\ldots,a_m$ and $a_{m+1},\ldots,a_{m+n}$, respectively, we obtain the last case of the lemma.

PROOF OF THEOREM 3.3. $S = S_{n_1} \times \cdots \times S_{n_r}$. Since $[\alpha] = \alpha S$ and

$$E = S_r \odot S = \{\gamma \in S_n : \exists \beta \in S_r : \gamma(i_1) < \cdots < \gamma(i_r) \quad \forall (i_1,\ldots,i_r) \in N_{\beta(1)} \times \cdots \times N_{\beta(r)}\},$$

the test statistic is thus a minimum of test statistics of the type of Theorem 3.2, over the $r!$ possible ordered alternatives.

For $\beta \in S_r$ let $A_{1,\beta} : F_{\beta(1)}(x) \ge \cdots \ge F_{\beta(r)}(x)$ be the corresponding ordered alternative. Then

$$E_\beta = e_\beta S = \{\tau \in S_n : \tau(N_{\beta(1)}) < \cdots < \tau(N_{\beta(r)})\}$$

is the set of all extremal permutations for $A_{1,\beta}$.

The minimum distance between the sets $[\alpha]$ and E_β can be calculated from the equation (see Critchlow (1986), Lemma A.3)

$$\min_{\substack{\pi\in\alpha S\\ \gamma\in e_\beta S}} M(\pi,\gamma) = \min_{\substack{\pi\in\alpha e_\beta^{-1} S_\beta\\ \gamma\in S_\beta}} M(\pi,\gamma),$$

where S_β denotes the subgroup $S_{n_{\beta(1)}} \times \ldots \times S_{n_{\beta(r)}}$.

For $\beta \in S_r$, analogously to Theorem 3.2, we calculate

$$\begin{aligned}\min_{\substack{\pi\in\alpha S\\ \gamma\in e_\beta S}} M(\pi,\gamma) &= \max_{1\le j\le r}\max_{1\le i\le n_{\beta(j)}}\left|a_{i\beta(j)} - \left(i+\sum_{k=1}^{j-1} n_{\beta(k)}\right)\right| \\ &= \max_{1\le j\le r}\left[\max\left\{\left|a_{1\beta(j)} - \left(1+\sum_{k=1}^{j-1} n_{\beta(k)}\right)\right|,\right.\right. \\ &\qquad \left.\left.\left|a_{n_{\beta(j)}\beta(j)} - \left(n_{\beta(j)}+\sum_{k=1}^{j-1} n_{\beta(k)}\right)\right|\right\}\right].\end{aligned}$$

Therefore

$$\begin{aligned}M(\alpha S, S_2\odot(S_{n_1}\times\cdots\times S_{n_r})) &= \min_{\beta\in S_r}\min_{\substack{\pi\in\alpha S\\ \gamma\in e_\beta S}} M(\pi,\gamma) \\ &= \min_{\beta\in S_r}\max_{1\le j\le r}\left[\max\left\{\left|a_{1\beta(j)} - \left(1+\sum_{k=1}^{j-1} n_{\beta(k)}\right)\right|,\right.\right. \\ &\qquad \left.\left.\left|a_{n_{\beta(j)}\beta(j)} - \left(n_{\beta(j)}+\sum_{k=1}^{j-1} n_{\beta(k)}\right)\right|\right\}\right].\end{aligned}$$

PROOF OF THEOREM 3.1. Theorem 3.1 is a special case of Theorem 3.3 for $t = 2$.

PROOF OF THEOREM 3.4 For the one-sample location problem with a one-sided alternative, the test statistic is a special case of the test statistics from Theorem 3.2. Thus for $r = 3$, $n_1 = m$, $n_2 = 1$ and $n_3 = m$, we have

$$\begin{aligned}&M(\alpha(S_m\times S_1\times S_m), S_m\times S_1\times S_m) \\ &\quad = \max\{|1-a_1|, |a_m - m|, |m+1-a_{m+1}|, \\ &\qquad\qquad |a_{m+1}-m-1|, |m+2-a_{m+2}|, |a_n - n|\}.\end{aligned}$$

Using the fact that for the one-sample location problem $a_{n+1-i} = n+1-a_i$, it follows that the last maximum is $a_m - m$.

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