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EXPONENTIAL MIXTURE REPRESENTATION OF GEOMETRIC STABLE DISTRIBUTIONS

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Abstract. We show that every strictly geometric stable (GS) random variable can be represented as a product of an exponentially distributed random variable and an independent random variable with an explicit density and distribution function. An immediate application of the representation is a straightforward simulation method of GS random variables. Our result generalizes previous representations for the special cases of Mittag-Leffler and symmetric Linnik distributions.

Key words and phrases: Heavy-tail distribution, Linnik distribution, Mittag-Leffler distribution, random summation, stable distribution.

1. Introduction and statement of results

Strictly geometric stable (GS) distributions, introduced in Klebanov *et al.* (1984), play an important role in heavy-tail modeling of economic data (see, e.g., Anderson and Arnold (1993), Mittnik and Rachev (1991, 1993), Kozubowski and Rachev (1994), Rachev and SenGupta (1993)) and appear as solutions to certain characterization problems in statistics (see, e.g., Pakes (1992), Baringhaus and Grubel (1997)). As their densities and distribution functions do not admit explicit forms (with few exceptions) strictly GS laws are usually described in terms of characteristic function (ch.f.),

$$\psi(t) = [1 + \lambda |t|^{\alpha} \exp(-i\pi\alpha\tau \operatorname{sign}(t)/2)]^{-1}, \tag{1.1}$$

where $0 < \alpha \leq 2$, $\lambda > 0$, and $|\tau| \leq \min(1, 2/\alpha - 1)$ (see, e.g., Klebanov *et al.* (1996)). We shall write $GS_{\alpha}(\lambda, \tau)$ to denote the GS distribution given by (1.1). The special case $\tau = 0$ leads to a symmetric distribution with ch.f.

$$\psi(t) = [1 + \lambda |t|^{\alpha}]^{-1}, \tag{1.2}$$

known as (symmetric) Linnik distribution since its introduction in Linnik (1963). The theory of symmetric Linnik distributions was developed in parallel to that of GS laws (see, e.g., Devroye (1990), Anderson (1992), Anderson and Arnold (1993), Kotz *et al.* (1995)). As strictly GS laws are generalizations of (1.2), they are also referred to as non-symmetric Linnik distributions (see, e.g., Erdogan (1995)). Another special case of (1.1) is the class of Mittag-Leffler distributions, introduced in Pillai (1990). These are probability distributions concentrated on $(0, \infty)$ with Laplace transform

$$l(s) = [1 + \lambda s^{\alpha}]^{-1}, \qquad s \geq 0, \tag{1.3}$$

and correspond to $\alpha \leq 1$ and $\tau = 1$ in (1.1). For $\alpha = 1$ we get an exponential distribution.

Every strictly GS r.v. Y admits the representation

$$Y \stackrel{d}{=} Z^{1/\alpha} X, \tag{1.4}$$

where Z is standard exponential, X is strictly stable with ch.f.

$$\varphi(t) = \exp\{-\lambda |t|^\alpha \exp(-i\pi\alpha\tau \operatorname{sign}(t)/2)\} \tag{1.5}$$

and denoted $S_\alpha(\lambda, \tau)$, and Z is independent of X (when writing equalities in distribution we follow the convention that all random variables that appear on the same side of an equation are independent). Relation (1.4), which was first proved for the symmetric case in Devroye (1990) and extended to the general case in Pakes (1992), provides a major tool in studying GS distributions through the theory of stable laws (see, e.g., Kozubowski (1994*a*, 1994*b*) and Ramachandran (1997) for recent results). However, except for a few special cases, neither densities nor distribution functions of stable laws admit explicit forms, and representations alternative to (1.4) should be of interest.

The main result of this paper is a new representation of strictly GS laws, that yields itself easily for practical applications. It generalizes and unifies recent results for the special cases of Mittag-Leffler and symmetric Linnik distributions (see Kotz and Ostrovskii (1996), Pakes (1998), Kozubowski (1998)). The representation involves a positive random variable W_ρ with the density function

$$g_\rho(x) = \frac{\sin(\pi\rho)}{\pi\rho[(x + \cos(\pi\rho))^2 + \sin^2(\pi\rho)]}, \qquad x \geq 0, \tag{1.6}$$

where $0 < \rho < 1$. By taking the weak limits, we also include the special cases $\rho = 1$ and $\rho = 0$, obtaining a unit mass at $x = 1$ for $\rho = 1$ and a distribution with a density $f_0(x) = (1 + x)^{-2}$ for $\rho = 0$.

Let $Y_{\alpha,\tau} \sim GS_\alpha(1, \tau)$ be strictly GS (since λ is essentially a scaling factor, we set $\lambda = 1$ in (1.1)). Denote $\rho_\pm = \frac{\alpha}{2}(1 \pm \tau)$. Note that $0 \leq \rho_\pm \leq 1$, as $|\tau| \leq \min(1, 2/\alpha - 1)$. Let $W_{\rho_\pm}$ be a positive r.v. with the density $g_{\rho_\pm}$ defined by (1.6). Further, define

$$W_{\alpha,\tau} = I W_{\rho_+} + (I - 1) W_{\rho_-}, \tag{1.7}$$

where I is an independent of $W_{\rho_\pm}$ indicator r.v. with

$$P(I = 1) = (1 + \tau)/2 \quad \text{and} \quad P(I = 0) = (1 - \tau)/2. \tag{1.8}$$

Finally, let $x^{\langle a \rangle}$ denote the signed power: $x^{\langle a \rangle} = |x|^a \operatorname{sign}(x)$. Then, the following representation holds.

THEOREM 1.1. *Let $Y_{\alpha,\tau} \sim GS_\alpha(1, \tau)$, where $0 < \alpha \leq 2$ and $|\tau| \leq \min(1, 2/\alpha - 1)$. Then,*

$$Y_{\alpha,\tau} \stackrel{d}{=} Z \cdot W_{\alpha,\tau}^{\pm\langle 1/\alpha \rangle}, \tag{1.9}$$

where Z is standard exponential and $W_{\alpha,\tau}$ is given by (1.7) and is independent of Z.

Stable distributions with ch.f. (1.5) admit a representation analogous to (1.9), where the role of exponential distribution is played by a stable subordinator.

THEOREM 1.2. *Suppose that $X_{\alpha,\tau} \sim S_\alpha(1,\tau)$, where $1 < \alpha \le 2$ and $|\tau| < 2/\alpha - 1$. Let $X_{1/\alpha,1}$ be the stable subordinator $S_{1/\alpha}(1,1)$, and let $W_{\alpha,\tau}$, given by (1.7), be independent of $X_{1/\alpha,1}$. Then,*

$$X_{\alpha,\tau} \stackrel{d}{=} X_{1/\alpha,1}^{-1/\alpha} \cdot W_{\alpha,\tau}^{\pm\langle 1/\alpha \rangle}. \tag{1.10}$$

We conclude this section with several remarks and then prove Theorems 1.1 and 1.2 in Section 2.

Remark 1.1. The random variable $W_{\alpha,\tau}$ given by (1.7) is related to *cutoffs* of two Cauchy distributions. Recall that a cutoff of a continuous r.v. X, denoted $(X)_+$, is defined as a non-negative r.v. with density $f_+(x) = f(x)/\rho$, where f is the density of X and $\rho = P(X \ge 0)$ (see Definition 2.1 in Section 2). Since for any $0 < \rho < 1$, the density of the Cauchy r.v. $X_{1,\tau} \sim S_1(1,\tau)$ with $\tau = 2\rho - 1$,

$$f_\rho(x) = \frac{\sin(\pi\rho)}{\pi[(x+\cos(\pi\rho))^2 + \sin^2(\pi\rho)]}, \qquad x \in R, \tag{1.11}$$

integrates to ρ on $(0,\infty)$, we see that W_ρ given by (1.6) is a cutoff of the Cauchy r.v. $X_{1,\tau}$:

$$(X_{1,\tau})_+ \stackrel{d}{=} W_\rho, \qquad \text{where} \quad \tau = 2\rho - 1. \tag{1.12}$$

Therefore, $W_{\alpha,\tau}$ is a mixture of $(X_{1,2\rho_+-1})_+$ and $-(X_{1,2\rho_--1})_+$. In the special case $\alpha = 1$ (and only in this case), the r.v. $W_{\alpha,\tau}$ given by (1.7) is a mixture of $(X_{1,\tau})_+$ and $-(X_{1,-\tau})_+$, and thus has a Cauchy distribution $S_1(1,\tau)$ itself. Then, the representation (1.9) reduces to the basic relation (1.4). Note also that in case $\rho = 1$, we have $W_1 \equiv 1$, and the Cauchy cutoff construction has the following interpretation: as $\rho \to 1$, the Cauchy distribution given by (1.11) converges to a unit mass at 1, and so does its cutoff, as can be verified by considering the d.f. corresponding to the density (1.6).

Remark 1.2. In the symmetric case, we have $\tau = 0$ so that $\rho_\pm = \alpha(1 \pm \tau)/2 = \alpha/2$. Consequently, $W_{\alpha,\tau}^{\pm\langle 1/\alpha \rangle}$ has the same distribution as $\delta \cdot W_{\alpha/2}^{1/\alpha}$, where $W_{\alpha/2}$ has the density (1.6), δ is the Rademacher (± 1 with probabilities 1/2), and δ and $W_{\alpha/2}$ are independent. Since $Z \cdot \delta$ has a Laplace distribution, formula (1.9) reduces to $Y_{\alpha,0} \stackrel{d}{=} Y_{2,0} \cdot W_{\alpha/2}^{\pm 1/\alpha}$. For $\alpha > 1$, the latter representation was discussed in Devroye (1996), who pointed out its relation to the representation of Linnik density derived in Kawata ((1972), pp. 396–397). In the general symmetric case, the representation is due to Kotz and Ostrovskii (1996).

Remark 1.3. In the case $\alpha < 1$ and $\tau = 1$, we obtain $W_{\alpha,\tau} \stackrel{d}{=} W_\alpha^{1/\alpha}$. Consequently, (1.9) reduces to the exponential mixture representation of Mittag-Leffler distributions,

$$Y_{\alpha,1} \stackrel{d}{=} ZW_\alpha^{1/\alpha}, \tag{1.13}$$

derived first in Pakes (1998), and then, independently, in Kozubowski (1998).

Remark 1.4. In the stable case, consider $\alpha = 2$. Here, we must have $\tau = 0$ and the stable distribution $S_\alpha(1,\tau)$ reduces to $N_{0,2}$ (the normal distribution with mean zero and variance equal to two). Further, $X_{1/2,1}$ has the Lévy distribution with density $h(x) = (2\sqrt{\pi})^{-1}x^{-3/2}\exp(-0.25/x)$ (see Samorodnitsky and Taqqu (1994), p. 10). In addition, we have $\rho_{\pm} = 1$ so that $W_{\rho_{\pm}} \equiv 1$ and the representation (1.10) produces the well-known relation $N_{0,2} \stackrel{d}{=} \delta X_{1/2,1}^{-1/2}$, where δ is Rademacher r.v. independent of $X_{1/2,1}$.

Remark 1.5. Writing (1.9) in terms of densities leads to the following representation of a strictly GS density,

$$p(\pm x) = \frac{\sin \pi\rho_{\pm}}{\pi} \int_0^\infty \frac{v^\alpha \exp(-vx)dv}{1 + v^{2\alpha} + 2v^\alpha \cos \pi\rho_{\pm}}, \qquad x > 0. \tag{1.14}$$

Formula (1.14) was derived by purely analytic methods in Erdogan (1995) and, in a slightly more general setting and $\alpha > 1$, in Klebanov *et al.* (1996). Our result provides an alternative proof of (1.14) and its interpretation in terms of random variables.

2. Proofs

Our proofs of Theorems 1.1 and 1.2 heavily use *cutoffs* of distributions, defined in Zolotarev (1986). We now recall the definition of a cutoff and collect its basic properties.

DEFINITION 2.1. Let X have a continuous distribution not entirely concentrated on the negative semi-axis. Let f and F denote the density and distribution function (d.f.) of X, respectively. A cutoff of X, denoted $(X)_+$, is defined as a non-negative r.v. with density $f_+(x) = f(x)/\rho$ and d.f. $F_+(x) = P(X \le x \mid X \ge 0) = (F(x) - F(0))/\rho$, where $\rho = P(X \ge 0)$.

The following properties of cutoffs are elementary and we present them without proofs. We follow the standard convention for equalities in distribution, that the variables that appear on the same side are assumed to be independent.

1. If $a > 0$, then

$$(aX)_+ \stackrel{d}{=} a(X)_+. \tag{2.1}$$

2. If $P(X \ge 0) = 1$, then

$$(X)_+ \stackrel{d}{=} X. \tag{2.2}$$

3. If X and Y are independent with $P(X \ge 0) = 1$ and $P(Y \ge 0) > 0$, then

$$(XY)_+ \stackrel{d}{=} X(Y)_+. \tag{2.3}$$

4. The following relations are equivalent:
(i) $X \stackrel{d}{=} Y$.
(ii) $(X)_+ \stackrel{d}{=} (Y)_+,\ (-X)_+ \stackrel{d}{=} (-Y)_+,\ P(X \ge 0) = P(Y \ge 0)$.

5. If a r.v. X has a continuous distribution and there exists a constant $c \ge 0$ such that $P(X \le -x) = cP(X \ge x)$ for all $x \ge 0$, then $(X)_+ \stackrel{d}{=} |X|$.

6. Suppose, that X_1 and X_2 are two independent, non-negative random variables, and

$$X \stackrel{d}{=} IX_1 + (I-1)X_2, \tag{2.4}$$

where I is an indicator r.v. independent of X_1 and X_2. Then,

$$(X)_+ \stackrel{d}{=} X_1, \quad \text{and} \quad (-X)_+ \stackrel{d}{=} X_2. \tag{2.5}$$

7. If $a > 0$, then

$$(\pm X^{\langle a \rangle})_+ \stackrel{d}{=} (\pm X)_+^a. \tag{2.6}$$

We also collect some results related to cutoffs of stable distributions. With slight change in notation, the Lemmas presented below follow from Zolotarev's (1986) results. We denote a generic stable $S_\alpha(1,\tau)$ r.v. (see (1.5)) by $X_{\alpha,\tau}$.

LEMMA 2.1. *Let* $1 \le \alpha \le 2$. *Then,*

$$(X_{1/\alpha,1})_+^{-1/\alpha} \stackrel{d}{=} (X_{\alpha,2/\alpha-1})_+. \tag{2.7}$$

For the proof of Lemma 2.1 see Theorem 3.2.5 of Zolotarev (1986).

LEMMA 2.2. *Let* $1 \le \alpha \le 2$ *and* $|\tau| \le 2/\alpha - 1$. *Then,*

$$(X_{\alpha,\tau})_+ \stackrel{d}{=} (X_{\alpha,2/\alpha-1})_+(X_{1,\alpha(\tau+1)-1})_+. \tag{2.8}$$

For the proof of Lemma 2.2 see the Corollary to Theorem 3.3.2 of Zolotarev (1986).

LEMMA 2.3. *Let* $1 < \alpha \le 2$ *and let* Z *have a standard exponential distribution. Then,*

$$Z \stackrel{d}{=} Z^{1/\alpha} X_{1/\alpha,1}^{-1/\alpha}. \tag{2.9}$$

PROOF. Denote $Z_\alpha = (X_{\alpha,2/\alpha-1})_+$. By relation (3.4.9) of Zolotarev (1986), we have

$$Z_\alpha Z_\alpha^{1/\alpha} Z_\alpha^{1/\alpha^2} \cdots \stackrel{d}{=} Z. \tag{2.10}$$

Raising both sides of equation (2.10) to the $1/\alpha$ power results in

$$Z_\alpha^{1/\alpha} Z_\alpha^{1/\alpha^2} \cdots \stackrel{d}{=} Z^{1/\alpha}. \tag{2.11}$$

Combining relations (2.10)–(2.11) produces $Z_\alpha Z^{1/\alpha} \stackrel{d}{=} Z$, and the result follows by Lemma 2.1. □

LEMMA 2.4. *Let* $0 < \alpha < 1$ *and* $|\tau| \le 1$. *Then,*

$$(X_{1,2\alpha-1})_+(X_{1,\tau})_+^\alpha \stackrel{d}{=} (X_{1,\alpha(1+\tau)-1})_+. \tag{2.12}$$

PROOF. Apply relation (3.3.7) of Theorem 3.3.2 of Zolotarev (1986), taking $\alpha = 1$, $\rho = \alpha$, and $\rho' = (1+\tau)/2$. □

LEMMA 2.5. *Let* $0 < \alpha \leq 1$ *and* $|\tau| \leq 1$. *Then,*

$$X_{\alpha,\tau} \stackrel{d}{=} X_{\alpha,1} X_{1,\tau}. \tag{2.13}$$

For the proof of Lemma 2.5 see the Corollary to Theorem 3.3.2 of Zolotarev (1986). We now prove our main theorems. We start with the result for stable laws.

PROOF OF THEOREM 1.2. First, note that by (1.7) and (1.12), $W_{\alpha,\tau}$ is a mixture of two cutoffs of Cauchy distributions:

$$W_{\alpha,\tau} \stackrel{d}{=} I(X_{1,\alpha(1+\tau)-1})_+ + (I-1)(X_{1,\alpha(1-\tau)-1})_+, \tag{2.14}$$

where the variables on the RHS of (2.14) are independent and $P(I=1) = (1+\tau)/2$, $P(I=0) = (1-\tau)/2$. Also, since Cauchy r.v. (and its cutoff) has the same distribution as its reciprocal, we have $W_{\alpha,\tau} \stackrel{d}{=} W_{\alpha,\tau}^{-1}$, so we restrict our attention to the case of positive power of $W_{\alpha,\tau}$ that appears on the RHS of (1.10). We shall utilize the Property 4 of cutoffs. We start by showing that the cutoffs of both sides of (1.10) have the same distributions. Indeed,

$$\begin{aligned}
(\text{RHS of } (1.10))_+ &= (X_{1/\alpha,1}^{-1/\alpha} W_{\alpha,\tau}^{\langle 1/\alpha \rangle})_+ \stackrel{d}{=}_1 X_{1/\alpha,1}^{-1/\alpha} (W_{\alpha,\tau}^{\langle 1/\alpha \rangle})_+ \\
&\stackrel{d}{=}_2 X_{1/\alpha,1}^{-1/\alpha} (W_{\alpha,\tau})_+^{1/\alpha} \stackrel{d}{=}_3 (X_{\alpha,2/\alpha-1})_+ (W_{\alpha,\tau})_+^{1/\alpha} \\
&\stackrel{d}{=}_4 (X_{\alpha,2/\alpha-1})_+ (X_{1,\alpha(1+\tau)-1})_+^{1/\alpha} \stackrel{d}{=}_5 (X_{\alpha,\tau})_+ \\
&= (\text{LHS of } (1.10))_+.
\end{aligned}$$

When showing equality 1, we used Property 3 of cutoffs, for equality 2, we used Property 7 of cutoffs, for 3, we applied Lemma 2.1, for 4, we used Property 6 of cutoffs and relation (2.14), and for 5, we applied Lemma 2.2.

Next, we show that if the two sides of (1.10) are multiplied by -1, then their cutoffs have the same distributions:

$$\begin{aligned}
(-\text{RHS of } (1.10))_+ &= (-X_{1/\alpha,1}^{-1/\alpha} W_{\alpha,\tau}^{\langle 1/\alpha \rangle})_+ \stackrel{d}{=}_1 X_{1/\alpha,1}^{-1/\alpha} (-W_{\alpha,\tau}^{\langle 1/\alpha \rangle})_+ \\
&\stackrel{d}{=}_2 X_{1/\alpha,1}^{-1/\alpha} (W_{\alpha,\tau})_+^{1/\alpha} \stackrel{d}{=}_3 (X_{\alpha,2/\alpha-1})_+ (-W_{\alpha,\tau})_+^{1/\alpha} \\
&\stackrel{d}{=}_4 (X_{\alpha,2/\alpha-1})_+ (X_{1,\alpha(1-\tau)-1})_+^{1/\alpha} \stackrel{d}{=}_5 (X_{\alpha,-\tau})_+ \\
&= (-\text{LHS of } (1.10))_+,
\end{aligned}$$

where equalities 1 through 5 are obtained the same way as before.

Finally, we note that $P(\text{LHS of } (1.10) \geq 0) = P(X_{\alpha,\tau} \geq 0) = (1+\tau)/2$, and so is $P(\text{RHS of } (1.10) \geq 0) = P(W_{\alpha,\tau}^{<1/\alpha>} \geq 0) = P(I=1) = (1+\tau)/2$. The result follows by Property 4 of cutoffs. □

We now prove the result for *GS* distributions.

PROOF OF THEOREM 1.1. We shall consider several cases.

Case 1. $1 < \alpha \le 2$. In view of the basic relation (1.4) between stable and GS distributions, Theorem 1.2, and Lemma 2.3, we have the following chain of equalities in distribution

$$Y_{\alpha,\tau} \stackrel{d}{=} Z^{1/\alpha} X_{\alpha,\tau} \stackrel{d}{=} Z^{1/\alpha} X_{1/\alpha,1}^{-1/\alpha} W_{\alpha,\tau}^{\pm\langle 1/\alpha\rangle} \stackrel{d}{=} Z W_{\alpha,\tau}^{\pm\langle 1/\alpha\rangle},$$

that proves the result in this case.

Case 2. $\alpha = 1$. Here the basic relation (1.4) produces $Y_{1,\tau} \stackrel{d}{=} Z X_{1,\tau}$, where $X_{1,\tau}$ has a Cauchy distribution $S_1(1,\tau)$. The result follows since the r.v. $W_{1,\tau}$ given by (1.7) has the Cauchy $S_1(1,\tau)$ distribution.

Case 3. $0 < \alpha < 1$. First, note the following chain of equalities in distribution

$$Y_{\alpha,\tau} \stackrel{d}{=}_1 Z^{1/\alpha} X_{\alpha,\tau} \stackrel{d}{=}_2 Z^{1/\alpha} X_{\alpha,1} X_{1,\tau} \stackrel{d}{=}_3 Y_{\alpha,1} X_{1,\tau} \stackrel{d}{=}_4 Z W_{\alpha}^{\pm 1/\alpha} X_{1,\tau},$$

where W_α has density (1.6) (with $\rho = \alpha$). In steps 1 and 3 we used the basic relation (1.4) of stable and GS distributions, in step 2 we used Lemma 2.5, and in step 4 we used the representation (1.13).

Thus, the theorem will be proved if we can show the following relation:

$$W_\alpha^{1/\alpha} X_{1,\tau} \stackrel{d}{=} W_{\alpha,\tau}^{\langle 1/\alpha\rangle}. \tag{2.15}$$

To show (2.15), we utilize cutoffs:

$$\begin{aligned}
(\text{LHS of } (2.15))_+ &\stackrel{d}{=}_1 W_\alpha^{1/\alpha} (X_{1,\tau})_+ \stackrel{d}{=}_2 (X_{1,2\alpha-1})_+^{1/\alpha} (X_{1,\tau})_+ \\
&= [(X_{1,2\alpha-1})_+ (X_{1,\tau})_+^{\alpha}]^{1/\alpha} \stackrel{d}{=}_3 (X_{1,\alpha(1+\tau)-1})_+^{1/\alpha} \\
&\stackrel{d}{=}_4 (W_{\alpha,\tau}^{\langle 1/\alpha\rangle})_+ \\
&= (\text{RHS of } (2.15))_+.
\end{aligned}$$

We used Property 3 of cutoffs in step 1, relation (1.12) in step 2, Lemma 2.4 in step 3, and relation (2.14) together with Properties 6 and 7 of cutoffs in step 4. Similarly, we show the equality in distribution of cutoffs of two sides of (2.15), after each is multiplied by -1:

$$\begin{aligned}
(-\text{LHS of } (2.15))_+ &\stackrel{d}{=}_1 W_\alpha^{1/\alpha} (X_{1,-\tau})_+ \stackrel{d}{=}_2 (X_{1,2\alpha-1})_+^{1/\alpha} (X_{1,-\tau})_+ \\
&= [(X_{1,2\alpha-1})_+ (X_{1,-\tau})_+^{\alpha}]^{1/\alpha} \stackrel{d}{=}_3 (X_{1,\alpha(1-\tau)-1})_+^{1/\alpha} \\
&\stackrel{d}{=}_4 (-W_{\alpha,\tau}^{\langle 1/\alpha\rangle})_+ \\
&= (-\text{LHS of } (2.15))_+.
\end{aligned}$$

The explanations for steps 1 through 4 are the same as before. We also utilized the well known property of stable laws: $-X_{\alpha,\tau} \stackrel{d}{=} X_{\alpha,-\tau}$. To finish the proof, note that

$$\begin{aligned}
P(\text{LHS of } (2.15) \ge 0) &= P(X_{1,\tau} \ge 0) = (1+\tau)/2 = P(I = 1) \\
&= P(\text{RHS of } (2.15) \ge 0).
\end{aligned}$$

Thus, relation (2.15) follows from Property 4 of cutoffs. The theorem holds. □

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