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START-UP DEMONSTRATION TESTS WITH REJECTION OF UNITS UPON OBSERVING d FAILURES

N. BALAKRISHNAN[1] AND P. S. CHAN[2]

[1] *Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1*
[2] *Department of Statistics, The Chinese University of Hong Kong, Shatin, Hong Kong, China*

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Abstract. The probability generating function of number of trials for the start-up demonstration test with rejection upon d failures is derived. The exact distribution of the number of trials is obtained. Some recurrence relations for the probabilities are also established. The average length of the test is derived. Some illustrative examples are finally presented.

Key words and phrases: Probability generating function, runs, start-up demonstration test.

1. Introduction

A start-up demonstration test requires a pre-specified number of consecutive successful start-ups of an equipment for acceptance. For example, a purchaser of a power generating equipment may require 20 consecutive successful start-ups for the acceptance of that equipment. In early research on start-up demonstration testing (for example, Hahn and Gage (1983) and Viveros and Balakrishnan (1993)), it was assumed that the testing of the equipment will continue until consecutive successes occur no matter how many failures are observed prior to it. This is clearly not practical. Some authors have also suggested pre-fixing the total number of trials and if the required number of consecutive successful start-ups are not achieved by that many trials, then the equipment be rejected. While this clearly avoids the difficulty mentioned above, it still has a disadvantage in that early rejection of a bad equipment is not possible in the testing scheme. In this paper, we therefore propose a modification to this start-up demonstration test which is as follows: If c consecutive successful start-ups are achieved before d failures, then the equipment under test will be accepted; if d failures occur before c consecutive successful start-ups, then the equipment will be rejected. We then study the distribution of the number of trials conducted in the experiment leading to either rejection or acceptance of the equipment under test. Let X denote this waiting time (number of trials until termination).

In Section 2, the probability generating function of X (the number of trials required to terminate the experiment) is derived. Its probability mass function is also obtained. Using the probability generating function, recurrence relations for the probabilities are derived in Section 3. These relations can be used effectively to compute the probabilities of X. In Section 4, we derive the moments of X and make some observations. In Section 5, we study the conditional distribution of the number of trials given either the unit is accepted or rejected. Finally, in Section 6, we present some illustrative examples.

It is important to mention here that the waiting time distribution for k consecutive successes amongst independent Bernoulli trials is called *geometric distribution of order*

k; see, for example, Johnson *et al.* (1992). This distribution and various modifications, extensions and generalizations of it have been studied in the literature by numerous authors. The waiting time discussed in this paper is called the "sooner" waiting time and its distribution has been discussed earlier by Ebneshahrashoob and Sobel (1990), Aki (1997), Aki and Hirano (1994, 1995), Koutras (1996), and Balasubramanian *et al.* (1993). In addition to presenting some new results in this direction, we have also motivated in this paper the work on such waiting time problems in the context of start-up demonstration testing. It needs to be mentioned here that, for the same purpose of facilitating an early rejection of a bad equipment under test, Koutras and Balakrishnan (1999) proposed a start-up demonstration testing procedure using a simple scan-based statistic. All these developments on the theory of runs and associated waiting time problems and their applications to start-up demonstration testing (amongst many others) have been detailed by Balakrishnan and Koutras (2000).

2. Probability generating function

Assume that the start-ups are independent events, and let
p = Probability of a successful start-up in any single trial,
q = Probability of a failure in a single trial, i.e., $= 1 - p$,
c = consecutive successes needed for acceptance,
d = number of failures required for rejection,
X = the total number of trials until termination of the experiment.
There are two possible cases that lead to termination of the experiment.

Case 1. A typical sequence of start-ups leading to the acceptance of the unit at the r-th start-up, with k failures, is given by

$$\underbrace{S\cdots S}_{r_1}F\underbrace{S\cdots S}_{r_2}F\cdots F\underbrace{S\cdots S}_{r_k}F\underbrace{S\cdots S}_{c}, \tag{2.1}$$

with $0 \le r_i \le c-1$, $i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} r_i + k + c = r$, and $0 \le k \le d-1$.
The probability for the sequence in (2.1) is

$$p^{r_1} q p^{r_2} q \cdots q p^{r_k} q p^c$$

which is a typical term in the coefficient of t^r in the expansion of

$$\begin{aligned} \pi_k(t) &= (1 + pt + \cdots + p^{c-1}t^{c-1})qt(1 + pt + \cdots + p^{c-1}t^{c-1})qt \\ &\quad \cdots qt(1 + pt + \cdots + p^{c-1}t^{c-1})qtp^c t^c \\ &= (1 + pt + \cdots + p^{c-1}t^{c-1})^k (qt)^k p^c t^c \\ &= p^c q^k t^{k+c} \left(\frac{1 - p^c t^c}{1 - pt} \right)^k . \end{aligned} \tag{2.2}$$

Case 2. A typical sequence of start-ups leading to the rejection of the unit at the r-th start-up is given by

$$\underbrace{S\cdots S}_{r_1}\underbrace{F}_{\text{1st}}\underbrace{S\cdots S}_{r_2}\underbrace{F}_{\text{2nd}}\cdots F\underbrace{S\cdots S}_{r_d}\underbrace{F}_{d\text{-th}}, \tag{2.3}$$

with $0 \le r_i \le c-1$, $i = 1, 2, \ldots, d$, and $\sum_{i=1}^{d} r_i + d = r$.

The probability of the sequence in (2.3) is

$$p^{r_1} q p^{r_2} q \cdots q p^{r_d} q$$

which is a typical term in the coefficient of t^r in the expansion of

$$\begin{aligned}\pi_d(t) &= (1 + pt + \cdots + p^{c-1}t^{c-1})^d (qt)^d \\ &= q^d t^d \left(\frac{1 - p^c t^c}{1 - pt}\right)^d .\end{aligned} \tag{2.4}$$

Then, the probability generating function of X is simply obtained as

$$\begin{aligned}\pi(t) &= \sum_{k=0}^{d} \pi_k(t) \\ &= \sum_{k=0}^{d-1} \pi_k(t) + \pi_d(t) \\ &= p^c t^c \sum_{k=0}^{d-1} \left\{ qt \left(\frac{1 - p^c t^c}{1 - pt}\right)\right\}^k + q^d t^d \left(\frac{1 - p^c t^c}{1 - pt}\right)^d .\end{aligned} \tag{2.5}$$

Now, we have

$$\begin{aligned}\left(\frac{1 - p^c t^c}{1 - pt}\right)^k &= (1 - p^c t^c)^k (1 - pt)^{-k} \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} p^{ic} t^{ic} \sum_{j=0}^{\infty} \binom{k + j - 1}{j} p^j t^j \\ &= \sum_{r=0}^{k(c-1)} t^r p^r B_k(r),\end{aligned}$$

where

$$B_k(r) = \sum_{i=0}^{\min(k, [r/c])} (-1)^i \binom{k}{i} \binom{k + r - ic - 1}{r - ic}. \tag{2.6}$$

Then, we can write from (2.5) the probability generating function of X as

$$\begin{aligned}\pi(t) &= p^c t^c \sum_{k=0}^{d-1} \left\{ qt \left(\frac{1 - p^c t^c}{1 - pt}\right)\right\}^k + q^d t^d \left(\frac{1 - p^c t^c}{1 - pt}\right)^d \\ &= \sum_r t^r \left[\sum_{k=0}^{d-1} q^k p^{r-k} B_k(r - k - c) + q^d p^{r-d} B_d(r - d)\right] \\ &= \sum_r p^r t^r \left[\sum_{k=0}^{d-1} \left(\frac{q}{p}\right)^k B_k(r - k - c) + \left(\frac{q}{p}\right)^d B_d(r - d)\right].\end{aligned} \tag{2.7}$$

From (2.7), we immediately obtain the probability mass function of X as

$$\begin{aligned}P(x) &= \Pr(X = x) \\ &= p^x \left[\sum_{k=0}^{d-1} \left(\frac{q}{p}\right)^k B_k(x - k - c) + \left(\frac{q}{p}\right)^d B_d(x - d)\right],\end{aligned} \tag{2.8}$$

$$x = \min(c, d), \min(c, d) + 1, \ldots, cd.$$

3. Recurrence relations

In the last section, we derived the probability generating function of X and also the probability mass function. In this section, we will make use of the probability generating function in (2.5) in order to establish some recurrence relations for the probabilities of X. Recall that the probability generating function is

$$\pi(t) = p^c t^c \sum_{k=0}^{d-1} \left\{ qt \left(\frac{1-p^c t^c}{1-pt} \right) \right\}^k + q^d t^d \left(\frac{1-p^c t^c}{1-pt} \right)^d .$$

After some algebraic manipulations, we have the following identity:

$$\begin{aligned} (3.1) \qquad & (1 - t + qp^c t^{c+1})\pi(t) \\ & = p^c t^c (1-pt) + q^d t^d (1-t)(1-pt)\{1 + pt + \cdots + p^{c-1} t^{c-1}\}^{d+1}. \end{aligned}$$

Comparing coefficients of t^x on both sides of (3.1) for $x = \min(c,d), \min(c,d)+1, \dots,$ we have

$$\begin{aligned} (3.2) \qquad & P(x) - P(x-1) + qp^c P(x-c-1) \\ & = \text{Coef. of } t^x \text{ in } \Bigg\{ p^c t^c (1-pt) \\ & \qquad + q^d t^d \{1 - (1+p)t + pt^2\} \sum_{i=0}^{(d+1)(c-1)} A_i p^i t^i \Bigg\}, \end{aligned}$$

where

$$(3.3) \qquad A_i = \text{Coef. of } z^i \text{ in } (1 + z + \cdots + z^{c-1})^{d+1}.$$

We will discuss the computational methods for A_i in the Appendix.

Now, let S be the set of values that X can take on. Then, there are three possible cases depending on the values of c and d.

Case 1. When $c + 1 < d$. Here, $S = \{c, c+1, \dots, c(d+2)+1\}$; then, (3.2) yields

$$\begin{aligned} & P(c) = p^c, \\ & P(c+1) - P(c) = -p^{c+1}, \\ & P(x) - P(x-1) + qp^c P(x-c-1) \\ & = \begin{cases} 0, & \text{if } \; c+2 \le x \le d-1, \\ q^d p^{x-d-1}\{A_{x-d-2} - (1+p)A_{x-d-1} + pA_{x-d}\}, & \\ & \text{if } \; d \le x \le (d+1)c+1, \\ 0, & \text{if } \; x \ge (d+1)c+2. \end{cases} \end{aligned}$$

Case 2. When $d = c + 1$. Here, $S = \{c, c+1, \dots, c(d+2)+1\}$; then, (3.2) yields

$$\begin{aligned} & P(c) = p^c, \\ & P(c+1) - P(c) = -p^{c+1} + q^{c+1}, \\ & P(x) - P(x-1) + qp^c P(x-c-1) \\ & = \begin{cases} q^{c+1} p^{x-c-2}\{A_{x-c-3} - (1+p)A_{x-c-2} + p^{x-c-1} A_{x-c-1}\}, & \\ & \text{if } \; c+2 \le x \le c(c+2)+1, \\ 0, & \text{if } \; x \ge c(c+2)+2. \end{cases} \end{aligned}$$

Case 3. When $c \geq d$. Here, $S = \{d, d+1, \ldots, c(d+2)+1\}$; then, (3.2) yields

$$P(x) - P(x-1) + qp^c P(x-c-1)$$
$$= \begin{cases} q^d p^{x-d-1}\{A_{x-d-2} - (1+p)A_{x-d-1} + pA_{x-d}\}, \\ \qquad \text{if} \quad x = d, d+1, \ldots, c-1, \\ p^c + q^d p^{c-d-1}(A_{c-d-2} - (1+p)A_{c-d-1} + pA_{c-d}), \\ \qquad \text{if} \quad x = c, \\ -p^{c+1} + q^d p^{c-d}(A_{c-d-1} - (1+p)A_{c-d} + pA_{c-d+1}), \\ \qquad \text{if} \quad x = c+1, \\ q^d p^{x-d-1}(A_{x-d-2} - (1+p)A_{x-d-1} + pA_{x-d}), \\ \qquad \text{if} \quad c+2 \leq x \leq (d+1)c+1, \\ 0, \qquad \text{if} \quad x \geq c(d+1)+2. \end{cases}$$

Using the above recurrence relations, all the probabilities associated with the waiting time variable X can be computed in a simple recursive manner.

4. Moments of X

The moments of X can be readily derived from the probability generating function of X in (2.5).

THEOREM 1. *The mean number of trials required to terminate the experiment is*

$$E(X) = \frac{1-p^c}{qp^c}\{1 - (1-p^c)^d\}. \tag{4.1}$$

PROOF. Define

$$G(t) = \frac{p^c t^c (1-pt)}{1 - t + qp^c t^{c+1}} \tag{4.2}$$

and

$$H(t) = (1-t)H^*(t), \tag{4.3}$$

where

$$H^*(t) = q^d t^d \left(\frac{1-p^c t^c}{1-pt}\right)^d \frac{1-p^c t^c}{1-t+qp^c t^{c+1}}. \tag{4.4}$$

Then, we note that

$$E(X) = G'(1) + H'(1).$$

In order to differentiate $G(t)$ with respect to t, we note that

$$\begin{aligned} \frac{G'(t)}{G(t)} &= \frac{d \log G(t)}{dt} \\ &= \frac{c}{t} - \frac{p}{1-pt} - \frac{-1+(c+1)qp^c t^c}{1-t+qp^c t^{c+1}} \end{aligned} \tag{4.5}$$

from which we readily observe

$$G'(1) = -\frac{1}{q} + \frac{1}{qp^c}.$$

Next, we note that

$$H'(1) = -H^*(1) = -\frac{(1-p^c)^{d+1}}{qp^c}. \tag{4.6}$$

$E(X)$ in (4.1) is obtained simply by adding the expressions in (4.5) and (4.6). □

From the above theorem, it is useful to make the following observations:

1. Case when $d = 1$ gives the known result of

$$E(X) = \frac{1-p^c}{q}.$$

2. When $d \to \infty$, X clearly has the geometric distribution of order c mentioned earlier in Section 1 for which it is known that

$$E(X) = \frac{1-p^c}{qp^c}.$$

3. Upon comparing this mean with that presented in (4.1), it is easily seen that, on an average, the waiting time has decreased by $(1-p)^{d+1}/qp^c$ due to the rejection of items upon observing d failures.

4. $E(X)$ monotonically increases to $(1-p^c)/qp^c$ with d.

Higher moments of X can be also derived from the probability generating function in a similar manner.

5. Conditional distributions

In this section, we will discuss the conditional distributions of the number of trials X, given that either the item is accepted or rejected.

Firstly, let Y be the number of trials, given that the equipment is accepted. Then, the probability generating function of Y is obtained from (2.5) immediately as

$$\pi_A(t) = \sum_{k=0}^{d-1} \pi_k(t) / \sum_{k=0}^{d-1} \pi_k(1) = \frac{p^c t^c \sum_{k=0}^{d-1} \left\{ qt \left(\frac{1-p^c t^c}{1-pt} \right) \right\}^k}{1-(1-p^c)^d}. \tag{5.1}$$

By writing once again

$$\left(\frac{1-p^c t^c}{1-pt} \right)^k = \sum_{r=0}^{k(c-1)} t^r p^r B_k(r),$$

we obtain from (5.1) that

$$\begin{aligned}
(5.2)\qquad \pi_A(t) &= \frac{p^c t^c}{1-(1-p^c)^d}\sum_{k=0}^{d-1}\sum_{r=0}^{k(c-1)} q^k t^k t^r p^r B_k(r) \\
&= \frac{1}{1-(1-p^c)^d}\sum_{s=c}^{dc} p^s t^s \left\{ \sum_{k=0}^{\min(d-1,s-c)} \left(\frac{q}{p}\right)^k B_k(s-k-c) \right\}.
\end{aligned}$$

From (5.2), we get an explicit expression for the probability mass function of Y as

$$\Pr(Y=y) = \frac{p^y}{1-(1-p^c)^d} \sum_{k=0}^{\min(d-1,y-c)} \left(\frac{q}{p}\right)^k B_k(y-k-c), \qquad y=c,c+1,\ldots,cd.$$

Once again, recurrence relations for the probabilities of Y can be established from (5.2). By writing

$$\begin{aligned}
\pi_A(t) &= \frac{p^c t^c}{1-(1-p^c)^d} \left\{ \frac{1-\left[qt\left(\frac{1-p^c t^c}{1-pt}\right)\right]^d}{1-\left[qt\left(\frac{1-p^c t^c}{1-pt}\right)\right]} \right\} \\
&= \frac{p^c t^c (1-pt)}{\{1-(1-p^c)^d\}\{1-t+qp^c t^{c+1}\}} \left\{ 1 - q^d t^d \sum_{i=0}^{(c-1)d} p^i t^i A_i(c-1,d) \right\},
\end{aligned}$$

we readily have

$$(5.3)\qquad \{1-t+qp^c t^{c+1}\}\pi_A(t) = \lambda t^c (1-pt) \left\{ 1 - q^d t^d \sum_{i=0}^{(c-1)d} p^i t^i A_i(c-1,d) \right\},$$

where $\lambda = p^c/\{1-(1-p^c)^d\}$ and $A_i(c-1,d)$ is as defined in (3.3).

From (5.3), we derive the following relations where we denote $P_A(r)$ for $\Pr(Y=r)$:

$$\begin{aligned}
&P_A(r) - P_A(r-1) + qp^c P_A(r-c-1) \\
&= \begin{cases} \lambda, & \text{if } r=c, \\ -p\lambda, & \text{if } r=c+1, \\ 0, & \text{if } c+2 \le r \le c+d-1, \\ \lambda q^d p^{r-c-d}\{A_{r-c-d-1}(c-1,d) - A_{r-c-d}(c-1,d)\}, & \\ \qquad \text{if } c+d+1 \le r \le cd. & \end{cases}
\end{aligned}$$

Next, let Z be the number of trials given that the item is rejected. Then, its probability generating function is obtained from (2.5) immediately as

$$\begin{aligned}
(5.4)\qquad \pi_R(t) &= \frac{\pi_d(t)}{\pi_d(1)} = \frac{\pi_d(t)}{(1-p^c)^d} \\
&= \frac{q^d t^d}{(1-p^c)^d}\left(\frac{1-p^c t^c}{1-pt}\right)^d.
\end{aligned}$$

Since

$$\left(\frac{1-p^c t^c}{1-pt}\right)^k = \sum_{r=0}^{k(c-1)} t^r p^r B_k(r),$$

we obtain from (5.4) that

$$\pi_R(t) = \frac{q^d}{(1-p^c)^d} \sum_{r=0}^{d(c-1)} t^{r+d} p^r B_d(r). \tag{5.5}$$

From (5.5), we immediately get

$$\Pr(Z = z) = \frac{q^d}{(1-p^c)^d} p^{z-d} B_d(z-d), \qquad z = d, d+1, \ldots, cd. \tag{5.6}$$

6. Illustrative examples

In Table 1, the values of the probability mass function and the expected value of the number of trials to terminate the experiment have been tabulated for the case $c = 10$, $d = 6$ when $p = 0.5(0.1)0.9$. Similarly, in Tables 2 and 3, the corresponding conditional probabilities (conditioned on acceptance and rejection of the equipment, respectively) have been tabulated. These tables may be used to answer different questions that will be of interest in a start-up demonstration test as displayed in the following examples.

Example 1. Suppose an experimenter tests an equipment for start-ups and requires 10 consecutive start-ups for acceptance, but decides to reject the equipment if it fails to start six times prior to achieving 10 consecutive start-ups. Then, from Table 1, we find the probability that the testing will terminate on or before the 20th trial, when the actual probability of a successful start-up in any trial is 0.8, to be

$$\begin{aligned}\Pr(X \le 20) &= 0.000064 + 0.000307 + \cdots + 0.051656 \\ &= 0.513025.\end{aligned}$$

Similarly, we find from Table 1 that the mean number of trials in this case is $E(X) = 20.54015$.

Example 2. In the last example considered, if we were interested in finding the probability that the testing terminated on or before the 20th trial, given that the equipment got accepted, it is found from Table 2 to be

$$\begin{aligned}\Pr(Y \le 20) &= 0.217288 + 0.043458 + \cdots + 0.042607 \\ &= 0.650277.\end{aligned}$$

Example 3. For the same example, we find from Table 3 the probability that the testing terminated on or before the 20th trial, given that the equipment got rejected, to be

$$\begin{aligned}\Pr(Z \le 20) &= 0.000127 + 0.000607 + \cdots + 0.060497 \\ &= 0.378950.\end{aligned}$$

Table 1. Probabilities and expected values of the number of trials to terminate the experiment.

$c = 10$, $d = 6$

	Prob. of success in a single trial				
X	0.50	0.60	0.70	0.80	0.90
6	0.015625	0.004096	0.000729	0.000064	0.000001
7	0.046875	0.014746	0.003062	0.000307	0.000005
8	0.082031	0.030966	0.007501	0.000860	0.000017
9	0.109375	0.049545	0.014003	0.001835	0.000041
10	0.124023	0.072933	0.050302	0.110677	0.348761
11	0.123535	0.082682	0.039350	0.026760	0.035017
12	0.113281	0.090708	0.048098	0.029226	0.035113
13	0.097168	0.093231	0.056023	0.032105	0.035247
14	0.079041	0.090960	0.062561	0.035294	0.035422
15	0.061584	0.085058	0.067369	0.038672	0.035643
16	0.046204	0.076620	0.070169	0.042063	0.035912
17	0.033485	0.066694	0.070826	0.045256	0.036225
18	0.023502	0.056235	0.069398	0.048035	0.036574
19	0.016012	0.046037	0.066127	0.050215	0.036945
20	0.010612	0.036671	0.061391	0.051656	0.037324
21	0.006855	0.028464	0.055387	0.049972	0.025534
22	0.004324	0.021579	0.048964	0.049289	0.024656
23	0.002666	0.015997	0.042350	0.047791	0.023727
24	0.001609	0.011607	0.035868	0.045548	0.022729
25	0.000951	0.008248	0.029762	0.042660	0.021644
26	0.000551	0.005743	0.024206	0.039260	0.020459
27	0.000313	0.003921	0.019308	0.035504	0.019166
28	0.000174	0.002626	0.015112	0.031553	0.017765
29	0.000095	0.001726	0.011611	0.027560	0.016261
30	0.000051	0.001114	0.008761	0.023658	0.014665
31	0.000027	0.000706	0.006492	0.019953	0.012992
32	0.000014	0.000440	0.004728	0.016574	0.011684
33	0.000007	0.000269	0.003382	0.013536	0.010381
34	0.000004	0.000161	0.002376	0.010868	0.009104
35	0.000002	0.000095	0.001641	0.008577	0.007875
36	0.000001	0.000055	0.001113	0.006653	0.006716
37	0.000000	0.000031	0.000741	0.005073	0.005642
38	0.000000	0.000018	0.000485	0.003800	0.004667
39	0.000000	0.000010	0.000312	0.002796	0.003798
40	0.000000	0.000005	0.000197	0.002019	0.003042
41	0.000000	0.000003	0.000122	0.001431	0.002399
42	0.000000	0.000001	0.000074	0.000995	0.001867
43	0.000000	0.000001	0.000044	0.000679	0.001426
44	0.000000	0.000000	0.000025	0.000453	0.001068
45	0.000000	0.000000	0.000015	0.000296	0.000784
46	0.000000	0.000000	0.000008	0.000189	0.000563
47	0.000000	0.000000	0.000004	0.000118	0.000395
48	0.000000	0.000000	0.000002	0.000072	0.000270
49	0.000000	0.000000	0.000001	0.000042	0.000180
50	0.000000	0.000000	0.000001	0.000024	0.000117
51	0.000000	0.000000	0.000000	0.000014	0.000074
52	0.000000	0.000000	0.000000	0.000007	0.000045
53	0.000000	0.000000	0.000000	0.000004	0.000027
54	0.000000	0.000000	0.000000	0.000002	0.000015
55	0.000000	0.000000	0.000000	0.000001	0.000008
56	0.000000	0.000000	0.000000	0.000000	0.000004
57	0.000000	0.000000	0.000000	0.000000	0.000002
58	0.000000	0.000000	0.000000	0.000000	0.000001
$E(X)$	11.95905	14.68573	18.11318	20.54015	17.25364

Table 2. Conditional Probabilities of the number of trials to terminate the experiment given the item is accepted.

$c = 10$, $d = 6$

	Prob. of success in a single trial				
Y	0.50	0.60	0.70	0.80	0.90
10	0.167075	0.169205	0.178830	0.217288	0.377498
11	0.083537	0.067682	0.053649	0.043458	0.037750
12	0.083537	0.067682	0.053649	0.043458	0.037750
13	0.083537	0.067682	0.053649	0.043458	0.037750
14	0.083537	0.067682	0.053649	0.043458	0.037750
15	0.083537	0.067682	0.053649	0.043458	0.037750
16	0.080927	0.066989	0.053519	0.043444	0.037749
17	0.074400	0.064910	0.053062	0.043388	0.037748
18	0.064611	0.061167	0.052104	0.043255	0.037743
19	0.053190	0.055928	0.050539	0.043005	0.037733
20	0.041769	0.049640	0.048348	0.042607	0.037716
21	0.031408	0.042440	0.044072	0.037366	0.024526
22	0.022801	0.035486	0.040396	0.035667	0.023167
23	0.016030	0.028920	0.036399	0.033772	0.021791
24	0.010942	0.022994	0.032224	0.031684	0.020395
25	0.007255	0.017837	0.028008	0.029416	0.018974
26	0.004678	0.013503	0.023889	0.026997	0.017527
27	0.002937	0.009984	0.019994	0.024469	0.016052
28	0.001798	0.007217	0.016423	0.021884	0.014552
29	0.001075	0.005105	0.013240	0.019296	0.013027
30	0.000628	0.003536	0.010478	0.016758	0.011484
31	0.000359	0.002399	0.008137	0.014317	0.009926
32	0.000201	0.001596	0.006210	0.012111	0.008820
33	0.000110	0.001040	0.004653	0.010091	0.007760
34	0.000059	0.000664	0.003423	0.008277	0.006754
35	0.000031	0.000416	0.002471	0.006682	0.005810
36	0.000016	0.000255	0.001752	0.005307	0.004936
37	0.000008	0.000154	0.001219	0.004146	0.004138
38	0.000004	0.000091	0.000833	0.003183	0.003420
39	0.000002	0.000053	0.000558	0.002402	0.002785
40	0.000001	0.000030	0.000367	0.001780	0.002236
41	0.000000	0.000017	0.000236	0.001295	0.001771
42	0.000000	0.000009	0.000149	0.000925	0.001388
43	0.000000	0.000005	0.000092	0.000648	0.001070
44	0.000000	0.000003	0.000056	0.000444	0.000809
45	0.000000	0.000001	0.000033	0.000299	0.000600
46	0.000000	0.000001	0.000019	0.000196	0.000436
47	0.000000	0.000000	0.000011	0.000126	0.000309
48	0.000000	0.000000	0.000006	0.000079	0.000214
49	0.000000	0.000000	0.000003	0.000048	0.000144
50	0.000000	0.000000	0.000002	0.000028	0.000095
51	0.000000	0.000000	0.000001	0.000016	0.000061
52	0.000000	0.000000	0.000000	0.000009	0.000038
53	0.000000	0.000000	0.000000	0.000005	0.000023
54	0.000000	0.000000	0.000000	0.000002	0.000013
55	0.000000	0.000000	0.000000	0.000001	0.000007

Table 3. Conditional Probabilities of the number of trials to terminate the experiment given the item is rejected.

$c = 10,\ d = 6$

	Prob. of success in a single trial				
Z	0.50	0.60	0.70	0.80	0.90
6	0.015717	0.004248	0.000866	0.000127	0.000013
7	0.047151	0.015292	0.003636	0.000607	0.000071
8	0.082514	0.032113	0.008909	0.001700	0.000223
9	0.110018	0.051381	0.016629	0.003628	0.000535
10	0.123770	0.069365	0.026191	0.006530	0.001083
11	0.123770	0.083238	0.036668	0.010448	0.001949
12	0.113456	0.091562	0.047057	0.015323	0.003216
13	0.097248	0.094178	0.056468	0.021014	0.004962
14	0.079014	0.091823	0.064233	0.027319	0.007257
15	0.061455	0.085702	0.069942	0.033997	0.010160
16	0.045999	0.076977	0.073293	0.040715	0.013688
17	0.033245	0.066760	0.074158	0.047081	0.017807
18	0.023261	0.056052	0.072642	0.052706	0.022426
19	0.015794	0.045670	0.069052	0.057258	0.027408
20	0.010429	0.036190	0.063837	0.060497	0.032579
21	0.006711	0.027946	0.057510	0.062286	0.037735
22	0.004215	0.021063	0.050572	0.062596	0.042663
23	0.002588	0.015518	0.043466	0.061487	0.047146
24	0.001554	0.011185	0.036552	0.059093	0.050974
25	0.000914	0.007893	0.030091	0.055598	0.053954
26	0.000527	0.005455	0.024266	0.051239	0.055939
27	0.000297	0.003696	0.019179	0.046284	0.056845
28	0.000165	0.002455	0.014866	0.040999	0.056649
29	0.000089	0.001601	0.011305	0.035634	0.055390
30	0.000048	0.001024	0.008439	0.030398	0.053158
31	0.000025	0.000643	0.006184	0.025458	0.050085
32	0.000013	0.000397	0.004449	0.020935	0.046333
33	0.000006	0.000240	0.003143	0.016902	0.042085
34	0.000003	0.000143	0.002180	0.013398	0.037530
35	0.000002	0.000083	0.001485	0.010428	0.032861
36	0.000001	0.000048	0.000993	0.007969	0.028250
37	0.000000	0.000027	0.000652	0.005978	0.023844
38	0.000000	0.000015	0.000420	0.004402	0.019752
39	0.000000	0.000008	0.000265	0.003181	0.016055
40	0.000000	0.000004	0.000165	0.002254	0.012797
41	0.000000	0.000002	0.000100	0.001565	0.009998
42	0.000000	0.000001	0.000060	0.001065	0.007651
43	0.000000	0.000001	0.000035	0.000709	0.005732
44	0.000000	0.000000	0.000020	0.000462	0.004201
45	0.000000	0.000000	0.000011	0.000294	0.003010
46	0.000000	0.000000	0.000006	0.000183	0.002105
47	0.000000	0.000000	0.000003	0.000111	0.001434
48	0.000000	0.000000	0.000002	0.000065	0.000951
49	0.000000	0.000000	0.000001	0.000037	0.000611
50	0.000000	0.000000	0.000000	0.000021	0.000381
51	0.000000	0.000000	0.000000	0.000011	0.000229
52	0.000000	0.000000	0.000000	0.000006	0.000132
53	0.000000	0.000000	0.000000	0.000003	0.000073
54	0.000000	0.000000	0.000000	0.000001	0.000039
55	0.000000	0.000000	0.000000	0.000001	0.000019
56	0.000000	0.000000	0.000000	0.000000	0.000008
57	0.000000	0.000000	0.000000	0.000000	0.000003

Appendix

We present here two methods for the computation of A_i.

Method 1. We have

$$\begin{aligned} A_i &= \text{Coef. of } z^i \text{ in } (1+z+z^2+\cdots+z^{c-1})^{d+1} \\ &= \text{Coef. of } z^i \text{ in } \left(\frac{1-z^c}{1-z}\right)^{d+1}. \end{aligned}$$

Consequently, we have the identity

$$\left(\frac{1-z^c}{1-z}\right)^{d+1} = \sum_{i=0}^{(c-1)(d+1)} A_i z^i$$

which can be rewritten as

$$(1-z)^{d+1} \sum_{i=0}^{(c-1)(d+1)} A_i z^i = (1-z^c)^{d+1};$$

thence,

$$\sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} z^j \sum_{i=0}^{(c-1)(d+1)} A_i z^i = \sum_{s=0}^{d+1} (-1)^s \binom{d+1}{s} z^{cs}.$$

Comparing coefficients of z^r on both sides, we get

$$\sum_{j=0}^{\min(d+1,r)} (-1)^j \binom{d+1}{j} A_{r-j} = \begin{cases} 0 & \text{if } r \text{ is not a multiple of } c \\ (-1)^{r/c} \binom{d+1}{r/c} & \text{if } r \text{ is a multiple of } c \end{cases}$$

using which A_i's can be computed recursively.

Method 2. Let $A_i(c-1,d+1) = \text{Coef. of } z^i \text{ in } (1+z+\cdots+z^{c-1})^{d+1}$. Then

$$\begin{aligned} A_i(c-1,d+1) = &\sum_{j=0}^{\min(c-1,i)} \text{Coef. of } z^j \text{ in } (1+z+\cdots+z^{c-1}) \\ &\times \text{Coef. of } z^{i-j} \text{ in } (1+z+\cdots+z^{c-1})^d \\ = &\sum_{j=0}^{\min(c-1,i)} A_{i-j}(c-1,d). \end{aligned}$$

Also, note that $A_i(c-1,1) = 1$ for $i = 0,1,\ldots,c-1$.

Employing this recurrence relation, A_i's can also be computed in a simple recursive manner.

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