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TESTING FOR UNIT ROOTS IN A NEARLY NONSTATIONARY SPATIAL AUTOREGRESSIVE PROCESS

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Abstract. The limiting distribution of the normalized periodogram ordinate is used to test for unit roots in the first-order autoregressive model $Z_{st} = \alpha Z_{s-1,t} + \beta Z_{s,t-1} - \alpha\beta Z_{s-1,t-1} + \epsilon_{st}$. Moreover, for the sequence $\alpha_n = e^{c/n}$, $\beta_n = e^{d/n}$ of local Pitman-type alternatives, the limiting distribution of the normalized periodogram ordinate is shown to be a linear combination of two independent chi-square random variables whose coefficients depend on c and d. This result is used to tabulate the asymptotic power of a test for various values of c and d. A comparison is made between the periodogram test and a spatial domain test.

Key words and phrases: First-order autoregressive process, unit roots, nearly nonstationary, periodogram ordinate, local Pitman-type alternatives, Ornstein-Uhlenbeck process.

1. *Introduction and main results*

Testing for a unit root in a first-order autoregressive time series model has received considerable attention beginning with the work by Dickey and Fuller (1979, 1981). Moreover, Bobkoski (1983), Phillips (1987), and Chan and Wei (1987) have investigated the near unit root problem. Most of the tests proposed are based on the time domain perspective since the spectral density of the process fails to exists in the unit root case. Recently, Akdi (1995) used the frequency domain approach to propose a test in terms of the periodogram ordinate of the process. The periodogram ordinate can be obtained using standard software such as SAS even though the spectral density fails to exist in the unit root case.

Consider the process $\{Z_{st} : 0 \leq s, t\}$ satisfying the spatial model

$$Z_{st} = \alpha Z_{s-1,t} + \beta Z_{s,t-1} - \alpha\beta Z_{s-1,t-1} + \epsilon_{st}, \tag{1.1}$$

An asymptotic test for unit roots based on the spatial periodogram ordinate of a modified Z-process is defined and investigated. The asymptotic power of the test is derived under local Pitman-type alternatives of the form $\alpha_n = e^{c/n}$, $\beta_n = e^{d/n}$, and shown to approach one as $c < 0$ and $d < 0$ decrease. Indeed, the limiting distribution of the normalized periodogram ordinate under the Pitman alternatives is shown to be a linear combination of two independent chi-square distributions whose coefficients depend on c and d.

Basu and Reinsel (1994) illustrate the feasibility of model (1.1) being nearly nonstationary with an example showing that one of the best fits to wheat-yield data is obtained by using a linear regression model with error structure at site (i, j) having the form of model (1.1). They indicate that the residual values obtained from an ordinary least squares fit exhibit a trend behavior which suggest that nonstationarity exists and, indeed, their estimated value of β is .947 (a near unit root). Moreover, Cullis and Gleeson

((1991), p. 1450, (4)) include the above model with $\alpha = \beta = 1$ in the class of models used to represent the error structure for the linear regression model used to analyze field data.

In order to investigate the asymptotic power of a test at the above mentioned sequence of local Pitman-type alternatives the following model

$$Z_{st}(n) = \alpha_n Z_{s-1,t}(n) + \beta_n Z_{s,t-1}(n) - \alpha_n \beta_n Z_{s-1,t-1}(n) + \epsilon_{st}, \tag{1.2}$$

$1 \le s, t \le n$, is needed as well as conditions (A.2)–(A.4) listed below.

(A.1) $\alpha = \beta = 1$

(A.2) $\alpha_n = e^{c/n}$, $\beta_n = e^{d/n}$, where c and d are nonzero constants (unknown)

(A.3) $\{\epsilon_{st}\}$ are i.i.d., mean zero, variance σ^2 and each has a finite fourth moment

(A.4) $Z_{st}(n) = 0$ when either $s \le 0$ or $t \le 0$

(A.5) $\{\bar{\alpha}_n\}$ and $\{\bar{\beta}_n\}$ are any selected initial estimators for which $\bar{\alpha}_n - 1 = O_P(n^{-1})$ and $\bar{\beta}_n - 1 = O_P(n^{-1})$ are valid under the assumptions of model (1.1) and (A.1), as well as under model (1.2) and (A.2).

Remark 1.1. Condition (A.4) is inserted primarily to simplify the exposition and can be relaxed considerably since only asymptotic results are considered here. Moreover, it can be shown that the following initial estimators obey (A.5):

$$\bar{\alpha}_n = \sum_{i=1}^{n}\sum_{j=1}^{n}(Z_{ij} - Z_{i,j-1})(Z_{i-1,j} - Z_{i-1,j-1}) \Big/ \sum_{i=1}^{n}\sum_{j=1}^{n}(Z_{i-1,j} - Z_{i-1,j-1})^2$$

$$\bar{\beta}_n = \sum_{i=1}^{n}\sum_{j=1}^{n}(Z_{ij} - Z_{i-1,j})(Z_{i,j-1} - Z_{i-1,j-1}) \Big/ \sum_{i=1}^{n}\sum_{j=1}^{n}(Z_{i,j-1} - Z_{i-1,j-1})^2.$$

Indeed, $\bar{\alpha}_n - 1 = O_P(n^{-3/2})$ and $\bar{\beta}_n - 1 = O_P(n^{-3/2})$ when $\alpha = \beta = 1$.

Fix positive integers k and ℓ and denote $I^2 = [0,1]\times[0,1]$. The periodogram ordinate is defined for the modified process

$$Y_{st}(n) = Z_{st}(n) - \frac{st}{n^2} Z_{nn}(n), \qquad 1 \le s, t \le n, \quad n \ge 1,$$

as this leads to a chi-square limiting distribution when the null hypothesis is valid. For ease of exposition, $Z_{st}(n)$ and $Y_{st}(n)$ are denoted simply by Z_{st} and Y_{st}. Define $\theta_k = 2\pi k$, $\theta_\ell = 2\pi \ell$, $\omega_k = \frac{\theta_k}{n}$, $\omega_\ell = \frac{\theta_\ell}{n}$ and denote the Fourier coefficients of the Y-process by $a_{nk\ell} = \frac{2}{n^2}\sum_{s,t=1}^{n}\cos(\omega_k s + \omega_\ell t) Y_{st}$ and $b_{nk\ell} = \frac{2}{n^2}\sum_{s,t=1}^{n}\sin(\omega_k s + \omega_\ell t) Y_{st}$. The periodogram ordinate of the Y-process is defined by

$$I_n(\omega_k, \omega_\ell) = \frac{n^2}{2}(a_{nk\ell}^2 + b_{nk\ell}^2).$$

Let $\lambda_1 \ge \lambda_2$ denote the eigenvalues of the asymptotic variance of $\frac{1}{2n\sigma}(a_{nk\ell}, b_{nk\ell})$ under the assumption that $\alpha_n = e^{c/n}$ and $\beta_n = e^{d/n}$. Observe that λ_1 and λ_2 depend on the choices of c and d.

The main results can now be stated.

THEOREM 1.1. *Let V_1 and V_2 denote two independent chi-square random variables each having one degree of freedom.*

(i) *Assume that the Z-process obeys model* (1.1), (A.1) *and* (A.3)–(A.4). *Then* $\frac{1}{2n^4\sigma^2} I_n(\omega_k, \omega_\ell) \xrightarrow{\mathcal{D}} \frac{3}{32\pi^4 k^2 \ell^2} \chi_2^2$ *(chi-square with 2 df).*

(ii) *Suppose that the Z-process satisfies model* (1.2) *and* (A.2)–(A.4). *Then* $\frac{1}{2n^4\sigma^2} I_n(\omega_k, \omega_\ell) \xrightarrow{\mathcal{D}} \lambda_1 V_1 + \lambda_2 V_2$.

Remark 1.2. In the AR(1) time series case, Akdi (1995) proved a version of Theorem 1.1 (i) above but did not consider (ii). However, (ii) is also valid in the time series case when constants are appropriately adjusted. The present paper deals with the modified version suggested by Professor Dickey in the time series case. All our results relate to the modified process. Choosing $k = \ell = 1$, the test considered here rejects $H_0 : \alpha = \beta = 1$ when the normalized periodogram ordinate is sufficiently small. In both the time series and the spatial case, the asymptotic power of the test is one at a fixed alternative with magnitude less than one; however, it is more difficult to obtain large asymptotic power under a sequence of Pitman alternatives.

Given the initial estimators of (α, β) in (A.5), let $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$, $n \geq 1$ denote a sequence of "Gauss-Newton estimators" (defined later) of (α, β) under the assumption of model (1.1). The following asymptotic result is the basis for defining the spatial domain test $\psi_n =: n^{3/2}(\hat{\theta}_n - (1,1))$ for testing $H_0 : \alpha = \beta = 1$.

THEOREM 1.2. (i) *Assume that model* (1.1), (A.1) *and* (A.3)–(A.5) *are satisfied. Then* $\psi_n \xrightarrow{\mathcal{D}} N(\underline{0}, \Gamma)$, *where* $\Gamma = \mathrm{diag}(2,2)$.

(ii) *Suppose that model* (1.2) *obeys* (A.2)–(A.5). *Then* $|\psi_{ni}| \xrightarrow{\mathcal{D}} \infty$, $i = 1, 2$, *where* $\psi_n = (\psi_{n1}, \psi_{n2})$.

2. Proof of Theorems

Fixed $t \in I^2 = [0,1] \times [0,1]$. The four quadrants of I^2 which have t as their origin are designated by $Q_1(\geq, \geq)$, $Q_2(<, \geq)$, $Q_3(<, <)$ and $Q_4(\geq, <)$. Define D_2 to be the set of all real-valued functions f on I^2 for which $\lim_{s \to t} f(s)$ exists when s belongs to a single quadrant and $\lim_{s \to t} f(s) = f(t)$ when $s \in Q_1$. Following Bickel and Wichura (1971), there is a metric on D_q which induces Skorohod's well-known topology when $q = 1$ and makes D_q separable, complete and whose Borel σ-field coincides with that generated by the coordinate mappings. Convergence in D_2 will be relative to the above mentioned metric.

Let $W(u,v)$ denote a Wiener process (Brownian sheet) on I^2; that is, a mean zero Gaussian process with $\mathrm{cov}(W(u,v), W(s,t)) = (u \wedge s) \cdot (v \wedge t)$. Define $a^0_{k\ell} = \int_{I^2} \cos(\theta_k u + \theta_\ell v) W(u,v) du dv + \frac{1}{\theta_k \theta_\ell} W(1,1)$ and $b^0_{k\ell} = \int_{I^2} \sin(\theta_k u + \theta_\ell v) W(u,v) du dv$. The following three lemmas are used to verify Theorem 1.1. Let I_2 denote the 2×2 identity matrix.

LEMMA 2.1. *Under the assumptions of Theorem* 1.1 (i), $\frac{1}{2n\sigma}(a_{nk\ell}, b_{nk\ell}) \xrightarrow{\mathcal{D}} (a^0_{k\ell}, b^0_{k\ell})$, *where the latter has distribution* $N(\underline{0}, \Sigma^0)$ *with* $\Sigma^0 = \frac{3}{32\pi^4 k^2 \ell^2} I_2$.

PROOF. Observe that since the Z-process obeys model (1.1), (A.1) and (A.4), $Y_{st} = \sum_{i=1}^{s} \sum_{j=1}^{t} \epsilon_{ij} - \frac{st}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_{ij}$. Define $U_n(u,v) = \cos(\omega_k [nu] + \omega_\ell [nv]) \cdot \frac{1}{n\sigma} \sum_{i=1}^{[nu]} \sum_{j=1}^{[nv]} \epsilon_{ij} - \frac{[nu] \cdot [nv]}{n^2} \cos(\omega_k [nu] + \omega_\ell [nv]) \cdot \frac{1}{n\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_{ij} =: R_n(u,v) - S_n(u,v)$, $(u,v) \in I^2$. Since $R_n(u,v) \xrightarrow{\mathcal{D}} \cos(\theta_k u + \theta_\ell v) W(u,v) = R(u,v)$ and $S_n(u,v) \xrightarrow{\mathcal{D}} uv \cos(\theta_k u + \theta_\ell v) W(1,1) = S(u,v)$ in D_2, it follows that $\{R_n\}$ and $\{S_n\}$ are each tight and thus $\{U_n\}$ is tight in D_2. Then $U_n \xrightarrow{\mathcal{D}} R - S$ in D_2 provided the finite-dimensional distributions of $\{U_n\}$ converge weakly to those of $R - S$.

Fix $(u, v) \in I^2$ and observe that

$$\begin{aligned} U_n(u,v) = &\cos(\omega_k[nu] + \omega_\ell[nv])\left(1 - \frac{[nu]\cdot[nv]}{n^2}\right)\cdot\frac{1}{n\sigma}\sum_{i=1}^{[nu]}\sum_{j=1}^{[nv]}\epsilon_{ij} \\ &- \frac{[nu]\cdot[nv]}{n^2}\cos(\omega_k[nu] + \omega_\ell[nv])\cdot\frac{1}{n\sigma}\sum_{i=[nu]+1}^{n}\sum_{j=1}^{n}\epsilon_{ij} \\ &- \frac{[nu]\cdot[nv]}{n^2}\cos(\omega_k[nu] + \omega_\ell[nv])\cdot\frac{1}{n\sigma}\sum_{i=1}^{[nu]}\sum_{j=[nv]+1}^{n}\epsilon_{ij} \\ =: &I_{n1} - I_{n2} - I_{n3}. \end{aligned}$$

Observe that $I_{n1} \xrightarrow{\mathcal{D}} (1 - uv)\cdot\cos(\theta_k u + \theta_\ell v)W(u,v)$, $I_{n2} \xrightarrow{\mathcal{D}} uv\cos(\theta_k u + \theta_\ell v)\cdot(W(1,1) - W(u,1))$ and $I_{n3} \xrightarrow{\mathcal{D}} uv\cos(\theta_k u + \theta_\ell v)(W(u,1) - W(u,v))$ in R. Since $\{I_{n1}\}$, $\{I_{n2}\}$ and $\{I_{n3}\}$ are independent sequences and $W(u,v)$, $W(1,1) - W(u,1)$ and $W(u,1) - W(u,v)$ are independent random variables, it follows that $U_n(u,v) \xrightarrow{\mathcal{D}} \cos(\theta_k u + \theta_\ell v)[(1 - uv)W(u,v) - uv(W(1,1) - W(u,1)) - uv(W(u,1) - W(u,v))] = \cos(\theta_k u + \theta_\ell v)W(u,v) - uv\cos(\theta_k u + \theta_\ell v)W(1,1) = R(u,v) - S(u,v)$ in R. Hence $U_n \xrightarrow{\mathcal{D}} R - S$ in D_2.

Define the continuous function $T : D_2 \to R$ by $T(f) = \int_{I^2} f(u,v)dudv$. It follows from the continuous mapping theorem (Billingsley (1968), p. 31) that $T(U_n) \xrightarrow{\mathcal{D}} \int_{I^2}\cos(\theta_k u + \theta_\ell v)W(u,v)dudv - W(1,1)\int_{I^2} uv\cos(\theta_k u + \theta_\ell v)dudv = \int_{I^2}\cos(\theta_k u + \theta_\ell v)\cdot W(u,v)dudv + \frac{1}{\theta_k\theta_\ell}W(1,1)$. However, $T(U_n) = \frac{a_{n-1,k,\ell}}{2n\sigma^2}$ and thus $\frac{a_{nk\ell}}{2n\sigma^2} \xrightarrow{\mathcal{D}} a^0_{k\ell}$ in D_2. The above argument can be extended to show that $\frac{\lambda_1 a_{nk\ell}}{2n\sigma^2} + \frac{\lambda_2 b_{nk\ell}}{2n\sigma^2} \xrightarrow{\mathcal{D}} \lambda_1 a^0_{k\ell} + \lambda_2 b^0_{k\ell}$ in R and thus $\frac{1}{2n\sigma}(a_{nk\ell}, b_{nk\ell}) \xrightarrow{\mathcal{D}} (a^0_{k\ell}, b^0_{k\ell})$ in R^2.

It was found using the software Mathematica that $\operatorname{var}\int_{I^2}\cos(\theta_k u + \theta_\ell v)\cdot W(u,v)dudv = \frac{3}{32\pi^4k^2\ell^2}$, $\operatorname{var}\int_{I^2}\sin(\theta_k u + \theta_\ell v)W(u,v)dudv = \frac{3}{32\pi^4k^2\ell^2}$ and $\operatorname{cov}(\int_{I^2}\cos(\theta_k u + \theta_\ell v)W(u,v)dudv, \int_{I^2}\sin(\theta_k u + \theta_\ell v)W(u,v)dudv) = 0$ and hence the entries of Σ^0 are easily verified. □

Recall that $W(u,v)$, $(u,v) \in I^2$, denotes a two-parameter Wiener process. Define the random elements W_n and J in D_2 by

$$W_n(u,v) = \sum_{i=1}^{[nu]}\sum_{j=1}^{[nv]}\alpha_n^{[nu]-i}\beta_n^{[nv]-j}\epsilon_{ij} \tag{2.1}$$

and

$$\begin{aligned} J(u,v) = W(u,v) &+ c\int_{[0,u]} W(x,v)e^{(u-x)c}dx \\ &+ d\int_{[0,v]} W(u,y)e^{(v-y)d}dy \\ &+ cd\int_{[0,u]\times[0,v]} W(x,y)e^{(u-x)c}e^{(v-y)d}dxdy. \end{aligned} \tag{2.2}$$

The J-process is a mean zero Gaussian process with $\operatorname{cov}(J(u,v), J(s,t)) = [\frac{e^{(u+s)c} - e^{|u-s|c}}{2c}]\cdot[\frac{e^{(v+t)d} - e^{|v-t|d}}{2d}]$; that is, J has the same covariance structure as the product of two-independent one-parameter Ornstein-Uhlenbeck processes (Breiman (1968), p. 349).

LEMMA 2.2. *Assume that conditions* (A.2)–(A.3) *are satisfied. Then* $\frac{1}{n\sigma}W_n \xrightarrow{\mathcal{D}} J$ *in* D_2, *where* W_n *and* J *are defined in* (2.1)–(2.2).

Remark 2.1. Lemma 2.2 is an extension of the corresponding result for the first-order autoregressive time series process satisfying $Y_t = \alpha_n Y_{t-1} + \epsilon_t$ with $\alpha_n = e^{c/n}$. This time series has been investigated by Bobkoski (1983), Chan and Wei (1987), and Phillips (1987) and it is known that the normalized Y-process, when embedded in $D_1 = D[0,1]$, converges weakly to the one-parameter Ornstein-Uhlenbeck process determined by $J_1(u) = W(u) + c\int_{[0,u]} e^{(u-x)c}W(x)dx$, $0 \le u \le 1$. Moreover, suppose that $X_t = \beta_n X_{t-1} + \delta_t$, with $\beta_n = e^{d/n}$, is another first-order autoregressive process and assume that $\{\epsilon_t\}$ and $\{\delta_t\}$ are independent, each i.i.d. with mean zero and finite second moment. Define $Z_{st} = X_s Y_t$ and observe that the product model can be expressed as $Z_{st} = \alpha_n Z_{s-1,t} + \beta_n Z_{s,t-1} - \alpha_n\beta_n Z_{s-1,t-1} + \epsilon_s\delta_t$ with

$$\operatorname{cov}(\epsilon_{s_1}\delta_{t_1}, \epsilon_{s_2}\delta_{t_2} = \begin{cases} \sigma_\epsilon^2\sigma_\delta^2, & \text{if } \ (s_1,t_1) = (s_2,t_2) \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the product model has the same covariance structures as model (1.2) when $\sigma_\epsilon^2\sigma_\delta^2 = \sigma^2$; however, the limiting distributions of the corresponding normalized Z_n-processes may differ. Indeed, according to Lemma 2.2 the normalized Z_n-process for model (1.2) converges weakly to a Gaussian process. However, if $\{\epsilon_t\}$ and $\{\delta_t\}$ are each Gaussian, then the limiting distribution of the normalized Z_n-process for the product model is a product of two independent Gaussian processes and therefore fails to be Gaussian. Asymptotic properties of Gauss-Newton estimators of (α_n, β_n) in model (1.2) have been given by Bhattacharyya *et al.* (1996) when $\alpha = \beta = 1$.

All integrals from henceforth are to be interpreted in the Riemann-Stieltjes sense as discussed in Hobson (1957) and Yeh (1963).

PROOF OF LEMMA 2.2. Define the random element X_n in D_2 by $X_n(u,v) = \frac{1}{n\sigma}\sum_{i=1}^{[nu]}\sum_{j=1}^{[nv]}\epsilon_{ij}$ and denote $g(x,y) = e^{(u-x)c}e^{(v-y)d}$, where (u,v) and (x,y) belong to I^2. Observe that

$$\frac{1}{n\sigma}e^{(u-i/n)c}e^{(v-j/n)d}\epsilon_{ij} = \int_{[(i-1)/n,i/n]\times[(j-1)/n,j/n]} g(x,y)dX_n(x,y)$$

and using model (1.2) and (A.2),

$$\begin{aligned}\frac{1}{n\sigma}W_n(u,v) = \frac{1}{n\sigma}\sum_{i=1}^{[nu]}\sum_{j=1}^{[nv]}(e^{([nu]-i)c/n}e^{([nv]-j)d/n} - e^{(u-i/n)c}e^{(v-j/n)d})\epsilon_{ij} \\ + \int_{[0,[nu]/n]\times[0,[nv]/n]} g(x,y)dX_n(x,y).\end{aligned}$$

Hence

$$\frac{1}{n\sigma}W_n(u,v) = \int_{[0,u]\times[0,v]} g(x,y)dX_n(x,y) + o_P(1). \tag{2.3}$$

The integral in (2.3) can be rewritten, using integration by parts, as a signed sum of integrals of the form $\int_A X_n(x,y)dg(x,y)$ over k-dimensional subregions A of $[0,u]\times[0,v]$, where $k=0,1,2$. Using the formula given by Hobson ((1957), p. 666),

$$\int_{[0,u]\times[0,v]} g(x,y)dX_n(x,y) = I_0 + I_1 + I_2, \qquad \text{where}$$
$$I_0 = X_n(u,v)g(u,v) - X_n(u,0)g(u,0) - X_n(0,v)g(0,v) + X_n(0,0)g(0,0)$$
$$= X_n(u,v),$$
$$I_1 = -\int_{[0,u]} X_n(x,v)dg(x,v) + \int_{[0,u]} X_n(x,0)dg(x,0) - \int_{[0,v]} X_n(u,y)dg(u,y)$$
$$+ \int_{[0,v]} X_n(0,y)dg(0,y)$$
$$= c\int_{[0,u]} X_n(x,v)g(x,v)dx + d\int_{[0,v]} X_n(u,y)g(u,y)dy \qquad \text{and}$$
$$I_2 = \int_{[0,u]\times[0,v]} X_n(x,y)dg(x,y) = cd\int_{[0,u]\times[0,v]} X_n(x,y)g(x,y)dxdy.$$

Thus, from (2.3),

$$\frac{1}{n\sigma}W_n(u,v) = X_n(u,v) + c\int_{[0,u]} X_n(x,v)g(x,v)dx + d\int_{[0,v]} X_n(u,y)g(u,y)dy$$
$$+ cd\int_{[0,u]\times[0,v]} X_n(x,y)g(x,y)dxdy + o_P(1)$$

and it follows from Billingsley ((1968), p. 31) that

$$\frac{1}{n\sigma}W_n(u,v) \xrightarrow{\mathcal{D}} W(u,v) + c\int_{[0,u]} W(x,v)e^{(u-x)c}dx$$
$$+ d\int_{[0,v]} W(u,y)e^{(v-y)d}dy + cd\int_{[0,u]\times[0,v]} W(x,y)e^{(u-x)c}e^{(v-y)d}dxdy$$

for each fixed $(u,v)\times I^2$. The above argument can be extended to show that the finite-dimensional distributions of $\{\frac{1}{n\sigma}W_n\}$ converge weakly to those of J.

It remains to show that the sequence $\{\frac{1}{n\sigma}W_n\}$ is tight in D_2. Define the increment of Z_n over the rectangular set $(s,t]$ by

$$Z_n(s,t] = \sum_{i=[ns_1]+1}^{[nt_1]} \sum_{j=[ns_2]+1}^{[nt_2]} \alpha_n^{[nt_1]-i}\beta_n^{[nt_2]-j}\epsilon_{ij},$$

where $s=(s_1,s_2)$ and $t=(t_1,t_2)$ belong to I^2. Furthermore, let $p=(p_1,p_2)$ and $q=(q_1,q_2)$ belong to I^2 and assume that $(p,q]$ is a rectangular set having only the line segment connecting (p_1,p_2) and (t_1,t_2) as a common edge with $(s,t]$. Tightness of $\{\frac{1}{n\sigma}W_n\}$ will follow by showing that $\frac{1}{n^4}E(Z_n^2(s,t]\cdot Z_n^2(p,q]) \le M\lambda(s,t]\cdot\lambda(p,q]$ for some constant M independent of $(s,t]$ and $(p,q]$, where λ denotes Lebesque measure (Bickel and Wichura (1971), Theorem 3).

It follows from (A.3) that $E(\epsilon_{ij}\epsilon_{i'j'}\epsilon_{ab}\epsilon_{a'b'}) = 0$ unless two pairs of indices coincide. Moreover, by (A.2), $\alpha_n^{2([nt_1]-i)} \le e^{2|c|}$ and $\beta_n^{2([nt_2]-j)} \le e^{2|d|}$ and hence there exists a finite constant M such that

$$\begin{aligned}\frac{1}{n^4}E(Z_n^2(s,t]\cdot Z_n^2(p,q]) &\le \frac{M}{n^4}([nt_1]-[ns_1])([nq_1]-[np_1])([nt_2]-[ns_2])([nq_2]-[np_2]) \\ &= M(t_1-s_1)(q_1-p_1)(t_2-s_2)(q_2-p_2) = M\lambda(s,t]\cdot\lambda(p,q]\end{aligned}$$

since the vertices of the rectangular sets may be taken to be $(\frac{k}{n}, \frac{\ell}{n})$ for some nonnegative integers k and ℓ (Bickel and Wichura (1971), p. 1665). Likewise, the moment inequality also holds when the two rectangular sets have any other common edge. Therefore $\{\frac{1}{n\sigma}W_n\}$ converges weakly in D_2 to J. The desired covariance function of the J-process can also be checked. □

Denote

$$a_{k\ell}^1 = \int_{I^2}\cos(\theta_k u + \theta_\ell v)J(u,v)dudv + \frac{J(1,1)}{\theta_k\theta_\ell}, \tag{2.4}$$

$$b_{k\ell}^1 = \int_{I^2}\sin(\theta_k u + \theta_\ell v)J(u,v)dudv \tag{2.5}$$

and let $\Sigma^1 = (\sigma_{ij}^1)$ denote the variance of $(a_{k\ell}^1, b_{k\ell}^1)$. Then

$$\begin{aligned}\sigma_{11}^1 = &\int_{I^4}\cos(\theta_k u+\theta_\ell v)\cos(\theta_k u+\theta_\ell v)\operatorname{cov}(J(u,v),J(s,t))dudvdsdt \\ &+\frac{1}{\theta_k^2\theta_\ell^2}\operatorname{var}J(1,1)+\frac{2}{\theta_k\theta_\ell}\int_{I^2}\cos(\theta_k u+\theta_\ell v)\operatorname{cov}(J(u,v),J(1,1))dudv.\end{aligned}$$

The first integral can be expressed in terms of two-fold integrals by using the addition formula for cosine to expand the integrand. The software Mathematica was then used to evaluate the integrals. For sake of simplicity, the values of σ_{ij}^1 are listed below when $c = d$. Denote $\theta = \theta_k = \theta_\ell$.

$$\begin{aligned}\sigma_{11}^1 = &\frac{(c^2+\theta^2+3c-4ce^c+ce^{2c})^2}{4(c^2+\theta^2)^4} - \frac{\theta^2(1-e^c)^4}{2(c^2+\theta^2)^4} \\ &+\frac{[c(c^2+\theta^2)-\theta^2(1-e^{2c})]^2}{4c^2(c^2+\theta^2)^4}+\frac{(1-e^{2c})^2}{4c^2\theta^4} \\ &+\frac{(1-e^c)^4}{2\theta^2(c^2+\theta^2)^2}-\frac{(1-e^{2c})^2}{2c^2(c^2+\theta^2)^2},\end{aligned} \tag{2.6}$$

$$\sigma_{12}^1 = \frac{-\theta(1-e^c)^2[3c^2-4c^2e^c+c^2e^{2c}+\theta^2(1-e^{2c})]}{2c(c^2+\theta^2)^4} \tag{2.7}$$

and

$$\begin{aligned}\sigma_{22}^1 = &\frac{[c(c^2+\theta^2)-\theta^2(1-e^{2c})]\cdot[c^2+\theta^2+3c-4ce^c+ce^{2c}]}{2c(c^2+\theta^2)^4} \\ &+\frac{\theta^2(1-e^c)^4}{2(c^2+\theta^2)^4}.\end{aligned} \tag{2.8}$$

It is straightforward to check that Σ^1 converges to Σ^0 as $c \to 0$ when $c = d$ (or, more generally, when $(c,d) \to (0,0)$).

LEMMA 2.3. *Assume that the conditions listed in Theorem* 1.1 (ii) *are satisfied. Given the notation defined in* (2.4)–(2.8), $\frac{1}{2n\sigma}(a_{nk\ell}, b_{nk\ell}) \xrightarrow{\mathcal{D}} (a^1_{k\ell}, b^1_{k\ell})$, *where* $(a^1_{k\ell}, b^1_{k\ell})$ *is a bivariate normal with mean* $\underline{0}$ *and variance* $\Sigma^1 = (\sigma^1_{ij})$.

PROOF. Since the process obeys model (1.2) and (A.2),

$$Z_{st} = \sum_{i=1}^{s}\sum_{j=1}^{t} \alpha_n^{s-i}\beta_n^{t-j}\epsilon_{ij} = \sum_{i=1}^{s}\sum_{j=1}^{t} e^{(s-i)c/n}e^{(t-j)d/n}\epsilon_{ij},$$

Moreover, $a_{nk\ell} = \frac{2}{n^2}\sum_{s,t=1}^{n}\cos(\omega_k s + \omega_\ell t)Y_{st}$ and thus

$$\begin{aligned} \frac{a_{nk\ell}}{2n\sigma} &= \frac{1}{n^3\sigma}\sum_{s,t=1}^{n}\sum_{i=1}^{s}\sum_{j=1}^{t}\cos(\omega_k s + \omega_\ell t)e^{(s-i)c/n}e^{(t-j)d/n}\epsilon_{ij} \\ &\quad - \frac{1}{n^3\sigma}\sum_{s,t=1}^{n}\sum_{i,j=1}^{n}\frac{st}{n^2}\cos(\omega_k s + \omega_\ell t)e^{(1-i/n)c}e^{(1-j/n)d}\epsilon_{ij}, \end{aligned} \tag{2.9}$$

Define

$$\begin{aligned} U_n(u,v) &= \frac{1}{n\sigma}\sum_{i=1}^{[nu]}\sum_{j=1}^{[nv]}\cos(\omega_k[nu] + \omega_\ell[nv])e^{([nu]-i)c/n}e^{([nv]-j)d/n}\epsilon_{ij} \\ &\quad - \frac{1}{n\sigma}\sum_{i,j=1}^{n}\frac{[nu]}{n}\cdot\frac{[nv]}{n}\cos(\omega_k[nu] + \omega_\ell[nv])e^{(1-i/n)c}e^{(i-j/n)d}\epsilon_{ij} \\ &=: R_n(u,v) - S_n(u,v), \qquad 0 \le u, \quad v \le 1. \end{aligned} \tag{2.10}$$

According to Lemma 2.2,

$$\begin{aligned} R_n(u,v) &\xrightarrow{\mathcal{D}} \cos(\theta_k u + \theta_\ell v)J(u,v) := R(u,v), \\ S_n(u,v) &\xrightarrow{\mathcal{D}} uv\cos(\theta_k u + \theta_\ell v)J(1,1) := S(u,v) \end{aligned}$$

in D_2 and thus $\{U_n\}$ is tight in D_2. Moreover, $U_n \xrightarrow{\mathcal{D}} R - S$ in D_2 provided the finite-dimensional distributions of $\{U_n\}$ converge weakly to those of $R - S$.

Fix $(u,v) \in I^2$ and denote $g(x,y) = e^{(u-x)c}e^{(v-y)d}$, $h(x,y) = e^{(1-x)c}e^{(1-y)d}$ and

$$X_n(x,y) = \frac{1}{n\sigma}\sum_{i=1}^{[nx]}\sum_{j=1}^{[ny]}\epsilon_{ij}.$$

Then

$$\begin{aligned} U_n(u,v) &= \cos(\theta_k u + \theta_\ell v)\int_{[0,u]\times[0,v]} g(x,y)\,dX_n(x,y) \\ &\quad - uv\cos(\theta_k u + \theta_\ell v)\int_{I^2} h(x,y)\,dX_n(x,y) + o_P(1). \end{aligned} \tag{2.11}$$

Integrating by parts (Hobson (1957), p. 666), $\int_{[0,u]\times[0,v]} g(x,y)dX_n(x,y) = I_0 + I_1 + I_2$, where

$$\begin{aligned} I_0 &= X_n(u,v)g(u,v) - X_n(u,0)g(u,0) - X_n(0,v)g(0,v) + X_n(0,0)g(0,0) \\ &= X_n(u,v), \\ I_1 &= -\int_{[0,u]} X_n(x,v)dg(x,v) + \int_{[0,u]} X_n(x,0)dg(x,0) \\ &\quad - \int_{[0,v]} X_n(u,v)g(x,v)dx + \int_{[0,v]} X_n(0,y)dg(0,y) \\ &= c\int_{[0,u]} X_n(x,v)g(x,v)dx + d\int_{[0,v]} X_n(u,y)g(u,y)dy \end{aligned}$$

and

$$I_2 = \int_{[0,u]\times[0,v]} X_n(x,y)dg(x,y) = cd\int_{[0,u]\times[0,v]} X_n(x,y)g(x,y)dxdy.$$

Hence

$$\begin{aligned} \int_{[0,u]\times[0,v]} g(x,y)dX_n(x,y) = X_n(u,v) &+ c\int_{[0,u]} e^{(u-x)c}X_n(x,v)dx \\ &+ d\int_{[0,v]} e^{(v-y)d}X_n(u,y)dy \\ &+ cd\int_{[0,u]\times[0,v]} e^{(u-x)c}e^{(v-y)d}X_n(x,y)dxdy. \end{aligned} \tag{2.12}$$

A similar argument shows that

$$\begin{aligned} \int_{I^2} h(x,y)dX_n(x,y) = X_n(1,1) &+ c\int_I e^{(1-x)c}X_n(x,1)dx \\ &+ d\int_I e^{(1-y)d}X_n(1,y)dy \\ &+ cd\int_{I^2} e^{(1-x)c}e^{(1-y)d}X_n(x,y)dxdy. \end{aligned} \tag{2.13}$$

It follows from (2.11)–(2.13) that $U_n = \phi(X_n) + o_P(1)$, where $\phi : D_2 \to R$ is defined by

$$\begin{aligned} \phi(f) = \cos(\theta_k u + \theta_\ell v)\Bigg[f(u,v) &+ c\int_{[0,u]} e^{(u-x)c}f(x,v)dx \\ &+ d\int_{[0,v]} e^{(v-y)d}f(u,y)dy \\ &+ cd\int_{[0,u]\times[0,v]} e^{(u-x)c}e^{(v-y)d}f(x,y)dxdy\Bigg] \\ - uv\cos(\theta_k u + \theta_\ell v)\Bigg[f(1,1) &+ c\int_I e^{(1-x)c}f(x,1)dx \\ &+ d\int_I e^{(1-y)d}f(1,y)dy \\ &+ cd\int_{I^2} e^{(1-x)c}e^{(1-y)d}f(x,y)dxdy\Bigg]. \end{aligned}$$

Since ϕ is continuous when restricted to the set C_2 of all continuous members of D_2, $W(C_2) = 1$ and $X_n \xrightarrow{\mathcal{D}} W$ in D_2, it follows (Billingsley (1968), p. 31) that

$$U_n(u,v) \xrightarrow{\mathcal{D}} \Bigg[W(u,v) + c\int_{[0,u]} e^{(n-x)c}W(y,v)dx + d\int_{[0,v]} e^{(v-y)d}W(u,y)dy + cd\int_{[0,u]\times[0,v]} e^{(u-x)c}e^{(v-y)d}W(x,y)dxdy\Bigg] \cdot \cos(\theta_k u + \theta_\ell v) - \Bigg[W(1,1) + c\int_I e^{(1-x)c}W(x,1)dx + d\int_I e^{(1-y)d}W(1,y)dy + cd\int_{I^2} e^{(1-x)c}e^{(1-y)d}W(x,y)dxdy\Bigg] \cdot uv\cos(\theta_k u + \theta_\ell v) = \cos(\theta_k u + \theta_\ell v)J(u,v) - uv\cos(\theta_k u + \theta_\ell v)J(1,1)$$

in R. The above argument can be extended to a finite linear combination of $U_n(u_i, v_i)$, $1 \le i \le k$, and thus it follows that the finite-dimensional distributions of $\{U_n\}$ converge weakly to those of $R-S$. Hence $U_n \xrightarrow{\mathcal{D}} R-S$ in D_2. Moreover, it follows from (2.9)–(2.10) that

$$\frac{a_{n-1,k\ell}}{2n\sigma} = \int_{I^2} U_n(u,v)dudv$$

and thus

$$\frac{a_{nk\ell}}{2n\sigma} \xrightarrow{\mathcal{D}} \int_{I^2} \cos(\theta_k u + \theta_\ell v)J(u,v)dudv - J(1,1)\int_{I^2} uv\cos(\theta_k u + \theta_\ell v)dudv = a^1_{k\ell}.$$

Extending this argument, $\lambda_1 \frac{a_{nk\ell}}{2n\sigma} + \lambda_2 \frac{b_{nk\ell}}{2n\sigma} \xrightarrow{\mathcal{D}} \lambda_1 a^1_{k\ell} + \lambda_1 b^1_{k\ell}$ and thus $\frac{1}{2n\sigma}(a_{nk\ell}, b_{nk\ell}) \xrightarrow{\mathcal{D}} (a^1_{k\ell}, b^1_{k\ell})$ in R^2. □

PROOF OF THEOREM 1.1. Verification of Theorem 1.1 (i–ii) follows easily from Lemmas 2.1 and 2.3. Indeed, in case (ii), fix $c < 0$ and $d < 0$ and let λ_1 and λ_2 denote the corresponding eigenvalues of Σ^1 and let Q be an orthogonal matrix such that $Q\Sigma^1 Q' = \Gamma = \text{diag}(\lambda_1, \lambda_2)$. According to Lemma 2.3, $R_{nk\ell} = \frac{1}{2\sigma n}(a_{nk\ell}, b_{nk\ell})Q' \xrightarrow{\mathcal{D}} N(\underline{0}, \Gamma)$ and hence

$$\frac{I_n(\omega_k, \omega_\ell)}{2\sigma^2 n^4} = \frac{1}{4\sigma^2 n^2}(a^2_{nk\ell} + b^2_{nk\ell}) = R_{nk\ell}R'_{nk\ell} \xrightarrow{\mathcal{D}} \lambda_1 V_1 + \lambda_2 V_2. \qquad \square$$

The spatial domain test ψ_n is defined in terms of "Gauss-Newton estimators". Given model (1.1), let $\theta = (\alpha, \beta)$ and denote the sequence of initial estimators given in (A.5) by $\bar\theta_n = (\bar\alpha_n, \bar\beta_n)$, $n \ge 1$. Define $f_{ij}(a,b) = aZ_{i-1,j} + bZ_{i,j-1} - abZ_{i-1,j-1}$, $F_{ij}(a,b) = (\frac{\partial f_{ij}}{\partial a}(a,b), \frac{\partial f_{ij}}{\partial b}(a,b)) = (Z_{i-1,j} - bZ_{i-1,j-1}, Z_{i,j-1} - aZ_{i-1,j-1})$ and $R_{ij}(a,b) = -(\alpha - a)(\beta - b)Z_{i-1,j-1}$. Then model (1.1) can be written in the form $Z_{ij} = f_{ij}(\bar\theta_n) + F_{ij}(\bar\theta_n)(\theta - \bar\theta_n)' + R_{ij}(\bar\theta_n) + \epsilon_{ij}$. Denote $A_n = \sum_{i=1}^n \sum_{j=1}^n F'_{ij}(\bar\theta_n)F_{ij}(\bar\theta_n)$. The sequence of "Gauss-Newton estimators" of $\theta = (\alpha, \beta)$ is defined to be

$$\hat\theta_n =: (\hat\alpha_n, \hat\beta_n) = \hat\delta_n + \bar\theta_n, \quad \text{where} \tag{2.14}$$

$$\hat\delta'_n =: A_n^{-1} \sum_{i=1}^n \sum_{j=1}^n F'_{ij}(\bar\theta_n)(Z_{ij} - f_{ij}(\bar\theta_n)), \qquad n \ge 1.$$

The spatial domain test for testing $H_0 : \alpha = \beta = 1$ is defined by

(2.15) $$\psi_n =: n^{3/2}(\hat{\theta}_n - (1,1)).$$

PROOF OF THEOREM 1.2. A proof of Theorem 1.2 (i) is given by Bhattacharyya *et al.* ((1996), Theorem 3.1) when (A.4) is replaced by the stronger assumption that $\bar{\alpha}_n - \alpha = O_P(n^{-3/2})$ and $\bar{\beta}_n - \beta = O_P(n^{-3/2})$, for $\alpha = \beta = 1$. However, the proof is still valid when $\bar{\alpha}_n - \alpha = O_P(n^{-1})$ and $\bar{\beta}_n - \beta = O_P(n^{-1})$.

Next, suppose that the assumptions of Theorem 1.2 (ii) are satisfied. It is shown by Bhattacharyya *et al.* ((1997), Theorem 1.1) that $n^{3/2}(\hat{\theta}_n - \theta_n) \xrightarrow{\mathcal{D}} N(\underline{0}, \Sigma)$, where $\Sigma = \operatorname{diag}(g_c^{-1}(1), g_d^{-1}(1))$, $g_\gamma(x) = (e^{2\gamma x} - 1 - 2\gamma x)/4\gamma^2$, $\gamma \neq 0$, and $\theta_n = (\alpha_n, \beta_n)$. Again, it was assumed that $\bar{\alpha}_n - \alpha_n = O_P(n^{-3/2})$ and $\bar{\beta}_n - \beta_n = O_P(n^{-3/2})$ but the proof is also valid when $\bar{\alpha}_n - \alpha_n = O_P(n^{-1})$ and $\bar{\beta}_n - \beta_n = O_P(n^{-1})$. Since $\psi_n = n^{3/2}(\hat{\theta}_n - (1,1)) = n^{3/2}(\hat{\theta}_n - \theta_n) + n^{3/2}(\theta_n - (1,1))$, it follows that $|\psi_{ni}| \xrightarrow{\mathcal{D}} \infty$, $i = 1, 2$, where $\psi_n = (\psi_{n1}, \psi_{n2})$, which completes the proof of Theorem 1.2. □

3. Comparison of tests

Consider the testing problem $H_0 : \alpha = \beta = 1$ vs. $H_1 : 0 < \alpha, \beta < 1$ given data $\{Z_{ij} : 1 \leq i, j \leq n\}$. A periodogram test is defined by

(3.1) $$\phi_n =: 16\pi^4 k^2 \ell^2 I_n(\omega_k, \omega_\ell)/\sigma^2 n^4.$$

According to Theorem 1.1 (i), $\phi_n \xrightarrow{\mathcal{D}} 3\chi_2^2$ when $\alpha = \beta = 1$, where χ_2^2 denotes a chi-square random variate with two degrees of freedom. Given a significance level $0 < \delta < 1$, choose $k = \ell = 1$ and let r_δ denote the real number for which $P\{\chi_2^2 \geq r_\delta\} = 1 - \delta$. Then, for n sufficiently large, the null hypothesis is rejected when $\phi_n < 3r_\delta$.

Given a sequence $\{(\alpha_n, \beta_n)\}$ where $\alpha_n = e^{c/n}$ and $\beta_n = e^{d/n}$, of local Pitman-type alternatives, it follows from Theorem 1.1 (ii) that $\phi_n \xrightarrow{\mathcal{D}} aV_1 + bV_2$, where $a = 32\pi^4\lambda$, $b = 32\pi^4\lambda_2$, V_1 and V_2 are independent chi-square random variables each having one degree of freedom, and λ_1 and λ_2 are eigenvalues of the matrix $\Sigma^1 = (\sigma_{ij}^1)$. Note that a and b are functions of c and d, and $P_{c,d}(x) = P\{aV_1 + bV_2 \leq x \mid c, d\}$ is used to compute the asymptotic power of ϕ_n. Numerical values of $P_{c,d}(x)$ are given in Table 1 for significance levels of $\delta = .025$ and $.05$, and several values of $c = d < 0$. Observe that the asymptotic power of ϕ_n increases as c decreases, and eventually becomes one.

Given the significance level $0 < \delta < 1$, a level δ spatial domain test ψ_n (see (2.15)) is determined from the asymptotic result $\psi_n \xrightarrow{\mathcal{D}} N(\underline{0}, \Gamma)$, where $\Gamma = \operatorname{diag}(2, 2)$ (under the null hypothesis), as shown in Theorem 1.2 (i). Moreover, Theorem 1.2 (ii) implies that the asymptotic power of ψ_n is one for each sequence $\{(e^{c/n}, e^{d/n})\}$ of local Pitman-type alternatives. Hence ψ_n is superior to ϕ_n based on the criterion of comparing the asymptotic power at the above sequence of local Pitman alternatives.

Finally, it should be mentioned that the asymptotic power of ϕ_n is one at each fixed alternative (α, β). Thus one needs a sequence of local Pitman-type alternatives in order to be able to distinguish between ϕ_n and ψ_n when relying on asymptotic results. It is not surprising that the spatial domain test has asymptotic power one at each sequence $\{(e^{c/n}, e^{c/n})\}$ of local Pitman-type alternatives, whereas one needs $c < 0$ sufficiently small for this result to be valid when using the periodogram test. The reason being that ψ_n is a parametric test, and hence should perform better under the assumptions of a parametric model. However, the advantage of the periodogram analysis is that the

Table 1. Asymptotic power: $P_{c,d}(x) = P\{aV_1 + bV_2 \leq x \mid c, d\}$.

c	d	a	b	$P_{cd}(.152)$	$P_{cd}(.309)$
−.001	−.001	2.9980014	2.9980007	.025	.050
−.01	−.01	2.9801341	2.9800709	.025	.051
−.1	−.1	2.812499	2.8071983	.027	.054
−.5	−.5	2.2408056	2.1778355	.034	.068
−1	−1	1.7549978	1.6460808	.044	.087
−5	−5	.3624993	.3226461	.199	.363
−10	−10	.0731931	.0647334	.668	.894
−15	−15	.0217135	.0186837	.977	.998
−20	−20	.0085121	.0070005	1.000	1.000

periodogram can always be computed (even without a model) and is thought to be less sensitive to model assumptions, and hence has wider applicability. Moreover, while the periodogram provides a means of detecting the seasonal periodicity, a consistent set of large periodogram ordinates and corresponding chi-square random variables (each having two degrees of freedom) can be used as a diagnostic tool to detect nonstationarity, and also long term dependence of the underlying stochastic process.

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