Ann. Inst. Statist. Math.
Vol. 52, No. 1, 15–27 (2000)

UNIFORMLY MORE POWERFUL, TWO-SIDED TESTS FOR HYPOTHESES ABOUT LINEAR INEQUALITIES

HUIMEI LIU

Department of Statistics, National Chung Hsing University, 67 Ming Sheng East Rd., Sec. 3, Taipei 10433, Taiwan, R.O.C.

(Received July 15, 1996; revised September 1, 1998)

Abstract. Let $\boldsymbol{X}$ have a multivariate, p-dimensional normal distribution ($p \geq 2$) with unknown mean $\boldsymbol{\mu}$ and known, nonsingular covariance $\boldsymbol{\Sigma}$. Consider testing H_0 : $\boldsymbol{b}_i'\boldsymbol{\mu} \leq 0$, for some $i = 1, \dots, k$, and $\boldsymbol{b}_i'\boldsymbol{\mu} \geq 0$, for some $i = 1, \dots, k$, versus $H_1 : \boldsymbol{b}_i'\boldsymbol{\mu} > 0$, for all $i = 1, \dots, k$, or $\boldsymbol{b}_i'\boldsymbol{\mu} < 0$, for all $i = 1, \dots, k$, where $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$, $k \geq 2$, are known vectors that define the hypotheses and suppose that for each $i = 1, \dots, k$ there is an $j \in \{1, \dots, k\}$ (j will depend on i) such that $\boldsymbol{b}_i'\boldsymbol{\Sigma}\boldsymbol{b}_j \leq 0$. For any $0 < \alpha < 1/2$. We construct a test that has the same size as the likelihood ratio test (LRT) and is uniformly more powerful than the LRT. The proposed test is an intersection-union test. We apply the result to compare linear regression functions.

Key words and phrases: Intersection-union test, likelihood ratio test, linear inequalities hypotheses, uniformly more powerful test.

1. Introduction

Let $\boldsymbol{X} = (X_1, \dots, X_p)'$ ($p \geq 2$) be a p-variate normal random variable with mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)'$ and known, nonsingular covariance matrix $\boldsymbol{\Sigma}$. We consider the problem of testing

$$H_0 : \boldsymbol{b}_i'\boldsymbol{\mu} \leq 0 \quad \text{for some } i = 1, \dots, k \qquad \text{and} \qquad \boldsymbol{b}_i'\boldsymbol{\mu} \geq 0 \quad \text{for some } i = 1, \dots, k$$

(1.1) versus

$$H_1 : \boldsymbol{b}_i'\mu > 0 \quad \text{for all } i = 1, \dots, k \qquad \text{or} \qquad \boldsymbol{b}_i'\boldsymbol{\mu} < 0 \quad \text{for all } i = 1, \dots, k.$$

Here $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$ ($k \geq 2$) are specified p-dimensional vectors that define the hypotheses. We assume H_1 is nonempty so the testing problem is meaningful. (We use the symbol H_1 to denote the set of $\boldsymbol{\mu}$ vectors specified by the hypothesis, as well as the statement of the hypothesis.) We also assume that the set $\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_k\}$ has no redundant vectors in it. That is, thete is no $\boldsymbol{b}_j$ such that $\{\boldsymbol{\mu} : \boldsymbol{b}_i'\boldsymbol{\mu} > 0, i = 1, \dots, k\} = \{\boldsymbol{\mu} : \boldsymbol{b}_i'\boldsymbol{\mu} > 0, i = 1, \dots, k, i \neq j\}$. Sasabuchi (1980) discusses conditions that are equivalent to our two assumptions.

In this paper, for any testing problem of the form (1.1), we propose a new test that has the same size as the size-α likelihood ratio test (LRT) and is uniformly more powerful than the LRT. First we consider hypotheses that have only two linear restrictions ($k = 2$). A new test, ϕ_d, is proposed for the case $\boldsymbol{b}_1'\boldsymbol{\Sigma}\boldsymbol{b}_2 \leq 0$. In the case, the rejection region of the new test is like Liu and Berger's (1995) in that it contains the rejection region of the LRT and an additional set, but the size of the new test is still α. So the new test is uniformly more powerful than the LRT. Berger (1989) proposed a more powerful test for the $\boldsymbol{b}_1'\boldsymbol{\Sigma}\boldsymbol{b}_2 \leq 0$ case. The test, ϕ_d, is different than Berger's test, and, in some cases, appears to be more powerful. Then, recognizing that for $k > 2$, H_1 can be written as the intersection of two sets each defined by two inequalities, the intersection-union method is used to obtain a test, ϕ_g, that is uniformly more powerful for the general problem (1.1). One of differences between the work presented here and the work of Liu and Berger (1995) is that we use various approach to show that the created two-sided test is a size-α test. Since we consider two-sided testing problem in this paper, the Theorem 2.1 of Berger (1989) can not be applied. The other difference is the set D, defined in Section 3, that makes the intersect-union method work.

Sasabuchi (1980) considered two-sided testing problem where both null and alternative hypotheses are determined by k linear inequalities. His problem is to test

$$\begin{aligned} H_{0S}: &\ \boldsymbol{b}_i'\boldsymbol{\mu} \geq 0 \quad \text{for all } i = 1, \ldots, k \text{ where equality holds} \\ &\ \text{for at least one value of } i \quad \text{and} \\ &\ \boldsymbol{b}_i'\boldsymbol{\mu} \leq 0 \quad \text{for all } i = 1, \ldots, k \text{ where equality holds} \\ &\ \text{for at least one value of } i. \end{aligned}$$

(1.2) versus

$$H_{1S}: \boldsymbol{b}_i'\boldsymbol{\mu} > 0 \quad \text{for all } i = 1, 2, \ldots, k \quad \text{or} \quad \boldsymbol{b}_i'\boldsymbol{\mu} < 0 \quad \text{for all } i = 1, 2, \ldots, k.$$

Sasabuchi (1980) showed that the size-α likelihood ratio test (LRT) of problem (1.2) for acute cones cases is the test that rejects H_{0S} if

$$\begin{aligned} Z_i = \frac{\boldsymbol{b}_i'\boldsymbol{X}}{(\boldsymbol{b}_i'\boldsymbol{\Sigma}\boldsymbol{b}_i)^{1/2}} \geq z_\alpha, &\quad \text{for all } i = 1, \ldots, k, \quad \text{or} \\ Z_i \leq -z_\alpha, \quad \text{for all } i = 1, \ldots, k, & \end{aligned}$$

where z_α is the upper 100α percentile of the standard normal distribution. Berger (1989) shows that, althouth $H_{0S} \subset H_0$ and H_0 is a much bigger set than H_{0S}, the size-α LRT in problem (1.1) for acute cones case is the same as Sasabuchi's (1980). The LRT has some optimal properties which are admissible (Cohen *et al.* (1983); Iwasa (1991)) and uniformly most powerful test among all monotone size-α tests (Lehmann (1952); Cohen *et al.* (1983)). But the LRT is a biased test (Lehmann (1952); Berger (1989); Liu and Berger (1995)). Iwasa (1991) also points out the LRT is d-admissible but not α-admissible in one-sided bivariate problem. The α-admissibility would guarantee the nonexistence of a uniformly more powerful test of size α, but the d-admissibility does not. So it is possible that we can find a nonmonotone test which is uniformly more powerful than the LRT, and several researchers have developed such tests. However, they most worked on one-sided testing problems.

Numerous researchers have found the uniformly more powerful tests for one-sided testing problems under various conditions (Gutmann (1987); Nomakuchi and Sakata (1987); Berger (1989); Iwasa (1991); Shirley (1992); Li and Sinha (1995); Liu and Berger (1995)). In the last one of these references gives a very general uniformly more powerful test for one-sided hypotheses in normal random vector. In their paper, they show LRT is uniformly dominated. In the same paper, Liu and Berger (1995) also give a restriction on the construction of a size-α test in one-sided testing problem.

To simplify computation, we consider the transformed version of the original problem that is similar to the one used by Sasabuchi (1980), Berger (1989) and Liu and Berger (1995). Let $\boldsymbol{\Gamma}$ be a $p \times p$ nonsingular matrix such that $\boldsymbol{\Gamma\Sigma\Gamma}' = \boldsymbol{I}_p$, the $p \times p$ identity matrix. So $\boldsymbol{\Gamma}^{-1}(\boldsymbol{\Gamma}^{-1})' = \boldsymbol{\Sigma}$. Make the transformation $\boldsymbol{Y} = \boldsymbol{\Gamma X}$. Then $\boldsymbol{Y} \sim N_p(\boldsymbol{\theta}, \boldsymbol{I}_p)$, where $\boldsymbol{\theta} = \boldsymbol{\Gamma\mu}$. Let $\|\boldsymbol{a}\| = (\boldsymbol{a}'\boldsymbol{a})^{1/2}$ denote the norm of a vector. Define $\boldsymbol{h}_i = \boldsymbol{b}_i'\boldsymbol{\Gamma}^{-1}/\|\boldsymbol{b}_i'\boldsymbol{\Gamma}^{-1}\|$.

Then $b_i'\mu = h_i'\theta \| b_i'\Gamma^{-1}\|$. Therefore, problem (1.1) is equivalent to observing Y and testing

$$H_0 : h_i'\theta \leq 0 \quad \text{for some } i = 1, \ldots, k \qquad \text{and} \qquad h_i'\theta \geq 0 \quad \text{for some } i = 1, \ldots, k.$$

(1.3) versus

$$H_1 : h_i'\theta > 0 \quad \text{for all } i = 1, \ldots, k \qquad \text{or} \qquad h_i'\theta < 0 \quad \text{for all } i = 1, \ldots, k.$$

We will use Y, h_i and θ through the rest of the paper. In terms of these variables, the size-α LRT of (1.1) or (1.3) is the test that rejects H_0 if $h_i'Y \geq z_\alpha$, for all $i = 1, \ldots, k$ or $h_i'Y \leq -z_\alpha$, for all $i = 1, \ldots, k$.

In Section 2 we propose a new test, ϕ_d, for the case $k = 2$ and $b_1'\Sigma b_2 \leq 0$. We compare the power of ϕ_d, Berger's test and the LRT in an example. In Section 3, we construct a uniformly more powerful, intersection-union test based on ϕ_d, for problem (1.1) when $k > 2$. Section 4, we consider an example which is concerned about testing whether linear regression functions are ordered.

2. Uniformly more powerful test

In this section, we will consider the testing problem (1.3) when $k = 2$ and $b_1'\Sigma b_2 = h_1'h_2\|b_1'\Gamma^{-1}\|\|b_2'\Gamma^{-1}\| \leq 0$, i.e., $h_1'h_2 \leq 0$. Let τ be the angle between the vectors h_1 and h_2. Since $\cos(\tau) = h_1'h_2 \leq 0$, τ is obtuse. But the angle in the cone $\mathcal{J} = \{\theta : h_1'\theta \leq 0, h_2'\theta \leq 0\}$ is $\xi = \pi - \tau$, which is acute. So we say H_1 are two acute cones when $h_1'h_2 \leq 0$. In this section we will describe a new test that is uniformly more powerful than the LRT when the alternative hypothesis is an acute cone. We start by defining the test, ϕ_d. Then we show that ϕ_d is a size-α test and is uniformly more powerful than the LRT.

Before describing the test, ϕ_d, we will define the functions and set which will be used to construct the rejection region for the test ϕ_d.

DEFINITION 2.1. For $-\infty < s \leq 0$, let L_s be the two dimensional set defined by

$$L_s = \left\{(u, v) : \frac{u + sv}{\sqrt{1 + s^2}} \geq z_\alpha, v \geq z_\alpha \text{ or } \frac{u + sv}{\sqrt{1 + s^2}} \leq -z_\alpha, v \leq -z_\alpha\right\}.$$

Let $c_s = (\sqrt{1 + s^2} - s)z_\alpha$.

L_s are two acute cones if $s < 0$. The vertices of two cones are (c_s, z_α) and $(-c_s, -z_\alpha)$. We will eventually express the LRT in terms of L_s. Throughout the rest of the paper, $\varphi(v)$ and $\Phi(v)$ denote the standard normal pdf and cdf, respectively.

DEFINITION 2.2. For any u, $-\infty < u < \infty$, define

$$P_s(u) = \alpha - \int_{L_s(u)} \varphi(v)dv$$

where $L_s(u) = \{v : (u, v) \in L_s\}$. Specifically, for $s = 0$,

$$P_s(u) = \begin{cases} \alpha, & -c_s < u < c_s = z_\alpha, \\ 0, & u \geq c_s = z_\alpha \text{ or } u \leq -c_s. \end{cases}$$

For $s < 0$,

$$P_s(u) = \begin{cases} \Phi\left(\dfrac{-\sqrt{1+s^2}z_\alpha - u}{s}\right), & u \le -c_s, \\ \alpha, & -c_s < u < c_s, \\ 1 - \Phi\left(\dfrac{\sqrt{1+s^2}z_\alpha - u}{s}\right), & u \ge c_s. \end{cases}$$

The specific formulas for $P_s(u)$, are easily verified by using the definition of L_s. $0 \le P_s(u) \le \alpha$ for all u. $P_0(u)$ is the limit of $P_s(u)$ as $s \to 0$. And, if $(U,V) \sim N_2((\mu,0),I_2)$, $P((U,V) \in L_s) = \int(\alpha - P_s(u))\varphi(u-\mu)du$. The line between (c_s, z_α) and $(-c_s, -z_\alpha)$, the vertices of L_s, has the equation $v = z_\alpha u/c_s = (\sqrt{1+s^2}+s)u$. We now define a set that contains this line, for $s \le 0$.

DEFINITION 2.3. For $s \le 0$ and $0 < d < 1$, let A_s be the set defined by

$$A_s = \{(u,v) : l_2^d(u) \le v \le l_1^d(u)\},$$

where

$$l_1^d(u) = \begin{cases} \Phi^{-1}\{\alpha + d \times P_s(u)\}, & -c_s \ge u \\ \Phi^{-1}\{\Phi((\sqrt{1+s^2}+s)u) + d \times P_s(u)\}, & -c_s < u < c_s, \\ \Phi^{-1}\{1 - (1-d) \times P_s(u)\}, & u \ge c_s \end{cases}$$

and

$$l_2^d(u) = \Phi^{-1}(\Phi(l_1^d(u)) - \alpha).$$

The following lemma is the key fact that will ensure that the probability of a type I error on the boundary points of H_0 is less than or equal to α for ϕ_d.

LEMMA 2.1. *Let $(U,V) \sim N_2((\mu,0),I_2)$. Let $s \le 0$ and $0 < \alpha < 1/2$. Then $P_{(\mu,0)}((U,V) \in A_s) \le \alpha$.*

PROOF.

$$\begin{aligned} &P_{(\mu,0)}((U,V) \in A_s) \\ &= \int_{-\infty}^{+\infty}\left(\int_{A_s(u)} \varphi(v)dv\right)\varphi(u-\mu)du \end{aligned} \tag{2.1}$$

where $A_s(u)$ is defined as following

$$A_s(u) = \{v : (u,v) \in A_s\} = \{v : l_2^d(u) \le v \le l_1^d(u)\}.$$

The expression in parentheses in (2.1) is clearly bounded above by α for all $u \in (-\infty,\infty)$. Since

$$\begin{aligned} \int_{A_s(u)} \varphi(v)dv &= \Phi(l_1^d(u)) - \Phi(l_2^d(u)) \\ &\le \Phi(l_1^d(u)) - \Phi(\Phi^{-1}(\Phi(l_1^d(u)) - \alpha)) = \alpha. \end{aligned}$$

So the expression in parentheses is bounded above by α, and, hence, $P_{(\mu,0)}((U,V) \in A_s) \leq \alpha$. □

Our new tests will be defined in terms of variables U_1, V_1, U_2 and V_2, that we now define.

DEFINITION 2.4. Let $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ be noncolinear vectors ($|\boldsymbol{h}_1'\boldsymbol{h}_2| < \|\boldsymbol{h}_1\|\|\boldsymbol{h}_2\| = 1 \times 1 = 1$). Let $\boldsymbol{g}_1 = \boldsymbol{h}_2 - (\boldsymbol{h}_1'\boldsymbol{h}_2)\boldsymbol{h}_1$ and $\boldsymbol{g}_2 = \boldsymbol{h}_1 - (\boldsymbol{h}_1'\boldsymbol{h}_2)\boldsymbol{h}_2$. ($\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are vectors spanned by $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ that are orthogonal to $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$; $\boldsymbol{g}_1'\boldsymbol{h}_1 = 0$, $\boldsymbol{g}_2'\boldsymbol{h}_2 = 0$.) Define $\boldsymbol{h}_i'\boldsymbol{y} = v_i$ and $\boldsymbol{g}_i'\boldsymbol{y}/\|\boldsymbol{g}_i\| = u_i$, $i = 1, 2$. Also define the corresponding random vectors $\boldsymbol{h}_i'\boldsymbol{Y} = V_i$ and $\boldsymbol{g}_i'\boldsymbol{Y}/\|\boldsymbol{g}_i\| = U_i$.

Note. $\|\boldsymbol{g}_1\| = \|\boldsymbol{g}_2\| = \sqrt{1-(\boldsymbol{h}_1'\boldsymbol{h}_2)^2}$. Since $\boldsymbol{g}_i'\boldsymbol{h}_i = 0$, we know that U_i and V_i are independent.

Now we define the test ϕ_d. In fact, we define a whole family of tests, indexed by the constant d, $0 < d < 1$, that appears in Definition 2.3.

DEFINITION 2.5. Consider the testing problem (1.3) for vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ that satisfy $\boldsymbol{h}_1'\boldsymbol{h}_2 \leq 0$. Fix d, $0 < d < 1$. Let $s = \boldsymbol{h}_1'\boldsymbol{h}_2(1-(\boldsymbol{h}_1'\boldsymbol{h}_2)^2)^{-1/2}$. For any α that satisfies $0 < \alpha < 1/2$, define ϕ_d as the test that rejects H_0 if $\boldsymbol{Y} \in S_1 \cap S_2$ where $S_1 = \{\boldsymbol{y} : (u_1, v_1) \in A_s\}$, $S_2 = \{\boldsymbol{y} : (u_2, v_2) \in A_s\}$.

Note. In fact, S_1 is defined by considering the subspace associated to a constraint as a coordinate axis and S_2 by considering the subspace associated to the another constraint as coordinate axis. From the definition, we know that $S_1 \cap S_2$ is symmetric to line $v_1 = z_\alpha u_1/c_s = (\sqrt{1+s^2}+s)u_1$ according to U_1-V_1 coordinate system ($v_2 = z_\alpha u_2/c_s = (\sqrt{1+s^2}+s)u_2$ according to U_2-V_2 coordinate system).

Examples of the sets L_s and A_s and lines $l_1^d(u)$ and $l_2^d(u)$ are shown in Fig. 1. In this figure, $s = -2$, $d = 1/2$ and $\alpha = 0.1$. The solid lines are $l_1^d(u)$ and $l_2^d(u)$. The line $l_1^d(u)$ lies above the upper boundary of L_s, which is given by the line $v = (\sqrt{1+s^2}z_\alpha - u)/s$ for $u \geq c_s$ and $v = -z_\alpha$ for $u \leq -c_s$. This is true since for $u \geq c_s$, $\Phi(l_1^d(u)) = 1-(1-d)P_s(u) = \Phi((\sqrt{1+s^2}z_\alpha - u)/s) + dP_s(u) > \Phi((\sqrt{1+s^2}z_\alpha - u)/s)$ and $l_1^d(u) = \Phi^{-1}\{\alpha + d \times P_s(u)\} > \Phi^{-1}(\alpha) = -z_\alpha$ for $u \leq -c_s$. And, for $u \geq c_s$, $l_2^d(u)$ is below the lower boundary of L_s because the lower boundary is $z_\alpha = \Phi^{-1}(1-\alpha) > l_2^d(u)$ for $u \geq c_s$ and $\Phi^{-1}(P_s(u)) > \Phi^{-1}(d \times P_s(u)) = l_2^d(u)$ for $u \leq -c_s$. Therefore, $L_s \subset A_s$.

The following lemma will show that the rejection region for the LRT is a subset of that for ϕ_d.

LEMMA 2.2. *Consider the testing problem* (1.3) *when* $k = 2$ *and* $0 < \alpha < 1/2$. *The rejection region for the size-*α *LRT is* $R_L = \{\boldsymbol{y} : \boldsymbol{h}_1'\boldsymbol{y} \geq z_\alpha \text{ and } \boldsymbol{h}_2'\boldsymbol{y} \geq z_\alpha\} \cup \{\boldsymbol{y} : \boldsymbol{h}_1'\boldsymbol{y} \leq -z_\alpha \text{ and } \boldsymbol{y}_2'\boldsymbol{y} \leq -z_\alpha\}$. *Let* $L_s^i = \{\boldsymbol{y} : (u_i, v_i) \in L_s\} \subset S_i$, $i = 1, 2$, *where* $s = \boldsymbol{h}_1'\boldsymbol{h}_2(1-(\boldsymbol{h}_1'\boldsymbol{h}_2)^2)^{-1/2}$. *Then* $L_s^i = R_L$ *for* $i = 1, 2$. *Hence, the rejection region for* ϕ_d, *namely* $S_1 \cap S_2$, *contains* R_L.

PROOF. For $i = 1$, $u_1 = \frac{\boldsymbol{g}_1'\boldsymbol{y}}{\|\boldsymbol{g}_1\|} = \frac{\boldsymbol{h}_2'\boldsymbol{y} - (\boldsymbol{h}_1'\boldsymbol{h}_2)\boldsymbol{h}_1'\boldsymbol{y}}{\sqrt{1-(\boldsymbol{h}_1'\boldsymbol{h}_2)^2}}$ and $v_1 = \boldsymbol{h}_1'\boldsymbol{y}$. Since $\frac{s}{\sqrt{1+s^2}} = \boldsymbol{h}_1'\boldsymbol{h}_2$, then $\|\boldsymbol{g}_1\| = \sqrt{1-(\boldsymbol{h}_1'\boldsymbol{h}_2)^2} = \frac{1}{\sqrt{1+s^2}}$. Hence,

$$\frac{u_1 + sv_1}{\sqrt{1+s^2}} = \boldsymbol{a}_1'\boldsymbol{y},$$

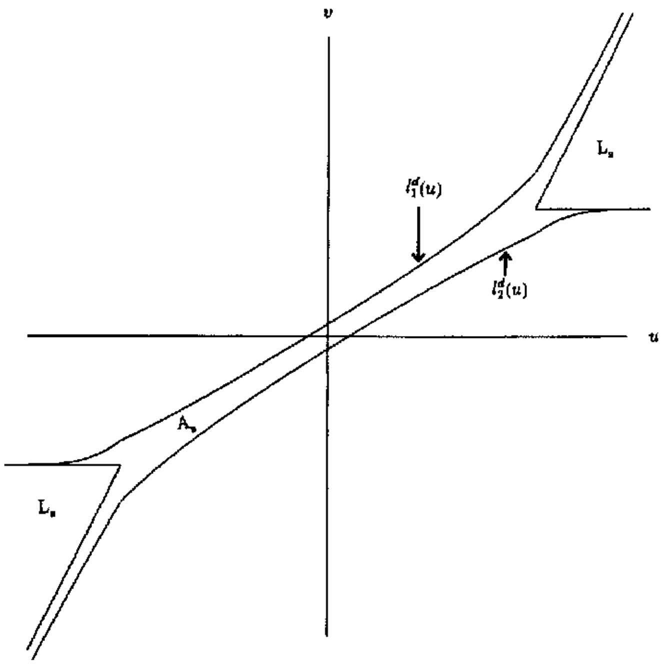

Fig. 1. The set A_s, L_s and functions $l_1^d(u)$ and $l_2^d(u)$.

where

$$a_1' = \left\{ \frac{h_2' - (h_1'h_2)h_1'}{\|g_1\|} \|g_1\| + (h_1'h_2)h_1' \right\} = h_2'.$$

Therefore, $L_s^1 = R_L$. For $i = 2$, similar algebra yields $\frac{u_2+sv_2}{\sqrt{1+s^2}} = h_1'y$ and $v_2 = h_2'y$. So $L_s^2 = R_L$, also. □

Another way to state Lemma 2.2 is to say that the three events, $\{Y \in R_L\}$, $\{(U_1, V_1) \in L_s\}$ and $\{(U_2, V_2) \in L_s\}$, are all the same event.

The following theorem is going to show that ϕ_d is a size-α test and uniformly more powerful than the LRT. The proof was suggested by a referee.

THEOREM 2.1. *For the testing problem* (1.3) *when* $k = 2$, *suppose that* $h_1'h_2 \leq 0$. *If* $0 < \alpha < 1/2$, *then* ϕ_d *has size exactly* α, *and* ϕ_d *is uniformly more powerful than the size-*α *LRT.*

PROOF. We can always consider, after making an orthogonal transformation, a situation in which h_1 and h_2 become in the form $(0, 1, 0, 0, \ldots)'$ and $(a_1, a_2, 0, 0, \ldots)'$ respectively, with $a_2 < 0$. The whole problem can be stated in terms of the only two first coordinates.

From Lemma 2.2 we know the rejection region of the size-α LRT, R_L is a subset of the rejection region of ϕ_d. Hence, ϕ_d is uniformly more powerful than the size-α LRT. Also,

(2.2) $$\text{the size of } \phi_d \geq \text{size of LRT} = \alpha.$$

From Lemma 2.1 we have got a set A_s, $s < 0$, which has a probability at most α as the mean belongs to a 'border subspace'. Consider now the line l_0 $(v = (\sqrt{1+s^2} + s)u)$

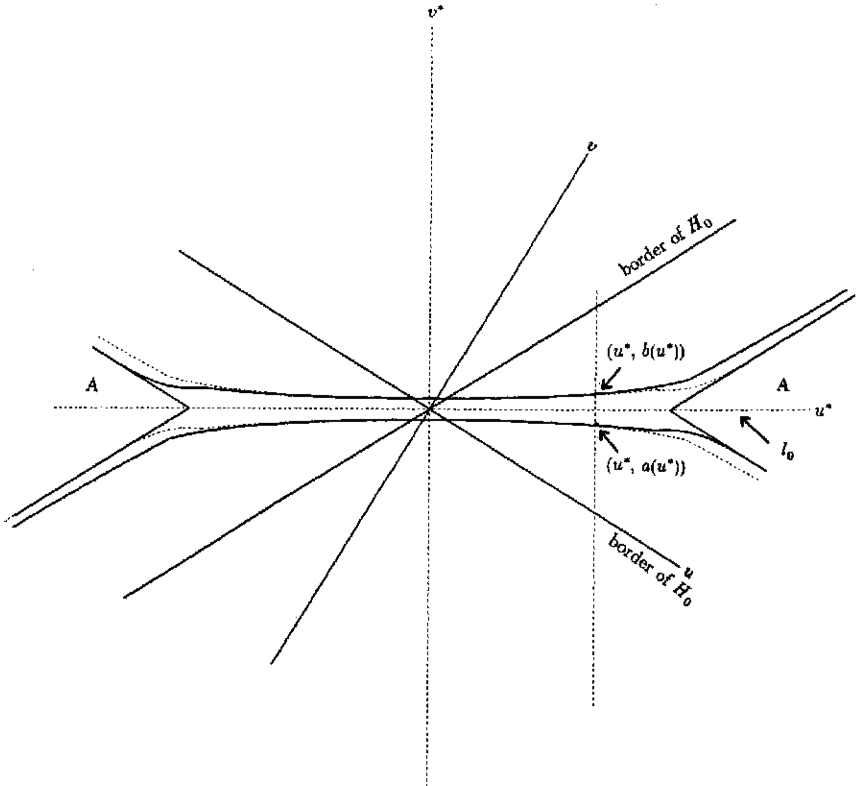

Fig. 2. $a(u^*)$ and $b(u^*)$.

and the set A_s^* symmetric to A_s respect to the line l_0. By symmetry, from Lemma 2.1 we also have a probability at most α for A_s^* as $\boldsymbol{\theta}$ belongs to the another 'border subspace'. Then the set A obtained as the intersection of A_s and A_s^*, satisfies $P_{\boldsymbol{\theta}}(A) \leq \alpha$ for every $\boldsymbol{\theta}$ in the border of H_0. Also A is symmetric respect to the line l_0.

Now, through a rotation that makes l_0 to become a coordinate axis, U^* axis (the corresponding orthogonal axis is V^* axis). Consider $\boldsymbol{\theta}$ ($\boldsymbol{\theta} = (\theta_1, \theta_2)') \in H_0$, and $\boldsymbol{\theta}^* = (\theta_1, \theta_2^*)'$ on the border of H_0. Let $(u^*, b(u^*))$ and $(u^*, a(u^*))$ be the boundary points of set A such that $b(u^*) > a(u^*)$ (see Fig. 2). Actually $a(u^*) = -b(u^*)$, since the set A is symmetric respect to the U^* axis. Then

$$(2.3) \qquad \begin{aligned} P_{\boldsymbol{\theta}} &= \int_{-\infty}^{\infty} \varphi(u^* - \theta_1) \int_{a(u^*)}^{b(u^*)} \varphi(v^* - \theta_2) dv^* du^* \\ &\leq \int_{-\infty}^{\infty} \varphi(u^* - \theta_1) \int_{a(u^*)}^{b(u^*)} \varphi(v^* - \theta_2^*) dv^* du^* \\ &= P_{\boldsymbol{\theta}^*}(A) \leq \alpha. \end{aligned}$$

Since, (2.3) is true for any $\boldsymbol{\theta} \in H_0$, the size of $\phi_d \leq \alpha$. With (2.2) this implies ϕ_d has size exactly α. □

Consider the testing problem with $\boldsymbol{h}_1' = (0, 1)$ and $\boldsymbol{h}_2' = (1/\sqrt{5}, -2/\sqrt{5})$, so that $(y_1, y_2) = (u_1, v_1)$ and $s = -2$. Let $d = 1/2$ and $\alpha = 0.1$. Then in Fig. 3, the solid lines

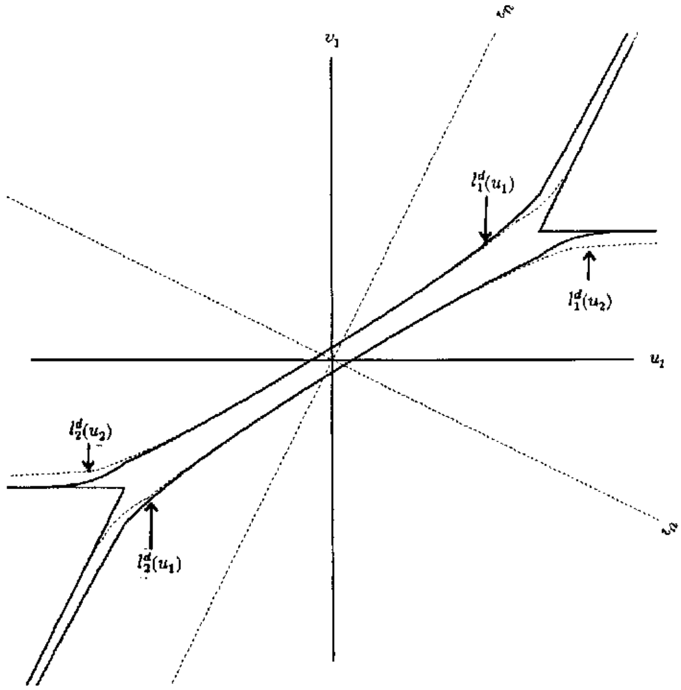

Fig. 3. Rejection region of ϕ_d when $s = -1/2$, $d = 1/2$ and $\alpha = 0.1$.

are $l_1^d(u_1)$ and $l_2^d(u_1)$ which are corresponding to $U_1 - V_1$ axes and the region between them is S_1. The dotted lines are $l_1^d(u_2)$ (lower line) and $l_2^d(u_2)$ (upper line) which are corresponding to $U_2 - V_2$ axes and the region between them is S_2. The rejection region is $S_1 \cap S_2$, and it contains $L_s = R_L$, the LRT's rejection region. In Fig. 4, the rejection region for Berger's (1989) test, ϕ_b, for the problem is shown. The union of the diamond shaped regions, $L_s \cup R_2 \cup \cdots \cup R_9$, is the rejection region for ϕ_b. Note that the rejection region for ϕ_b is almost completely contained in the rejection region for ϕ_d. In fact, ϕ_d may be uniformly more powerful than ϕ_b. In general, as s decreases, the containment of ϕ_b in ϕ_d comes closer and closer to reality.

Example 2.1. Suppose Y_1 and Y_2 are independent and $Y_i \sim N_1(\theta_i, 1)$. Consider $\boldsymbol{h}_1' = [0, 1]$, $\boldsymbol{h}_2' = [1/\sqrt{1+s^2}, s/\sqrt{1+s^2}]$, $s < 0$, so that we are testing $H_0 : \theta_2 \leq 0$ or $\theta_1 + s\theta_2 \leq 0$ and $\theta_2 \geq 0$ or $\theta_1 + s\theta_2 \geq 0$ against $H_1 : \theta_2 > 0$ and $\theta_1 + s\theta_2 > 0$ or $\theta_2 < 0$ and $\theta_1 + s\theta_2 < 0$. Here we selected $\alpha = 0.1$, $s = -2.0$ and $d = 1/2$ (as in Fig. 2) to compute the power of ϕ_d and LRT. Let $\beta_L(\boldsymbol{\theta})$ and $\beta_{\phi_d}(\boldsymbol{\theta})$ be the power functions of the LRT and ϕ_d, respectively. Values of these two functions for certain $\boldsymbol{\theta}$ values are in Table 1. These values are calculated by two steps. First, we calculate the cross-sectional probability $\int_{A(u)} \varphi(v - \theta_2)dv = f(u, \theta_2)$ which is a function of u and θ_2. Second, we calculate $\int_{-\infty}^{+\infty} f(u, \theta_2)\varphi(u - \theta_1)du$ using the trapezoidal rule with 300 points. The first part of the tables are for values of $\boldsymbol{\theta}' = (\theta, 0)$, $\theta \geq 0$. These values are on the boundary of H_0, so the powers are less than $\alpha = 0.1$. If a test is unbiased, then the power is equal to α for the values of $\boldsymbol{\theta}$ which are on the boundary of H_0. Here we can see that the LRT and ϕ_d are biased, but the difference between α and the power of ϕ_d is considerably smaller than that between α and the power of the LRT. The second part of the table is for values of $\boldsymbol{\theta}' = ((\sqrt{1+s^2} - s)\theta, \theta)$ which are on the line from the origin to the vertex (c_s, z_α). For

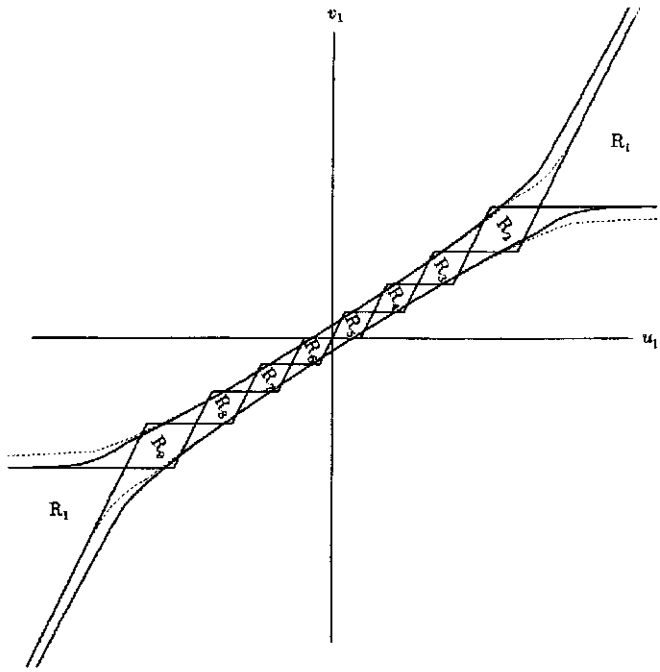

Fig. 4. Rejection region of ϕ_d and ϕ_b when $s = -2$, $d = 1/2$ and $\alpha = 0.1$.

Table 1. Power of LRT, ϕ_b and ϕ_d for $s = -2.0$, $d = 1/2$ and $\alpha = 0.1$.

	θ								
	0	1	2	3	4	5	6	7	8
$\beta_L(\theta,0)$	0.000	0.000	0.000	0.000	0.002	0.014	0.040	0.071	0.089
$\beta_b(\theta,0)$	0.053	0.053	0.053	0.053	0.053	0.050	0.054	0.073	0.090
$\beta_{\phi_d}(\theta,0)$	0.078	0.089	0.091	0.095	0.092	0.090	0.090	0.092	0.096
$\beta_L(4.236\theta,\theta)$	0.000	0.010	0.528	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_b(4.236\theta,\theta)$	0.053	0.177	0.528	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_{\phi_d}(4.236\theta,\theta)$	0.078	0.253	0.636	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_L(2.118\theta,\theta)$	0.000	0.000	0.135	0.481	0.691	0.830	0.920	0.968	0.989
$\beta_b(2.118\theta,\theta)$	0.053	0.115	0.151	0.481	0.691	0.830	0.920	0.968	0.989
$\beta_{\phi_d}(2.118\theta,\theta)$	0.078	0.219	0.422	0.528	0.691	0.830	0.920	0.968	0.989

example, $\beta_{\phi_d}(4.236,1)/\beta_L(4.236,1) \approx 25.3$ and $\beta_{\phi_d}(4.236,1) > \alpha > \beta_L(4.236,1)$. $\beta_{\phi_d}(\boldsymbol{\theta})$ is clearly bigger than $\beta_L(\boldsymbol{\theta})$ for $\theta \leq 3$. The largest difference is 0.108. The bottom of the table is for values of $\boldsymbol{\theta}' = (0.5(\sqrt{1+s^2}-s)\theta, \theta)$. $\beta_{\phi_d}(\boldsymbol{\theta})$ is clearly larger than $\beta_L(\boldsymbol{\theta})$ for $\theta \leq 3$. As s increases, there is less space to add to the rejection region of the LRT. So we can not improve the power as much when s is large.

3. A more powerful test in the general problem

We will now describe a size-α test that is uniformly more powerful than the size-α LRT for the general problem (1.3) and $0 < \alpha < 1/2$. We will denote this test by ϕ_g. The intersection-union method will be used to construct ϕ_g. A summary of this method may be found in Sections 8.2.4 and 8.3.5 of Casella and Berger (1990) or in Berger (1982).

To use the intersection-union method, $H_1 : \boldsymbol{h}_i'\boldsymbol{\theta} > 0$ for all $i = 1, \ldots, k$, or $\boldsymbol{h}_i'\boldsymbol{\theta} < 0$ for all $i = 1, 2, \ldots, k$, must be written as a intersection. Let $D = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n)\}$ for a rearrangement $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ which satisfies $\boldsymbol{h}_{i_j}' \boldsymbol{h}_{i_{j+1}} \leq 0$ for all $j = 1, 2, \ldots, n-1$. To construct a more powerful test, any such division of $\{1, \ldots, k\}$ will work, but different divisions will lead to different tests.

For each $(i, j) \in D$, consider testing $H_{0ij} : (\boldsymbol{h}_i'\boldsymbol{\theta} \leq 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} \geq 0)$ or $(\boldsymbol{h}_i'\boldsymbol{\theta} \geq 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} \leq 0)$ versus $H_{1ij} : (\boldsymbol{h}_i'\boldsymbol{\theta} > 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} > 0)$ or $(\boldsymbol{h}_i'\boldsymbol{\theta} < 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} < 0)$. If $\boldsymbol{h}_i'\boldsymbol{h}_j \leq 0$, let C_{ij} denote the size-α rejection region of ϕ_d (for some d) from Section 2. Since $H_1 = \bigcap_{(i,j)\in D} H_{1ij}$, we can define an intersection-union test based on the C_{ij}.

DEFINITION 3.1. For the testing problem (1.3) and $0 < \alpha < 1/2$, let ϕ_g be the test that rejects H_0 if $\boldsymbol{Y} \in \bigcap_{(i,j)\in D} C_{ij}$.

THEOREM 3.1. *For $0 < \alpha < 1/2$, the test ϕ_g is a size-α test of H_0 versus H_1, and ϕ_g is uniformly more powerful than the size-α LRT.*

PROOF. Since each of C_{ij} is a size-α rejection region for testing H_{0ij}, by Theorem 1 in Berger (1982), ϕ_g has size $\leq \alpha$. But, the size-α LRT's rejection region is

$$R_L = \{\boldsymbol{y} : \boldsymbol{h}_i'\boldsymbol{y} \geq z_\alpha, i = 1, \ldots, k\} \subset \{\boldsymbol{y} : \boldsymbol{h}_i'\boldsymbol{y} \geq z_\alpha \text{ and } \boldsymbol{h}_j'\boldsymbol{y} \geq z_\alpha\} \subset C_{ij},$$

for every $(i, j) \in D$. Hence R_L is contained in the rejection region of ϕ_g, the size of $\phi_g \geq$ size of the LRT $= \alpha$, and ϕ_g is uniformly more powerful than the LRT. □

ϕ_g is, in fact, strictly more powerful than the LRT because ϕ_g's rejection region contains an open set that is not in R_L. Let $\boldsymbol{y}$ denote a point satisfying $\boldsymbol{h}_i'\boldsymbol{y} = z_\alpha$, $i = 1, \ldots, k$. (If $k \geq p$, thete is only one such $\boldsymbol{y}$. If $k < p$, there are many such $\boldsymbol{y}$'s.) Every C_{ij} contains an open set that contains the line from $\boldsymbol{y}$ to the origin. So the intersection of the C_{ij}, ϕ_g's rejection region, contains an open set containing this line, and this open set is not in R_L.

4. Example-comparing linear regression functions

In this section, we will present an example by applying the above results to test whether two or more linear regression functions are ordered. First, we consider regression models for two populations; $E(Y_1) = \alpha_1 + \beta_1 x$ and $E(Y_2) = \alpha_2 + \beta_2 x$. Because these regression functions are linear in x, to determine whether $\alpha_1 + \beta_1 x$ is greater or smaller than $\alpha_2 + \beta_2 x$ on a finite interval $I = [x_*, x^*]$, one might test the hypotheses

$$H_{012} : \alpha_1 + \beta_1 x^* \leq \alpha_2 + \beta_2 x^*, \alpha_1 + \beta_1 x_* \geq \alpha_2 + \beta_2 x_* \quad \text{or}$$
$$\alpha_1 + \beta_1 x^* \geq \alpha_2 + \beta_2 x^*, \alpha_1 + \beta_1 x_* \leq \alpha_2 + \beta_2 x_*$$

(4.1) versus

$$H_{112} : \alpha_1 + \beta_1 x^* > \alpha_2 + \beta_2 x^*, \alpha_1 + \beta_1 x_* > \alpha_2 + \beta_2 x_* \quad \text{or}$$
$$\alpha_1 + \beta_1 x^* < \alpha_2 + \beta_2 x^*, \alpha_1 + \beta_1 x_* < \alpha_2 + \beta_2 x_*.$$

These hypotheses are equivalent to problem (1.1), where $\boldsymbol{b}_1' = [1,0]$, $\boldsymbol{b}_2' = [0,1]$, $\boldsymbol{\mu}' = [\mu_1, \mu_2]$, $\mu_1 = \alpha_1 + \beta_1 x^* - \alpha_2 - \beta_2 x^*$ and $\mu_2 = \alpha_1 + \beta_1 x_* - \alpha_2 - \beta_2 x_*$. Hence this testing problem is actually a special case of problem (1.1) for $k = 2$. Results derived for the problem (1.1) can be applied to this comparison of regression linear problem.

Given two independent sets of bivariate observations $(X_{11}, Y_{11}), \ldots, (X_{1n_1}, Y_{1n_1})$ and $(X_{21}, Y_{21}), \ldots, (X_{2n_2}, Y_{2n_2})$. We make the usual assumptions that given the $\boldsymbol{X}$ values the $\boldsymbol{Y}$'s are independent and normal with $E(Y_{ij} \mid X_{ij} = x_{ij}) = \alpha_i + \beta_i x_{ij}$ and $\mathrm{Var}(Y_{ij} \mid X_{ij} = x_{ij}) = \sigma^2$, where (α_i, β_i), $i = 1, 2$ and $\sigma^2 > 0$. Let W_1 and W_2 denote the difference distance between the two estimated least squares regression lines at x^* and x_*, respectively. Hence

$$(W_1, W_2) \sim N_2((\mu_1, \mu_2), \boldsymbol{\Sigma}_w),$$

where $\mu_1 = \alpha_1 + \beta_1 x^* - \alpha_2 - \beta_2 x^*$, $\mu_2 = \alpha_1 + \beta_1 x_* - \alpha_2 - \beta_2 x_*$ and $\boldsymbol{\Sigma}_w = \sigma^2 \boldsymbol{\Lambda}_{n_1 n_2 \bar{x}_i}$, where the diagonal elements of $\boldsymbol{\Lambda}_{n_1 n_2 \bar{x}_i}$ are $(q_{n_1 n_2} + \frac{(\bar{x}_1 - x_*)^2}{\sum_j (x_{1j} - \bar{x}_1)^2} + \frac{(\bar{x}_2 - x_*)^2}{\sum_j (x_{2j} - \bar{x}_2)^2}, q_{n_1 n_2} + \frac{(\bar{x}_1 - x^*)^2}{\sum_j (x_{1j} - \bar{x}_1)^2} + \frac{(\bar{x}_2 - x^*)^2}{\sum_j (x_{2j} - \bar{x}_2)^2})$ and off diagonal elements of the matrix are $q_{n_1 n_2} + \frac{(\bar{x}_1 - x_*)(\bar{x}_1 - x^*)}{\sum_j (x_{1j} - \bar{x}_1)^2} + \frac{(\bar{x}_2 - x_*)(\bar{x}_2 - x^*)}{\sum_j (x_{2j} - \bar{x}_2)^2}$ and $q_{n_1 n_2} = \frac{1}{n_1} + \frac{1}{n_2}$. The condition for ϕ_d is $\boldsymbol{b}_1' \boldsymbol{\Sigma}_w \boldsymbol{b}_2 \leq 0$. In this example

$$\begin{aligned} \boldsymbol{b}_1' \boldsymbol{\Sigma}_w \boldsymbol{b}_2 &= \mathrm{Cov}(W_1, W_2) \\ &= \left\{ \frac{1}{n_1} + \frac{1}{n_2} + \frac{(\bar{x}_1 - x_*)(\bar{x}_1 - x^*)}{\sum_j (x_{1j} - \bar{x}_1)^2} + \frac{(\bar{x}_2 - x_*)(\bar{x}_2 - x^*)}{\sum_j (x_{2j} - \bar{x}_2)^2} \right\} \times \sigma^2. \end{aligned}$$

Hence the condition is

$$\begin{aligned} &\boldsymbol{b}_1' \boldsymbol{\Sigma}_w \boldsymbol{b}_2 \leq 0 \\ &\quad \Leftrightarrow \frac{1}{n_1} + \frac{1}{n_2} + \frac{(\bar{x}_1 - x_*)(\bar{x}_1 - x^*)}{\sum_j (x_{1j} - \bar{x}_1)^2} + \frac{(\bar{x}_2 - x_*)(\bar{x}_2 - x^*)}{\sum_j (x_{2j} - \bar{x}_2)^2} \leq 0, \\ &\quad \Leftrightarrow \frac{1}{n_1} + \frac{1}{n_2} \leq \frac{(\bar{x}_1 - x_*)(x^* - \bar{x}_1)}{\sum_j (x_{1j} - \bar{x}_1)^2} + \frac{(\bar{x}_2 - x_*)(x^* - \bar{x}_2)}{\sum_j (x_{2j} - \bar{x}_2)^2}. \end{aligned} \tag{4.2}$$

The condition (4.2) is equivalent to $s \leq 0$. In fact, as the interval $I = [x_*, x^*]$ is narrowed and the number of the observations becomes big, then $s > 0$, and condition (4.2) is not satisfied. Hence ϕ_d does not exit. But as the interval I is widened, the values of s become negative. Therefore the uniformly more powerful test, ϕ_d, exists. We can improve on the LRT in this regression problem.

Now we consider that there are k (> 2) simple linear regression functions and we want to test whether those functions are ordered. That is, the null and alternative hypotheses can be stated as following,

$$\begin{aligned} H_0 &: \mu_{2i-1} \leq 0 \text{ or } \mu_{2i} \leq 0 \quad \text{for some } i = 1, \ldots, k-1 \quad \text{and} \\ &\quad \mu_{2i-1} \geq 0 \text{ or } \mu_{2i} \geq 0 \quad \text{for some } i = 1, \ldots, k-1. \\ H_1 &: \mu_{2i-1} > 0, \mu_{2i} > 0 \quad \text{for all } i = 1, \ldots, k-1 \quad \text{or} \\ &\quad \mu_{2i-1} < 0, \mu_{2i} < 0 \quad \text{for all } i = 1, \ldots, k-1 \end{aligned}$$

where $\mu_{2i-1} = \alpha_{i+1} + \beta_{i+1} x_* - \alpha_i - \beta_i x_*$, $\mu_{2i} = \alpha_{i+1} + \beta_{i+1} x^* - \alpha_i - \beta_i x^*$. Then $\boldsymbol{W} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}_w)$, with $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_{2(k-1)})'$, where $\mu_{2i-1} = \alpha_{i+1} + \beta_{i+1} x_* - \alpha_i - \beta_i x_*$

and $\mu_{2i} = \alpha_{i+1} + \beta_{i+1}x^* - \alpha_i - \beta_i x^*$ for $i = 1, 2, \ldots, k-1$. Here we can choose set D as $\{(1,2), (2,3), \ldots, (2k-3, 2k-2)\}$. The conditions that ϕ_g exists are $\mathrm{Cov}(W_i, W_j) \leq 0$ for $(i,j) \in D$. The conditions are equivalent to

$$\frac{1}{n_{2i-1}} + \frac{1}{n_{2i}} \leq \frac{(\bar{x}_{2i-1} - x_*)(x^* - \bar{x}_{2i-1})}{\sum_j (x_{2i-1,j} - \bar{x}_{2i-1})^2} + \frac{(\bar{x}_{2i} - x_*)(x^* - \bar{x}_{2i})}{\sum_j (x_{2i,j} - \bar{x}_{2i})^2}, \quad i = 1, 2, \ldots, k-1.$$

$$\frac{1}{n_i} \leq \frac{(\bar{x}_i - x_*)(x^* - \bar{x}_i)}{\sum_j (x_{ij} - \bar{x}_i)^2}, \quad i = 2, \ldots, k-1, \quad k \geq 3.$$

The above conditions are similar to condition (4.2) as the interval I is widened, the values of s become negative. Therefore the uniformly more powerful test, ϕ_g, exists. Then the LRT can be dominated in comparing two or more simple linear regression functions.

5. Comment

Menéndez *et al.* (1992) points out the tests which are uniformly more powerful than the LRT including the present one occur on one common situation—the size of the LRT is attained at an infinite boundary point of the null hypothesis. In the obtuse cone case, for two-sided testing hypotheses, the size of LRT occurs at a finite boundary point of the null hypothesis. Hence the method we use in Section 2 can not apply to obtuse cone case.

If D' is defined as same as in Section 4 of the Liu and Berger's (1995), then the intersection of H_{1ij}, where $(i,j) \in D'$, will not equal to the alternative hypotheses, H_1, of the problem (1.1) for $k \geq 4$. We explain it for $k = 4$. Actually, we can rewrite H_1 as $H_1^+ \cup H_1^-$, where $H_1^+ : \boldsymbol{h}_i'\boldsymbol{\theta} > 0$ for all $i = 1, \ldots, k$, and, $H_1^- : \boldsymbol{h}_i'\boldsymbol{\theta} < 0$ for all $i = 1, \ldots, k$. Similarly, H_{1ij} can also be rewritten as $H_{1ij}^+ \cup H_{1ij}^-$, where $H_{1ij}^+ : \boldsymbol{h}_i'\boldsymbol{\theta} > 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} > 0$, and, $H_{1ij}^- : \boldsymbol{h}_i'\boldsymbol{\theta} < 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} < 0$. Consider $k = 4$ and choose $D' = \{(1,2), (3,4)\}$. Then $\bigcap_{(i,j)\in D'} H_{1ij} = H_{112} \cap H_{134} = H_1 \cup (H_{112}^+ \cap H_{134}^-) \cup (H_{112}^- \cap H_{134}^+) \neq H_1$. But D' could work for $k = 3$. In fact, D' is exactly same as D, defined in Section 3, for $k = 3$.

Acknowledgements

Special thanks are given to my academic advisor Dr. Roger L. Berger. The author thanks the referees and editor for suggestions that greatly improved the presentation of the article. The author also thanks a referee who gives the suggestion for the proof of Theorem 2.1.

References

Berger, R. L. (1982). Multiparameter hypothesis testing and acceptance sampling, *Technometrics*, **24**, 295–300.

Berger, R. L. (1989). Uniformly more powerful tests for hypotheses concerning linear inequalities and normal means, *J. Amer. Statist. Assoc.*, **84**, 192–199.

Casella, G. and Berger, R. L. (1990). *Statistical Inference*, Wadsworth & Brook/Cole, Pacific Grove, California.

Cohen, A., Gatsonis, C. and Marden, J. I. (1983). Hypothesis tests and optimality properties in discrete multivariate analysis, *Studies in Econometrics*, *Time Series and Multivariate Statistics* (eds. S. Karlin, T. Amemiye and L. A. Goodman), 379–405, Academic Press, New York.

Gutmann, S. (1987). Tests uniformly more powerful than uniformly most powerful monotone tests, *J. Statist. Plann. Inference*, **17**, 279–292.

Iwasa, M. (1991). Admissibility of unbiased tests for a composite hypothesis with a restricted alternative, *Ann. Inst. Statist. Math.*, **43**, 657–665.

Lehmann, E. L. (1952). Testing multiparameter hypotheses, *Ann. Math. Statist.*, **23**, 541–552.

Li, T. and Sinha, B. K. (1995). Tests of ordered hypotheses for gamma scale parameters, *J. Statist. Plann. Inference*, **45**, 387–397.

Liu, H. and Berger, R. L. (1995). Uniformly more powerful tests for one-sided hypotheses about linear inequalities, *Ann. Statist.*, **23**, 55–72.

Menéndez, J. A., Rueda, C. and Salvador, B. (1992). Dominance of likelihood ratio tests under cone constraints, *Ann. Statist.*, **20**, 2087–2099.

Nomakuchi, K. and Sakata, T. (1987). A note on testing two-dimensional normal mean, *Ann. Inst. Statist. Math.*, **39**, 489–495.

Sasabuchi, S. (1980). A test of a multivariate normal mean with composite hypotheses determined by linear inequalities, *Biometrika*, **67**, 429–439.

Shirley, A. G. (1992). Is the minimum of several location parameters positive?, *J. Statist. Plann. Inference*, **31**, 67–79.