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ON RATES AND LIMIT DISTRIBUTIONS

J. PFANZAGL

Mathematical Institute, University of Cologne, Albertus-Magnus-Platz, D - 50923 Cologne, Germany

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Abstract. For regular parametric models, estimators converge uniformly at a rate $n^{-1/2}$, and the limit distribution is normal with mean 0. The situation is different if the best possible rate is $n^{-\alpha}$, with $\alpha \in (0, 1/2)$, as common for nonparametric models. In this case, uniformly attainable normal limit distributions with mean 0 are impossible.

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1. Introduction

Let $(X, \mathcal{A})$ be a measurable space, and $\mathfrak{P}$ a family of probability measures $P \mid \mathcal{A}$. The problem is to estimate a functional $\kappa : \mathfrak{P} \to \mathbb{R}$, based on a sample of size n, distributed according to P^n, with $P \in \mathfrak{P}$ unknown. For $n \in N$ let $\kappa^{(n)} : X^n \to \mathbb{R}$ be an estimator.

For parametric models we are used to the existence of estimator sequences with the following properties. (i) They converge with the rate $n^{1/2}$, uniformly on compact subsets of the parameter space, to a normal distribution, (ii) this normal distribution is maximally concentrated among all limit distributions which can be attained locally uniformly.

The situation is not so favourable in the nonparametric case. Checking the literature, one finds rate bounds, say c_n, $n \in \mathbb{N}$, for various models. That means: If for some estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$, the sequence of standardized errors, $\hat{c}_n(\kappa^{(n)} - \kappa(P))$, is under P^n asymptotically bounded, uniformly on $\mathfrak{P}$, then $\hat{c}_n$, $n \in \mathbb{N}$, cannot converge to infinity quicker than c_n, $n \in \mathbb{N}$. In other words: c_n, $n \in \mathbb{N}$, is the best possible rate for the convergence of $\kappa^{(n)}$ to $\kappa(P)$. A rate bound is not necessarily attainable. To show that c_n, $n \in \mathbb{N}$, is, in fact, a possible rate, it would be necessary to find an estimator sequence, say $\kappa_0^{(n)}$, $n \in \mathbb{N}$, such that $c_n(\kappa_0^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, is under P^n asymptotically bounded, uniformly on $\mathfrak{P}$. More often than not, the authors are satisfied with something quite different, namely: The existence of an estimator sequence converging with the rate c_n, $n \in \mathbb{N}$, to a limit distribution pointwise. That means: The distribution of $c_n(\kappa_0^{(n)} - \kappa(P))$ under P^n converges to a limit distribution for every $P \in \mathfrak{P}$.

Since (local) uniformity is a constitutive element in the definition of a rate bound, convergence to a limit distribution does not establish that this rate is attainable, unless the convergence to the limit distribution is (locally) uniform.

The reader should keep in mind that there may be several estimators attaining the rate bound c_n, $n \in \mathbb{N}$, differing by an amount which is stochastically of the order $O(c_n)$. Hence limit distributions attained with the rate c_n, $n \in \mathbb{N}$, —if any—are not unique.

The careful distinction between the convergence rate of $\kappa^{(n)}$ to κ, and the rate at which the distribution of $\kappa^{(n)} - \kappa(P)$ under P^n converges to a limit distribution is without much use in case of regular parametric models. Here, the distribution of $\kappa^{(n)} - \kappa(P)$ converges with the rate $n^{1/2}$ uniformly to a limit, and this implies that the rate $n^{1/2}$ is attainable. In nonparametric models, the uniform rate bounds for the convergence of $\kappa^{(n)}$ to $\kappa(P)$ are usually of type $c_n = n^\alpha$ with $\alpha \in (0, 1/2)$. Even if there is an estimator sequence converging to κ at this rate, this does not imply convergence at this rate to a limit distribution, let alone uniform convergence.

Rate bounds for the convergence to κ can be obtained by different methods. In the present paper, we make no assumptions about the origin of the rate bounds. Within this restricted framework, one can show that limit distributions attained with a convergence rate $n^\alpha L(n)$ with $\alpha \in (0, 1/2)$ and L a slowly varying function cannot be uniformly attained if they have expectation zero and a finite absolute moment of order $(1-\alpha)^{-1}$. Under more specific assumptions on the nature of the rate bounds it will be shown elsewhere (see Pfanzagl (1998)) that uniformly attainable limit distributions and confidence intervals with uniform covering probabilities are impossible if $\alpha \in (0, 1/2)$.

As a side result we obtain that limit distributions are necessarily normal if they are (i) locally uniformly attained with a rate $n^{1/2}L(n)$ and (ii) maximally concentrated on arbitrary intervals containing 0.

Our method of proof is not suitable to deal with the case $\alpha > 1/2$. It is, however, clear that comparable results are not to be expected. For the purpose of illustration we mention the family of uniform distributions over $(0, \vartheta)$, say P_ϑ. The sequence $\vartheta - \max\{x_1, \ldots, x_n\}$ converges with the rate n to a non-normal limit distribution, namely the exponential distribution. According to Millar ((1983), pp. 156–157) this distribution is maximally concentrated among all uniformly attainable limit distributions.

Theorem 2.1 in Section 2 is the main result. Applications to the case $\alpha = 1/2$ and $\alpha \in (0, 1/2)$ are given in Sections 3 and 4, respectively. Section 5 contains a detailed discussion of the concept of a rate bound. Various technical lemmas and the proof of Theorem 2.1 are given in Section 6.

2. The main result

We shall use the following notation. For any probability measure $P \mid \mathcal{A}$ and a measurable function $h : X \to \mathbb{R}$, $P \circ h$ denotes the induced distribution on $\mathbb{B}$, defined by $P \circ h(B) := P(h^{-1}B)$, $B \in \mathbb{B}$.

The theorem in this section refers to estimator sequences $\kappa^{(n)}$, $n \in \mathbb{N}$, the standardized distributions of which, $P^n \circ c_n(\kappa^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, converge weakly to a limit distribution, say $Q_P \mid \mathbb{B}$.

Warning. We call c_n the *rate of convergence*. This is in agreement with the terminology used by some authors (see e.g. Bickel *et al.* ((1993), p. 176) or Akahira and Takeuchi ((1995), p. 77)), whereas other authors would use the term "rate of convergence" for $1/c_n$.

2.1 *The definition of "uniform weak convergence"*

It is not entirely clear how to define uniform weak convergence of a sequence of distributions $Q_{P,n} \mid \mathbb{B}$, $n \in \mathbb{N}$, to a distribution $Q_P \mid \mathbb{B}$. In view of later applications we restrict our considerations to the case of nonatomic limit distributions. This enables the use of the Kolmogorov distance. (A straightforward extension is to the case where the discontinuity points of Q_P are the same for every $P \in \mathfrak{P}$, using the Lévy distance.)

Throughout the following, $\mathfrak{P}_n \subset \mathfrak{P}$ is a nonincreasing sequence (including the cases $\mathfrak{P}_n \equiv \mathfrak{P}$ and $\mathfrak{P}_n \downarrow \{P\}$).

For nonatomic limit distributions, a natural definition of "weak convergence, uniformly on $\mathfrak{P}_n$, $n \in \mathbb{N}$" is

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} |Q_{P,n}(-\infty, u] - Q_P(-\infty, u]| = 0 \quad \text{for every } u \in \mathbb{R}. \tag{2.1}$$

For technical reasons, we need a stronger concept of "uniform weak convergence", based on a distance function for probability measures on $\mathbb{B}$ with the following property.

$$D(P * R, Q * R) \le D(P, Q) \tag{2.2'}$$

and

$$D(P \circ (u \to au), Q \circ (u \to au)) = D(P, Q) \quad \text{for} \quad a \ge 1. \tag{2.2''}$$

It is straightforward to show that the Kolmogorov distance, defined by

$$D(P, Q) := \sup_{u\in\mathbb{R}} |P(-\infty, u] - Q(-\infty, u]| \tag{2.3}$$

has this property (as does the Lévy distance).

DEFINITION 2.1. A family $\mathfrak{Q}$ of distributions on $\mathbb{B}$ is *equicontinuous* if the family of distribution functions $F_Q(u) := Q(-\infty, u]$, $Q \in \mathfrak{Q}$, is equicontinuous on $\mathbb{R}$, and if

$$\lim_{u\to\infty} \inf_{Q\in\mathfrak{Q}} F_Q(u) = 1, \quad \text{and} \quad \lim_{u\to-\infty} \sup_{Q\in\mathfrak{Q}} F_Q(u) = 0.$$

Recall that a function $h : \mathbb{R} \to \mathbb{R}$ is *uniformly integrable* on $\mathfrak{Q}$ if

$$\lim_{v\to\infty} \sup_{Q\in\mathfrak{Q}} \int |h(u)| 1_{[v,\infty)}(|h(u)|) Q(du) = 0. \tag{2.4}$$

LEMMA 2.1. *If Q_P is nonatomic for $P \in \mathfrak{P}$, and $\{Q_P : P \in \mathfrak{P}\}$ is equicontinuous, then* (2.1) *implies*

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} D(Q_{P,n}, Q_P) = 0. \tag{2.5}$$

PROOF. The usual proof (see e.g. Petrov (1995), p. 17, Theorem 1.11) carries over. There is just one point which requires some more attention: The selection of a *finite* number of points u_i, $i = 1, \ldots, m$, such that $Q_P(-\infty, u_1] < \varepsilon$, $Q_P(u_m, \infty) < \varepsilon$, and $Q_P(u_i, u_{i+1}] < \varepsilon$ for every $P \in \mathfrak{P}$. For this purpose, define

$$u_{i+1} := \sup\{u \in \mathbb{R} : Q_P(-\infty, u] \leq Q_P(-\infty, u_i] + \varepsilon \text{ for } P \in \mathfrak{P}\}.$$

We have to show that the sequence u_i, $i = 1, 2, \ldots$ thus defined tends to infinity, so that one can stop at u_m for which $Q_P(-\infty, u_m] > 1 - \varepsilon$ for $P \in \mathfrak{P}$. Assume that, on the contrary, $u_i \uparrow v < \infty$. Since $v > u_{i+1}$, there exists P_i such that $Q_{P_i}(-\infty, v] > Q_{P_i}(-\infty, u_i] + \varepsilon$, i.e. $Q_{P_i}(u_i, v] > \varepsilon$. Since $u_i \uparrow v$, this contradicts the equicontinuity. □

Remark 2.1. The relations considered so far refer to arbitrary sequences $Q_{P,n}$, $n \in \mathbb{N}$. Our interest is in sequences of a special type, namely

$$Q_{P,n} = P^n \circ c_n(\kappa^{(n)} - \kappa(P)). \tag{2.6}$$

For such sequences, uniformly attained limit distributions are necessarily continuous functions of P. More precisely, assume that $P \to \kappa(P)$ is continuous at P_0 with respect to the sup-metric, and that $P^n \circ c_n(\kappa^{(n)} - \kappa(P)) \Rightarrow Q_P$ uniformly (in the sense of Definition 2.1) on some neighbourhood of P_0 which is open in the sup-metric. Then $P \to Q_P$ is continuous at P_0 (with respect to the sup-metric on $\mathfrak{P}$ and the topology of weak convergence on $\{Q_P : P \in \mathfrak{P}\}$).

A result of this type occurs in Rao ((1963), p. 196, Lemma 2(i)) and Wolfowitz ((1965), p. 254, Lemma 2). For a proof of this general version see Pfanzagl and Wefelmeyer ((1982), p. 163, Proposition 9.4.1).

Remark 2.2. To assume that the limit distributions Q_P are nonatomic is justified by the fact that this is the case in all examples from the literature. Under natural conditions it is necessarily so: Uniformly attained limit distributions have a Lebesgue-density. More precisely, assume there exists a parametric subfamily $P_\vartheta \in \mathfrak{P}$, say $\vartheta \in (-1, 1)$, such that, with $P_n = P_{c_n^{-1}a}$, the following relations are true for every $a \in \mathbb{R}$.

$$P_n^n \circ c_n(\kappa^{(n)} - \kappa(P_n)) \Rightarrow Q_{P_0},$$
$$c_n(\kappa(P_n) - \kappa(P_0)) \to aK_0,$$
$$P_n^n,\ n \in \mathbb{N}, \text{ is contiguous with respect to } P_0^n,\ n \in \mathbb{N}.$$

Then $Q_{P_0} << \lambda$.

This follows from Pfanzagl ((1994), p. 229, Proposition 7.1.11).

THEOREM 2.1. *Assume that* $\kappa^{(n)}$, $n \in \mathbb{N}$, *is a sequence of estimators such that* $P^n \circ c_n(\kappa^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, *converges weakly to a limit distribution, say* Q_P, *uniformly on* $\mathfrak{P}_n$, $n \in \mathbb{N}$, *i.e.*

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} D(P^n \circ c_n(\kappa^{(n)} - \kappa(P)), Q_P) = 0. \tag{2.7}$$

Assume that

$$c_n = n^\alpha L(n), \tag{2.8}$$

with $\alpha \in (0, \frac{1}{2}]$, *and* L *a slowly varying function.*

Assume that the limit distributions Q_P *have expectation* 0.

Then the following holds true.

(i) *If* $\alpha \in (0, 1/2)$ *and* $|u|^{1/(1-\alpha)}$ *is uniformly integrable on* $\{Q_P : P \in \mathfrak{P}\}$, *then there exists an estimator sequence* $\hat{\kappa}^{(n)}$, $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\hat{\kappa}^{(n)} - \kappa(P)| < u\} = 1 \quad \textit{for every } u > 0. \tag{2.9}$$

This implies in particular that $\hat{\kappa}^{(n)}$, $n \in \mathbb{N}$, *converges to* $\kappa(P)$ *at a rate better than* c_n, $n \in \mathbb{N}$.

(ii) *If* u^2 *is uniformly integrable on* $\{Q_P : P \in \mathfrak{P}\}$, *and if*

$$\sigma^2(P) := \int u^2 Q_P(du) \tag{2.10}$$

is bounded away from 0 *on* $\mathfrak{P}$, *then the following stronger assertion is true.*

There exists an estimator sequence $\hat{\kappa}^{(n)}$, $n \in \mathbb{N}$, *and a rate* $\hat{c}_n$, $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} D(P^n \circ \hat{c}_n(\hat{\kappa}^{(n)} - \kappa(P)), N_{(0,\sigma^2(P))}) = 0, \tag{2.11}$$

with $\hat{c}_n = c_n$ *if* $\alpha = 1/2$, *and with* $\hat{c}_n/c_n \to \infty$ *if* $\alpha \in (0, 1/2)$.

For virtually all estimator sequences occurring in the literature, the error of the estimator, $\kappa^{(n)} - \kappa(P)$, can be standardized such that the limit distribution is independent of P. More precisely, there exists a function $K : \mathfrak{P} \to (0, \infty)$, such that $P^n \circ c_n K(P)(\kappa^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, converges to a limit distribution Q, not depending on P. To say the same in a different way: $P^n \circ c_n(\kappa^{(n)} - \kappa(P))$ converges to the limit distribution

$$Q_P = Q \circ (u \to u/K(P)). \tag{2.12}$$

For limit distributions of this special type, the conditions of the theorem on the limit distributions, namely the equicontinuity of $u \to Q_P(-\infty, u]$ (which is needed to derive (2.5) from the simpler uniform convergence condition (2.1)), and the uniform integrability of $|u|^s$ on $\{Q_P : P \in \mathfrak{P}\}$ become quite simple.

PROPOSITION 2.1. *Assume that* Q_P *is of the special type* (2.12), *with* $\{K(P) : P \in \mathfrak{P}\}$ *bounded and bounded away from* 0.

Then the following is true.

(i) *The family* $\{Q_P : P \in \mathfrak{P}\}$ *is equicontinuous* (*in the sense of Definition* 2.1) *if* Q *is nonatomic.*

(ii) $|u|^s$ *is uniformly integrable on* $\{Q_P : P \in \mathfrak{P}\}$ *if* $\int |u|^s Q(du) < \infty$.

(iii) $\inf_{P\in\mathfrak{P}} \int |u|^s Q_P(du) > 0$ *if* $\int |u|^s Q(du) > 0$.

PROOF. (i) Let F denote the distribution function of Q. Let Λ be a compact subset of $(0, \infty)$. We shall show that the family of distribution functions $u \to F(\lambda u)$, $\lambda \in \Lambda$, is equicontinuous in the sense of Definition 2.1. Since $\inf_{P\in\mathfrak{P}} K(P) > 0$, this implies the equicontinuity of $\{Q_P : P \in \mathfrak{P}\}$.

Assume that $\lim_{n\to\infty} u_n = u_0$. If $\limsup_{n\to\infty} \sup_{\lambda\in\Lambda} |F(\lambda u_n) - F(\lambda u_0)| > 0$, there exists an infinite subsequence $\lambda_n \in \Lambda$, $n \in \mathbb{N}_0$, such that

$$\lim_{n\in\mathbb{N}_0} |F(\lambda_n u_n) - F(\lambda_n u_0)| > 0. \tag{2.13}$$

Let λ_n, $n \in \mathbb{N}_1 \subset \mathbb{N}_0$ be a convergent subsequence, say $\lim_{n\in\mathbb{N}_1} \lambda_n = \lambda_0$. Since

$$\lim_{n\in\mathbb{N}_1} \lambda_n u_n = \lim_{n\in\mathbb{N}_1} \lambda_n u_0 = \lambda_0 u_0,$$

we have

$$\lim_{n\in\mathbb{N}_1} |F(\lambda_n u_n) - F(\lambda_n u_0)| = 0,$$

in contradiction to (2.13). The relations

$$\lim_{u\to\infty} \inf_{\lambda\in\Lambda} F(\lambda u) = 1 \quad \text{and} \quad \lim_{u\to-\infty} \sup_{\lambda\in\Lambda} F(\lambda u) = 0$$

are proved similarly.

(ii) If $f : [0, \infty) \to \mathbb{R}$ is nondecreasing and $K(P) \geq \lambda_0 > 0$, we have

$$\int f(u) Q_P(du) = \int f\left(\frac{u}{K(P)}\right) Q(du) \leq \int f\left(\frac{u}{\lambda_0}\right) Q(du).$$

The assertion follows for $f(u) = |u|^s 1_{[v_n \infty)}(u)$.

(iii) Obvious. □

Remark 2.3. Some authors consider approximations by distributions including a bias-term. If we restrict ourselves to approximations by a normal distribution, that means the approximation of $P^n \circ c_n(\kappa^{(n)} - \kappa(P))$ by $N_{(\mu_n(P), \sigma^2(P))}$ (in the sense of (2.7)). To be suitable for the computation of confidence bounds, $\mu_n(P)$

has to be estimable in the sense that there is an estimator sequence $\mu^{(n)}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} P^n\{|\mu^{(n)} - \mu_n(P)| > u\} = 0 \qquad \text{for every } u > 0. \tag{2.14}$$

If this is the case, one may introduce the estimator sequence $\hat{\kappa}^{(n)} = \kappa^{(n)} - c_n^{-1}\mu^{(n)}$, for which $P^n \circ c_n(\hat{\kappa}^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, converges uniformly in the sense of (2.7) to $N_{(0,\sigma^2(P))}$. If $c_n = n^\alpha L(n)$ with $\alpha \in (0, 1/2)$ is a $\mathfrak{P}_n$-uniform rate bound, this is impossible according to Theorem 2.1 (ii). Hence $\mu_n(P)$ cannot be estimable in the sense of (2.14) in this case. See Example 4.6 for an application.

2.2 *Why uniform convergence?*

Limit distributions are an instrument for selecting estimators if the sample size is large. The second—and more important—use of limit distributions is for the construction of approximate confidence intervals, and this is the point where uniformity on $\mathfrak{P}$ becomes operationally significant. It is needed to obtain confidence intervals with a covering probability which is uniform on $\mathfrak{P}$.

Let $t_{P,\beta} \in \mathbb{R}$ be defined by

$$Q_P(-\infty, t_{P,\beta}] = \beta.$$

If there exists an estimator sequence $t_\beta^{(n)}$ which converges to $t_{P,\beta}$, uniformly on $\mathfrak{P}$, i.e.

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}} P^n\{|t_\beta^{(n)} - t_{P,\beta}| > u\} = 0 \qquad \text{for every } u > 0 \tag{2.15}$$

we obtain from the uniform convergence of $P^n \circ c_n(\kappa^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, to Q_P in the sense of (2.7) by a uniform version of Slutsky's Theorem that

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}} |P^n\{\kappa^{(n)} - c_n^{-1} t_\beta^{(n)} < \kappa(P)\} - \beta| = 0.$$

3. The case $\alpha = 1/2$

Roughly speaking, Theorem 2.1 (ii) says the following: If there is an estimator sequence converging at the rate $n^{1/2}L(n)$ uniformly on $\mathfrak{P}_n$ to some limit distribution with finite variance, then there also exists an estimator sequence converging at the same rate uniformly on $\mathfrak{P}_n$ to a normal distribution with the same variance. This implies the following:

PROPOSITION 3.1. *Assume the conditions specified in Theorem* 2.1 (ii).

If for some $P \in \mathfrak{P}$ the limit distribution Q_P is maximally concentrated (*on all intervals containing* 0) *among all limit distributions attained uniformly on $\mathfrak{P}$ with a rate $n^{1/2}L(n)$, then $Q_P = N_{(0,\sigma^2(P))}$.*

PROOF. According to Theorem 2.1 (ii), $N_{(0,\sigma^2(P))}$ is a limit distribution, uniformly attainable on $\mathfrak{P}_n$. Since $N_{(0,\sigma^2(P))}[-u', u''] \leq Q_P[-u', u'']$ for $u', u'' \geq 0$, we have for $u > 0$

$$N_{(0,\sigma^2(P))}(0, u] \leq Q_P(0, u] \quad \text{and} \tag{3.1'}$$

$$N_{(0,\sigma^2(P))}[-u, 0) \leq Q_P[-u, 0), \quad \text{hence} \tag{3.1''}$$

$$\int_0^\infty u^2 N_{(0,\sigma^2(P))}(du) \leq \int_0^\infty u^2 Q_P(du) \quad \text{and} \tag{3.2'}$$

$$\int_{-\infty}^0 u^2 N_{(0,\sigma^2(P))}(du) \leq \int_{-\infty}^0 u^2 Q_P(du). \tag{3.2''}$$

Since

$$\int_{-\infty}^{+\infty} u^2 N_{(0,\sigma^2(P))} du = \sigma^2(P) = \int_{-\infty}^{+\infty} u^2 Q_P(du),$$

this implies equality in (3.2′) and (3.2″), hence also equality in (3.1′) and (3.1″). □

That maximally concentrated limit distributions are normal follows from the convolution theorem for families fulfilling an LAN-condition. Proposition 3.1 arrives at a similar conclusion by a different approach. (For parametric families and $c_n = n^{1/2}$, this result occurs in Pfanzagl (1994), p. 294, Proposition 8.5.10.)

For regular parametric families, the rate bound is $n^{1/2}$, and asymptotically optimal estimator sequences are easy to obtain. In spite of this, one occasionally meets with estimator sequences converging at the optimal rate $n^{1/2}$ to a nonnormal limit distribution. As an example we mention estimators of the center of symmetry of a distribution on $\mathbb{B}$: The Bickel-Hodges estimator ((1967), Section 3) and the estimator based on the Kolmogorov-distance (see Rao *et al.* (1975), Theorem 4, p. 866). One does not need the theorem to learn that these estimator sequences are asymptotically inefficient. To establish the existence of an estimator sequence which is (pointwise) asymptotically optimal in the family of all symmetric distributions on $\mathbb{B}$ was one of the outstanding tasks set to asymptotic theory. (See Beran (1974) and Stone (1975) for the best results now available.)

A rate bound $n^{1/2}(\log n)^{1/2}$ occurs for location parameter families $p(x - \vartheta)$, $\vartheta \in \mathbb{R}$, if $p(x) = 0$ for $x < 0$ and $\lim_{x \downarrow 0} p'(x) \in (0, \infty)$. Woodroofe ((1972), Theorem 2.1, p. 115) shows that the maximum likelihood estimator for the location parameter converges at this rate to a normal limit distribution. Weiß and Wolfowitz ((1973), Theorem, p. 945) show that this limit distribution is maximally concentrated (see also Akahira and Takeuchi ((1981), Sections 2.4 and 2.5) and Smith (1985)).

We conclude this section with two nonparametric examples where the rate bound is $n^{1/2}L(n)$.

Example 3.1. Wicksell's problem. For a general discussion of this problem see Stoyan *et al.* ((1987), Section 11.4). Groeneboom and Jongbloed (1995) obtained the rate bound $n^{1/2}(\log n)^{-1/2}$, based on the loss function $|u|$, and they prove it to be attainable. Golubev and Levit (1998) obtain not only the rate bound, but also the best possible uniformly attainable limit distribution (Theorem 2, p. 2411). This is a normal distribution with mean 0. They also show (Theorem 1, p. 2411) that this best possible limit distribution is, in fact, locally uniformly attained by certain kernel-type estimators. These results are obtained for a large class of symmetric loss functions. They yield, in particular, assertions about the concentration in symmetric intervals and fit, therefore, into the framework of the present paper. Presumably, the local uniformity obtained in Theorem 1 of Golubev and Levit could be strenghtened to global uniformity (in the sense of (2.1)), thus confirming the result of Theorem 2.1 (ii) according to which maximally concentrated limit distributions which are uniformly attainable at a rate $n^{1/2}L(n)$ are necessarily normal.

The following example is taken from Golubev and Levit (1996).

Example 3.2. Let $\mathfrak{P}_\gamma$ denote the set of all distributions on $\mathbb{B}$, the density p of which admits an analytic continuation to $S_\gamma := \{x + iy : |y| < \gamma\}$ fulfilling certain regularity conditions. Let $\kappa(P) = p^{(m)}(x_0)$, where $p^{(m)}$ is the m-th derivative of p, and x_0 is fixed. Golubev and Levit show that, with the rate $c_n = n^{1/2}(\log n)^{-(2m+1)/2}$,

$$\liminf_{n\to\infty} \sup_{P\in\mathfrak{P}_\gamma} c_n^2 \int (\kappa^{(n)} - \kappa(P))^2 dP^n \geq \sigma^2(P_0),$$

with

$$\sigma^2(P_0) = \frac{p_0(x_0)}{\pi(2m+1)(2\gamma)^{2m+1}}$$

for every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$; and that there is an estimator sequence $\hat{\kappa}^{(n)}$ such that

$$P^n \circ c_n(\hat{\kappa}^{(n)} - \kappa(P)) \Rightarrow N_{(0,\sigma^2(P_0))},$$

uniformly on a sequence of neighbourhoods $\mathfrak{P}_n$ shrinking to P_0, with

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} c_n^2 \int (\hat{\kappa}^{(n)} - \kappa(P))^2 dP^n = \sigma^2(P_0).$$

(See Theorem 1, p. 359 and Theorem 3, p. 360.)

4. The case $\alpha \in (0, 1/2)$

The results available in the literature differ in various respects, such as the neighbourhoods used in the definition of local uniformity, and the loss functions used to measure the quality of estimators. Moreover, not all authors take the uniformity serious if it comes to the question whether a rate bound is attainable or not. This will be illustrated by the following examples.

Example 4.1. Estimating the value of a smooth density at a point.

In Section 3 we mentioned a model, considered by Golubev and Levit (1996*a*), where the uniform rate bound is of the type $n^{1/2}L(n)$. There are numerous papers establishing uniform rate bounds of the type $n^\alpha L(n)$, with $\alpha \in (0, 1/2)$ depending on the kind of smoothness condition.

Of particular interest are results of Ibragimov and Has'minskii insofar as they give uniform rate bounds for the convergence of $\kappa^{(n)}$ to $\kappa(P)$ which are uniformly attained. Let $\mathfrak{P}_{m,c}$ denote the family of all distributions on $\mathbb{R}$, the Lebesgue-density of which admits m derivatives, $m = 0, 1, \ldots$ and for which the m-th derivative fulfills the condition

$$|p^{(m)}(x) - p^{(m)}(y)| \le c|x - y| \quad \text{for} \quad x, y \in \mathbb{R}.$$

The problem is to estimate $p(x_0)$.

According to Ibragimov and Has'minskii ((1981), p. 237, Theorem 5.1) the following relation holds for every estimator sequence $p^{(n)}$ of $p(x_0)$

$$\text{(4.1)} \qquad \liminf_{n\to\infty} \sup_{P\in\mathfrak{P}_{m,c}} \int \ell(n^{(m+1)/(2m+3)}|p^{(n)} - p(x_0)|)dP^n > 0.$$

Moreover (see p. 236, Theorem 4.2), there are estimator sequences $p_0^{(n)}$, $n \in \mathbb{N}$, such that

$$\text{(4.2)} \qquad \limsup_{n\to\infty} \sup_{P\in\mathfrak{P}_{m,c}} \int \ell(n^{(m+1)/(2m+3)}|p_0^{(n)} - p(x_0)|)dP^n < \infty.$$

These theorems hold for a rather large class of loss functions ℓ.

Applied with $\ell(y) = 1_{[u,\infty)}(|y|)$, relation (4.1) yields that $n^{(m+1)/(2m+3)}$ is a rate bound for uniform convergence on $\mathfrak{P}_{m,c}$, for every $c > 0$. (More precisely, (5.2) holds for every $u > 0$.) From relation (4.2), applied with $\ell(y) = |y|$, one obtains with the help of Lemma 5.1 (i) that this rate bound is attained, uniformly on $\mathfrak{P}_{m,c}$.

Example 4.2. Estimating the mode.

Let $\mathfrak{P}$ be the family of all distributions on $\mathbb{R}$ with unimodal density, and let $\kappa(P)$ denote the mode of P. Has'minskii ((1979), p. 94, Corollary) shows that $n^{1/5}$ is a rate bound for uniform convergence on a certain subfamily of $\mathfrak{P}$. The best result now available is Theorem 5.5 in Donoho and Liu ((1991), pp. 653–654) which establishes $n^{1/5}$ as an attainable rate bound for uniform convergence of $\kappa^{(n)}$ to $\kappa(P)$ on the subfamily fulfilling the conditions

$$p(x) \le M \quad \text{for } x \in \mathbb{R} \quad \text{and}$$
$$|p(x) - p(\kappa(P))| \le C(x - \kappa(P))^2 \quad \text{for } |x - \kappa(P)| \le \delta$$

(with M, C and δ fixed).

Eddy ((1980), p. 873, Corollary 2.2) gives asymptotically normal estimator sequences with rates better than $n^{1/5}$. As an example we mention the following: If the density has a 4th bounded derivative, then there exists an estimator sequence which converges at the rate $n^{2/7}$ to a normal distribution (with mean $\neq 0$).

Of particular interest are three models with the following features in common: The uniform rate bound is $n^{1/3}$, and there exists an estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$, such that

$$P^n \circ n^{1/3} K(P)(\kappa^{(n)} - \kappa(P)) \Rightarrow Q_* \quad \text{for} \quad P \in \mathfrak{P}. \tag{4.3}$$

$Q_* \mid \mathbb{B}$ is defined as the distribution of the last time where the process $W(u) - u^2$, $u \in \mathbb{R}$, reaches its maximum. (W denotes the symmetric Brownian motion on $\mathbb{R}$ with $W(0) = 0$.) Q_* is symmetric about 0 and has moments of all orders (see Groeneboom (1989), Corollary 3.4, p. 94). From Proposition 2.1 it is clear that such a limit distribution cannot be uniformly attained if $\{K(P) : P \in \mathfrak{P}\}$ is bounded and bounded away from 0.

Now we consider the three cases in more detail.

Example 4.3. Estimating the value of a decreasing density at a point.

Let $\mathfrak{P}$ be the family of all distributions on $[0, \infty)$ with decreasing density with one bounded derivative. The problem is to estimate $p(x_0)$, for x_0 fixed. According to Kiefer ((1982), Section 2) the rate bound for estimator sequences of $p(x_0)$ uniformly on $\mathfrak{P}$ is $n^{1/3}$ provided $p'(x_0) < 0$. According to Prakasa Rao ((1969), p. 35, Theorem 6.3) the maximum likelihood estimator $p^{(n)}$ attains this rate pointwise. More precisely, relation (4.3) holds with $K(P) = (4p(x_0)|p'(x_0)|)^{-1/3}$.

Using the loss function $\ell(u) = |u|$, Birgé (1987) shows that $n^{1/3}$ is an attainable rate bound (even for finite n) for families of unimodal densities with bounded support.

Example 4.4. The interval censoring problem.

Assume that $(x, \delta) \in [0, \infty) \times \{0, 1\}$ is distributed with density

$$p_{F,G}(x) = F(x)^{\delta}(1 - F(x))^{1-\delta} g(x),$$

where F, G are unknown distribution functions, and f, g the respective densities. The problem is to estimate $F(x_0)$ for x_0 fixed.

According to Groeneboom ((1987), pp. 5–6) or Groeneboom and Wellner ((1992), p. 89, Theorem 5.1) there exists an estimator sequence such that relation (4.3) holds with

$$K(P_{F.G}) = (4F(x_0)(1 - F(x_0)) f(x_0) g(x_0))^{-1/3}.$$

That $n^{1/3}$ is a rate bound in the sense of (5.2) could be shown as in Groeneboom and Wellner ((1992), pp. 20–21). (They show, in fact, that

$$\limsup_{n \to \infty} \inf_{F \in \mathfrak{F}_n} n^{1/3} \int |F^{(n)} - F(x_0)| dP^n_{F,G} > 0,$$

where $\mathfrak{F}_n$ is a certain nonparametric neighbourhood of a given distribution function F_0.)

Example 4.5. The deconvolution problem.

The problem is to estimate the distribution function F of a probability measure $P \mid \mathbb{B}$ at a given point based on n i.i.d. observations from $P * Q$, with Q known. The rate bounds occurring in this problem depend on the regularity conditions imposed on P and Q. We restrict ourselves to the case where the most complete results are available, that is: Q is the uniform distribution on $(0,1)$, the support of P is a subset of $(0,1)$, and P has a positive density f at x_0. According to van Es ((1991), p. 104, Theorem 4.1) there exists an estimator sequence such that relation (4.3) holds with

$$K(P) = (4F(x_0)(1 - F(x_0))f(x_0))^{-1/3}.$$

(In van Es and van Zuijlen ((1996), p. 88, Theorem 1.5) this is proved under the assumption that the support of P is bounded, but not necessarily a subset of $(0,1)$.)

That $n^{1/3}$ is a rate bound follows from van Es, p. 112, Example 4.1, based on the loss function $|u|$.

Example 4.6. Estimation of the extreme value index.

For the sake of simplicity, we consider the following special case. Let $\mathfrak{B}$ denote the class of all functions $b : (0,\infty) \to \mathbb{R}$ fulfilling $b(x) \le x^{-\varrho}$ for $x > 1$, where $\varrho > 0$ is a known constant. For $\gamma > 0$ and $b \in \mathfrak{B}$ let $P_{\gamma,b}$ denote the probability measure on $\mathbb{B}_+$, the distribution function of which is determined by

$$F_{\gamma,b}^{-1}\left(1 - \frac{1}{t}\right) = t^\gamma \exp\left[\int_1^t \frac{b(x)}{x} dx\right] \quad \text{for} \quad t > 1.$$

Given a closed and bounded interval $I \subset (0,\infty)$, let $\mathfrak{P} := \{P_{\gamma,b} : \gamma \in I, b \in \mathfrak{B}\}$. The problem is to estimate γ, the extreme value index. (Notice that both, γ and b, are identifiable.)

According to a result of Hall and Welsh ((1984), p. 1080, Theorem 1), the sequence $c_n = n^{\varrho/(2\varrho+1)}$ is a $\mathfrak{P}$-uniform rate bound for estimator sequences of γ. (See Drees (1998), Theorem 2.1 and Lemma 2.1 for a presentation which fits better into the framework outlined above.) According to Theorem 3 in Hall and Welsh ((1984), p. 1083) this bound is uniformly attained by the Hill estimator, say $\gamma^{(n)}$, based on the order statistics $x_{i:n}$, $i = 1, \ldots, k_n$, with $k_n = [n^{2\varrho/(2\varrho+1)}]$. A more precise assertion about the asymptotic performance of this estimator can be obtained from Csörgö *et al.* (1985). Theorem 1, p. 1053, applied with $\lambda_n = n^{-1/(2\varrho+1)}$ and $K(u) = 1_{(0,1)}(u)$, implies

$$P_{\gamma,b}^n \circ (n^{\varrho/(2\varrho+1)}\gamma^{-1}(\gamma^{(n)} - \gamma) - \mu_n(\gamma,b)) \Rightarrow N_{(0,1)} \tag{4.4}$$

with $\mu_n(\gamma,b) = \gamma^{-1}n^{\varrho/(2\varrho+1)} \int_0^1 b(\frac{n^{1/(2\varrho+1)}}{x})dx$. Because of $|b(x)| \le x^{-\varrho}$, we have

$$|\mu_n(\gamma,b)| \le \frac{1}{\gamma(\varrho+1)}.$$

According to a remark of Drees in the proof of his Theorem 2.2, the convergence in (4.4) is uniform on $\mathfrak{P}$. This implies that $\gamma^{(n)}$, $n \in \mathbb{N}$, attains the $\mathfrak{P}$-uniform rate bound $n^{\varrho/(2\varrho+1)}$. Since this rate bound is of the type n^α with $\alpha = \varrho/(2\varrho+1) \in (0, \frac{1}{2})$, we obtain from Remark 2.3 that there is no estimator sequence $\mu^{(n)}$ such that $\mu^{(n)} - \mu_n(\gamma, b)$, $n \in \mathbb{N}$, converges to 0, uniformly on $\mathfrak{P}$. Hence, in spite of its $\mathfrak{P}$-uniformity, relation (4.4) cannot be used for the computation of confidence intervals with $\mathfrak{P}$-uniform covering probabilities.

5. The concept of a "rate bound"

As outlined in Section 2, a limit distribution can be utilized for the computation of confidence intervals with uniform covering probability on $\mathfrak{P}$ only if it is uniformly attained on $\mathfrak{P}$. In contrast to this, *bounds* for the asymptotic performance of estimator sequences (be it the rate of convergence or the concentration of the limit distibution) depend on local properties of the family $\mathfrak{P}$ and the functional κ. This is the reason for admitting sequences $\mathfrak{P}_n$, $n \in \mathbb{N}$ of probability measures in the following considerations, including sequences $\mathfrak{P}_n \equiv \mathfrak{P}$, as well as sequences $\mathfrak{P}_n$ shrinking to a given $P_0 \in \mathfrak{P}$.

Searching the literature for concepts of an optimal rate (for the convergence of an estimator sequence to a limit distribution) leaves us without a definite answer. It appears that the definitions usually reflect features specific for the model under investigation, that is: The definition is adjusted to what can be proved.

The definition that a rate c_n, $n \in \mathbb{N}$, is attainable seems to be generally accepted.

DEFINITION 5.1. c_n, $n \in \mathbb{N}$, is an *attainable rate* if there exists an estimator sequence $\kappa_0^{(n)}$, $n \in \mathbb{N}$, such that for every sequence $u_n \to \infty$,

$$\lim_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa_0^{(n)} - \kappa(P)| < u_n\} = 1. \tag{5.1}$$

(See Farrell (1972), p. 172, relation (1.4); Stone (1980), p. 1348, relation (1.3); Kiefer (1982), p. 420; Hall and Welsh (1984), Section 3, pp. 1083–1084.) An exception is Stone (1982) who requires (5.1) with $u_n \equiv u_0$, for some $u_0 > 0$. Akahira and Takeuchi ((1995), p. 77, Definition 3.5.1) use for c_n (in a relation equivalent to (5.1)) the term "consistency with order c_n, $n \in \mathbb{N}$". With $c_n = n^{1/2}$, relation (5.1) occurs as uniform $\sqrt{n}$-consistency in Bickel *et al.* ((1993), p. 18, Definition 2).

That $P^n \circ c_n(\kappa_0^{(n)} - \kappa(P))$, $n \in \mathbb{N}$, converges to a limit distribution Q_P, uniformly on $\mathfrak{P}_n$, does not automatically imply that the rate c_n, $n \in \mathbb{N}$, is attained in the sense of Definition 5.1. It follows, if $Q_P(-u_n, u_n) \to 1$ for $u_n \to \infty$ uniformly on $\mathfrak{P}_n$.

The following concept of a rate bound serves a distinct purpose: To show that a certain estimator sequence converges to $\kappa(P)$ at the best possible rate.

DEFINITION 5.2. c_n, $n \in \mathbb{N}$, is a *rate bound* if for every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$, there exists $u_0 > 0$ such that

$$\limsup_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u_0\} < 1. \tag{5.2}$$

If the definition of attainability is almost unequivocally accepted, the situation is different with respect to the concept of a rate bound. Here are a few examples, rewritten with our notations.

Stone ((1980), p. 1348) requires (5.2) for every $u > 0$ and, in addition, his condition (1.2) which is equivalent to the following.

For every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$, and every sequence $u_n \to 0$,

$$\lim_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u_n\} = 0. \tag{5.3}$$

The same condition occurs in Kiefer ((1982), p. 420). Stone ((1982), p. 393) has changed his mind. Now he requires (5.3) for $u_n \equiv u_0$ for some $u_0 > 0$.

Remark 5.1. That condition (5.3) is, in fact, stronger than (5.2) can be easily seen as follows: Let $\kappa^{(n)}$, $n \in \mathbb{N}$, be an estimator sequence such that $P^n \circ c_n(\kappa^{(n)} - \kappa(P)) \Rightarrow Q_P$, a nonatomic distribution. Let k_n, $n \in \mathbb{N}$, tend to infinity slowly enough so that $k_n/n \to 0$, and define

$$\hat{\kappa}^{(n)}(x_1, \ldots, x_n) = \frac{1}{2}\kappa^{(k_n)}(x_1, \ldots, x_{k_n}) + \frac{1}{2}\kappa^{(n-k_n)}(x_{k_n+1}, \ldots, x_n).$$

Let $\hat{c}_n := c_{k_n}$. Then $\hat{c}_n(\hat{\kappa}^{(n)} - \kappa(P))$ fulfills (5.2), but not (5.3) (if $k_n/n \to 0$ implies $c_{k_n}/c_{n-k_n} \to 0$).

The concept of a rate bound as defined by (5.2) is comparatively weak and, above all, it is exactly what one needs to establish the optimality of an attainable rate.

PROPOSITION 5.1. *If c_n, $n \in \mathbb{N}$, is a rate bound, and c'_n, $n \in \mathbb{N}$, an attainable rate, then* $\limsup_{n\to\infty} c'_n/c_n < \infty$.

Remark 5.2. Let $\hat{c}_n$, $n \in \mathbb{N}$, be an attainable rate bound. By Proposition 5.1 we have $\limsup_{n\to\infty} \hat{c}_n/c'_n < \infty$ for any rate bound c'_n, $n \in \mathbb{N}$, and $\limsup_{n\to\infty} c''_n/\hat{c}_n < \infty$ for any attainable rate c''_n. Hence, if $\hat{c}_n$ and c_n, $n \in \mathbb{N}$, are two attainable rate bounds, we have $0 < \liminf \hat{c}_n/c_n$ and $\limsup \hat{c}_n/c_n < \infty$. That means: An attainable rate bound is unique in an asymptotic sense.

PROOF. By definition (5.1) we have

$$\lim_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c'_n|\kappa_0^{(n)} - \kappa(P)| < u_n\} = 1 \quad \text{if} \quad u_n \to \infty.$$

If c_n is a rate bound in the sense of (5.2), there exists $u_0 > 0$ such that

$$\limsup_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa_0^{(n)} - \kappa(P)| < u_0\} < 1.$$

This implies

$$\limsup_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c'_n|\kappa_0^{(n)} - \kappa(P)| < u_0 c'_n/c_n\}$$
$$= \limsup_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa_0^{(n)} - \kappa(P)| < u_0\} < 1,$$

hence $\limsup_{n\to\infty} c'_n/c_n < \infty$. □

Now we discuss a weaker version of (5.2), namely:
For every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$, there exists $u_0 > 0$ such that

$$\liminf_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u_0\} < 1. \tag{5.4}$$

If we consider asymptotic concepts as approximate descriptions of properties for large samples, then it would be enough to have a certain favourable property for an infinite subsequence from which the sample size can be chosen. Hence it would hardly be justifiable to call a rate "optimal" if it can possibly be improved for infinitely many sample sizes. If c_n is a rate bound in the sense of (5.2), i.e. with lim sup, then, according to Proposition 5.1, there is no estimator sequence attaining a rate c'_n such that $\limsup_{n\to\infty} c'_n/c_n = \infty$, i.e. there is no estimator sequence which converges at the better rate c'_n for infinitely many sample sizes. If c_n is a rate bound in the weaker sense of (5.4), then an analogous version of Proposition 5.1 implies that there is no estimator sequence with a rate c'_n such that $\lim_{n\to\infty} c'_n/c_n = \infty$, i.e. no estimator sequence which converges at the better rate for every sample size. If condition (5.4) does not exclude the possibility of an improvement along an infinite subsequence, this does, of course, not imply that such an improvement is feasible. With the stronger condition (5.2) we know for sure that this is impossible.

The problems with "lim sup" versus "lim inf" result from the fact that our general considerations refer to arbitrary rate-sequences (say a sequence $c_n = n^{1/2}$ for $n = 2m$ and $c_n = n^{1/4}$ for $n = 2m+1$). We have no natural example for which an improved rate along an infinite subsequence of sample sizes is possible, whereas an improved rate for all sample sizes is not. To demonstrate that the distinction between "lim sup" and "lim inf" is not totally vacuous, we consider the following artificial:

Example 5.1. Assume $\mathfrak{P}_n \downarrow P_0$. Let $\kappa_0^{(n)}$ be an estimator sequence which is asymptotically optimal in the sense that for every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$,

$$\limsup_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{n^{1/2}|\kappa^{(n)} - \kappa(P)| < u\} \tag{5.5}$$
$$\le \lim_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{n^{1/2}|\kappa_0^{(n)} - \kappa(P)| < u\} = Q_{P_0}(-u, u)$$

for every $u > 0$.

Let $c_n = n^{1/2}$ for $n = 2m$, and $c_n = n^{1/4}$ for $n = 2m+1$. From (5.6), we have for every estimator sequence $\kappa^{(n)}$

$$\liminf_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u\} \leq Q_{P_0}(-u,u) < 1$$

for every $u > 0$. Hence c_n, $n \in \mathbb{N}$, is a rate bound in the sense of definition (5.4). As against that,

$$\limsup_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa_0^{(n)} - \kappa(P)| < u\} = 1$$

for every $u > 0$, so that c_n, $n \in \mathbb{N}$, is not a rate bound in the sense of definition (5.2). The rate bound c_n can be improved at the subsequence $n = 2m+1$, from $c_n = n^{1/4}$ to $c_n = n^{1/2}$.

So far, we have justified our favourite definition (5.2) by its usefulness for establishing the optimality of a certain attainable rate. We conclude this section by the following proposition which helps to better understand the meaning of condition (5.2) from the intuitive point of view.

PROPOSITION 5.2. *c_n, $n \in \mathbb{N}$, is a rate bound in the sense of definition* (5.2) *if and only if the following holds true*:

There is no estimator sequence $\kappa^{(n)}$ such that for some subsequence $\mathbb{N}_0 \subset \mathbb{N}$

$$\lim_{n\in\mathbb{N}_0} \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u\} = 1 \qquad \textit{for every } u > 0.$$

PROOF. Let $F_n : (0,\infty) \to [0,1]$ be a sequence of nondecreasing functions. Then

$$\limsup_{n\to\infty} F_n(u) = 1 \qquad \text{for every } u > 0 \tag{5.6}$$

is equivalent to the following:

There exists an infinite subset $\mathbb{N}_0 \subset \mathbb{N}$ such that

$$\lim_{n\in\mathbb{N}_0} F_n(u) = 1 \qquad \text{for every } u > 0. \tag{5.7}$$

Since $\lim_{n\in\mathbb{N}_0} F_n(u) \leq \limsup_{n\to\infty} F_n(u)$ for every $u > 0$, (5.7) implies (5.6). By (5.6), for every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$, $n_m > n_{m-1}$, such that $F_{n_m}(\frac{1}{m}) > 1 - 1/m$. Since F_n is nondecreasing, (5.7) holds with $\mathbb{N}_0 = \{n_m : m \in \mathbb{N}\}$.

Applied with $F_n(u) = \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u\}$, this yields the assertion. □

Remark 5.3. Some authors characterize rate bounds in a slightly different way. Farrell ((1972), p. 173) and Hall and Welsh ((1984), p. 1080) define c_n, $n \in \mathbb{N}$, as a rate bound if for every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$, and every sequence a_n, $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \inf_{P\in\mathfrak{P}_n} P^n\{|\kappa^{(n)} - \kappa(P)| < a_n\} = 1 \tag{5.8}$$

implies $\lim_{n\to\infty} c_n a_n = \infty$.

It is easy to see that this definition is equivalent to a slightly stronger version of (5.2), the one with "for every $u > 0$" in place of "for some $u_0 > 0$". The original version of (5.2) is equivalent to (5.8) with $\liminf_{n\to\infty} c_n a_n > 0$, the weaker version (5.4) is equivalent to $\limsup_{n\to\infty} c_n a_n > 0$. (Hint: It is convenient to prove such relations for arbitrary sequences of nondecreasing functions $G_n : (0,\infty) \to [0,1]$ and to apply them for $G_n(u) := \inf_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| < u\}$.)

In Section 4, Theorem 2.1 has been used to show that certain limit distributions are not uniformly attainable. This impossibility result depends on the definition of a uniform rate bound given by (5.2), i.e. one expressed in terms of probabilities. Many results on rates available in literature are useless for our purposes since they are based on loss functions. As an example we mention Ibragimov and Has'minskii ((1981), Sections iv 4 and 5), where $\ell : [0,\infty) \to [0,\infty)$ is symmetric about 0 and nondecreasing on $(0,\infty)$. Based on a given loss function, the common definitions are

c_n, $n \in \mathbb{N}$, is a *rate bound*, if for every estimator sequence $\kappa^{(n)}$, $n \in \mathbb{N}$,

$$\liminf_{n\to\infty} \sup_{P\in\mathfrak{P}_n} \int \ell(c_n(\kappa^{(n)} - \kappa(P)))dP^n > 0 \tag{5.9}$$

c_n, $n \in \mathbb{N}$, is an *attainable rate* if there exists an estimator sequene $\kappa_0^{(n)}$, $n \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \sup_{P\in\mathfrak{P}_n} \int \ell(c_n(\kappa_0^{(n)} - \kappa(P)))dP^n < \infty. \tag{5.10}$$

With an arbitrary loss function it is not even clear that rates c_n' with $c_n'/c_n \to \infty$ are not attainable if c_n is a rate bound. Hence we restrict ourselves to mentioning the following relations which might be useful in connection with our definitions (5.1) and (5.2).

LEMMA 5.1. *For $\ell(u) = |u|$ the following is true:*
(i) (5.10) *implies* (5.1),
(ii) (5.2) *implies* (5.9).

PROOF. The assertions follow easily from

$$\sup_{P\in\mathfrak{P}_n} c_n \int |\kappa^{(n)} - \kappa(P)|dP^n = \sup_{P\in\mathfrak{P}_n} \int P^n\{c_n|\kappa^{(n)} - \kappa(P)| > u\}du$$
$$\geq u_0 \sup_{P\in\mathfrak{P}_n} P^n\{c_n|\kappa^{(n)} - \kappa(P)| > u_0\}$$

for every $n \in \mathbb{N}$ and $u_0 > 0$. □

6. Lemmas and proofs

PROOF OF THEOREM 2.1. With $Q_{P,k}$ defined by (2.6), let

$$\delta_k := \sup_{P\in\mathfrak{P}_k} D(Q_{P,k}, Q_P). \tag{6.1}$$

With a distance function D fulfilling (2.2′) and (2.2″) we have

$$\sup_{P\in\mathfrak{P}_k} D(Q_{P,k}^{*m}, Q_P^{*m}) \le m\delta_k \quad \text{for} \quad m,k\in\mathbb{N}. \tag{6.2}$$

Since $\lim_{k\to\infty}\delta_k = 0$ by assumption, there exists $k_1(m)\in\mathbb{N}$, $m\in\mathbb{N}$, such that $m\delta_k \le 1/m$ for $k \ge k_1(m)$. Since L is slowly varying, the "Uniform Convergence Theorem" for slowly varying functions (see e.g. Seneta (1976), p. 2, Theorem 1.1) implies that $\frac{L(\lambda k)}{L(k)} \to 1$ as $k\to\infty$, uniformly for $\lambda\in[m,2m]$. Hence there exists $k_2(m)$ such that

$$\left|\frac{L(\lambda k)}{L(k)} - 1\right| < 1/m \quad \text{for} \quad \lambda\in[m,2m] \text{ and } k \ge k_2(m). \tag{6.3}$$

Let now $k(m) := \max\{k_1(m), k_2(m)\}$. W.l.g. we may choose $k(m+1) > k(m)$ for $m\in\mathbb{N}$.

For $n\in\mathbb{N}$ let m_n be defined as the largest $m\in\mathbb{N}$ such that $mk(m)\le n$, and let $r_n\in\{0,1,\ldots\}$ be the largest r such that $m_n(k(m_n)+r)\le n$. Let $k_n := k(m_n)+r_n$. We have

$$m_n k_n \le n < m_n(k_n+1). \tag{6.4}$$

Notice that m_n as well as k_n, $n\in\mathbb{N}$, tend to infinity. Since $k_n \ge k(m_n)$ we have $m_n\delta_{k_n} \le 1/m_n$. Using $\mathfrak{P}_n \subset \mathfrak{P}_{k_n}$, we obtain from (6.2) that

$$\lim_{n\to\infty}\sup_{P\in\mathfrak{P}_n} D(Q_{P,k_n}^{*m_n}, Q_P^{*m_n}) = 0. \tag{6.5}$$

For each n, we partition $x_1,\ldots,x_n$ into m_n groups of size k_n, and define

$$\hat\kappa^{(n)}(x_1,\ldots,x_n) := m_n^{-1}\sum_{i=1}^{m_n}\kappa^{(k_n)}(x_{(i-1)k_n+1},\ldots,x_{ik_n}). \tag{6.6}$$

In the definition of $\hat\kappa^{(n)}$, only the variables $(x_1,\ldots,x_{m_nk_n})$ are used. Observe that the number of variables neglected, i.e. $x_{m_nk_n+1},\ldots,x_n$, is smaller than m_n (since $n < m_n(k_n+1)$), hence a vanishing fraction of n: We have $m_n/n < 1/k(m_n)$.

The following relations hold for any $\beta\in(0,1/2]$. Below they will be applied for $\beta=\alpha$ and $\beta=1/2$. From (6.6),

$$\begin{aligned} &\hat c_n(\hat\kappa^{(n)}(x_1,\ldots,x_n) - \kappa(P)) \\ &\quad = m_n^{-(1-\beta)}\sum_{i=1}^{m_n} c_{k_n}(\kappa^{(k_n)}(x_{(i-1)k_n+1},\ldots,x_{ik_n}) - \kappa(P)), \end{aligned} \tag{6.7}$$

with $\hat{c}_n := m_n^\beta k_n^\alpha L(k_n)$. Hence

$$c_n/\hat{c}_n = \vartheta_n m_n^{\alpha-\beta}, \tag{6.8}$$

with

$$\vartheta_n = n^\alpha m_n^{-\alpha} k_n^{-\alpha} L(n) L(k_n)^{-1}. \tag{6.9}$$

Relation (6.4) implies

$$1 \le n^\alpha m_n^{-\alpha} k_n^{-\alpha} < (1 + k_n^{-1})^\alpha. \tag{6.10}$$

Since $k_n \ge k(m_n)$, relation (6.3) implies

$$\left| \frac{L(\lambda k_n)}{L(k_n)} - 1 \right| < 1/m_n \quad \text{for} \quad \lambda \in [m_n, 2m_n].$$

Applied with $\lambda = n/k_n$ we obtain

$$\left| \frac{L(n)}{L(k_n)} - 1 \right| < 1/m_n. \tag{6.11}$$

From (6.9)–(6.11),

$$\lim_{n\to\infty} \vartheta_n = 1. \tag{6.12}$$

Let

$$H_m(u_1, \ldots, u_m) := m^{-(1-\beta)} \sum_{i=1}^m u_i.$$

From (6.5) we obtain

$$\lim_{n\to 0} \sup_{P \in \mathfrak{P}_n} D(Q_{P,k_n}^{m_n} \circ H_{m_n}, Q_P^{m_n} \circ H_{m_n}) = 0.$$

Together with (6.7) this implies

$$\lim_{n\to\infty} \sup_{P \in \mathfrak{P}_n} D(P^n \circ \hat{c}_n(\hat{\kappa}^{(n)} - \kappa(P)), Q_P^{m_n} \circ H_{m_n}) = 0. \tag{6.13}$$

Ad (i). Assume now that $\alpha \in (0, 1/2)$ and that $|u|^{1/(1-\alpha)}$ is Q_P-integrable, uniformly on $\mathfrak{P}$. From Lemma 6.2, applied with $Q = Q_P$, $\mathfrak{Q} := \{Q_P : P \in \mathfrak{P}\}$ and $r = 1/(1-\alpha)$, we obtain from (6.20) that

$$\lim_{m\to\infty} \sup_{P \in \mathfrak{P}} Q_P^m \left\{ \left| m^{-(1-\alpha)} \sum_{i=1}^m u_i \right| > u \right\} = 0 \quad \text{for every } u > 0.$$

For $\beta = \alpha$ we therefore obtain from (6.13) that

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} P^n\{\hat{c}_n|\hat{\kappa}^{(n)} - \kappa(P)| > u\} = 0 \quad \text{for every } u > 0. \tag{6.14}$$

Since $\beta = \alpha$ implies $c_n/\hat{c}_n = \vartheta_n \to 1$, this relation holds with $\hat{c}_n$ replaced by c_n, which is (2.9).

Ad (ii). Assume now that u^2 is Q_P-integrable, uniformly on $\mathfrak{P}$. From Lemma 6.1, applied with $Q = Q_P$ and $\mathfrak{Q} := \{Q_P : P \in \mathfrak{P}\}$, we obtain from (6.18) and (6.13) for $\beta = \frac{1}{2}$ that

$$\lim_{n\to\infty} \sup_{P\in\mathfrak{P}_n} D(P^n \circ \hat{c}_n(\kappa^{(n)} - \kappa(P)), N_{(0,\sigma^2(P))}) = 0. \tag{6.15}$$

Since (see (6.8) and (6.12)) $m_n^{1/2-\alpha} c_n/\hat{c}_n \to 1$, this proves (2.11). □

LEMMA 6.1. *Let $\mathfrak{Q}$ be a family of distributions $Q \mid \mathbb{B}$ with the following properties.*

$$\int u Q(du) = 0 \quad \textit{for} \quad Q \in \mathfrak{Q}, \tag{6.16}$$

u^2 is uniformly integrable on $\mathfrak{Q}$, and

$$\inf_{Q\in\mathfrak{Q}} \sigma^2(Q) > 0. \tag{6.17}$$

Then the distribution of $m^{-1/2}\sum_{i=1}^m u_i$ under Q^m converges to $N_{(0,\sigma^2(Q))}$, uniformly on $Q \in \mathfrak{Q}$, i.e.

$$\lim_{m\to\infty} \sup_{Q\in\mathfrak{Q}} D\left(Q^m\left((u_1,\ldots,u_m) \to m^{-1/2}\sum_{i=1}^m u_i\right), N_{(0,\sigma^2(Q))}\right) = 0. \tag{6.18}$$

PROOF. From Theorem 5.6 in Petrov ((1995), p. 151), applied with

$$g(u) = \tau_m 1_{[\tau_m,\infty)}(|u|) + |u| 1_{(0,\tau_m)}(|u|)$$

we obtain that, with a universal constant A,

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left| Q^m\left\{ m^{-1/2}\sum_{i=1}^m u_i \le t\right\} - N_{(0,\sigma^2(Q))}(-\infty, t]\right| \\
&\quad \le A\left(\frac{1}{\sigma^2(Q)}\int u^2 1_{[\tau_m,\infty)}(|u|)Q(du) \right.\\
&\qquad \left. + \frac{m^{-1/2}}{\sigma^3(Q)}\int |u|^3 1_{[0,\tau_m)}(|u|)Q(du)\right)\\
&\quad \le A\left(\frac{1}{\sigma^2(Q)}\int u^2 1_{[\tau_m,\infty)}(|u|)Q(du) + \frac{m^{-1/2}\tau_m}{\sigma(Q)}\right).
\end{aligned}$$

The assertion follows for $\tau_m = m^{1/4}$, say. □

The following is a uniform version of (the easier part of) the Lemma of Marcinkiewicz and Zygmund (see Chow and Teicher (1978), p. 122, Theorem 2).

LEMMA 6.2. *Let $\mathfrak{Q}$ be a family of distributions $Q \mid \mathbb{B}$ with the following properties.*

$$\int uQ(du) = 0 \quad \text{for} \quad Q \in \mathfrak{Q}, \tag{6.19}$$

and $|u|^r$ is uniformly integrable on $\mathfrak{Q}$ for some $r \in (1,2)$.

Then $m^{-1/r}\sum_{i=1}^m u_i$, $m \in \mathbb{N}$, converges under Q^m stochastically to 0, uniformly on $\mathfrak{Q}$, i.e.

$$\lim_{m\to\infty} \sup_{Q\in\mathfrak{Q}} Q^m\left\{\left|m^{-1/r}\sum_{i=1}^m u_i\right| > u\right\} = 0 \quad \text{for every } u > 0. \tag{6.20}$$

PROOF. Since

$$\lim_{m\to\infty} \sup_{Q\in\mathfrak{Q}} \varepsilon^{-r}\int |u|^r 1_{[m^{1/r}\varepsilon,\infty)}(|u|)Q(du) = 0$$

for every $\varepsilon > 0$, there exists $\varepsilon_m \downarrow 0$ such that

$$\lim_{m\to\infty} \sup_{Q\in\mathfrak{Q}} \varepsilon_m^{-r}\int |u|^r 1_{(m^{1/r}\varepsilon_m,\infty)}(|u|)Q(du) = 0.$$

Let

$$h_m(u) := u1_{[0,m^{1/r}\varepsilon_m]}(|u|).$$

We have

$$\mu_m(Q) := \int h_m(u)Q(du) = -\int u1_{(m^{1/r}\varepsilon_m,\infty)}(|u|)Q(du),$$

hence

$$\begin{aligned}|\mu_m(Q)| &\le \int |u|1_{(m^{1/r}\varepsilon_m,\infty)}(|u|)Q(du)\\ &\le m^{-(r-1)/r}\varepsilon_m^{-(r-1)}\int |u|^r 1_{(m^{1/r}\varepsilon_m,\infty)}(|u|)Q(du).\end{aligned}$$

This implies

$$\lim_{m\to\infty} m^{(r-1)/r}\sup_{Q\in\mathfrak{Q}}|\mu_m(Q)| = 0.$$

Hence

$$\lim_{m\to\infty} \sup_{Q\in\mathfrak{Q}}\left|m^{-1/r}\sum_{i=1}^m h_m(u_i) - m^{-1/r}\sum_{i=1}^m (h_m(u_i) - \mu_m(Q))\right| = 0. \tag{6.21}$$

Moreover,

$$Q^m\left\{\sum_{i=1}^m u_i \neq \sum_{i=1}^m h_m(u_i)\right\} \le mQ\{|u| > m^{1/r}\varepsilon_m\}$$
$$\le \varepsilon_m^{-r}\int |u|^r 1_{(m^{1/r}\varepsilon_m,\infty)}(|u|)Q(du).$$

Hence

$$\lim_{m\to\infty}\sup_{Q\in\mathfrak{Q}} Q^m\left\{\sum_{i=1}^m u_i \neq \sum_{i=1}^m h_m(u_i)\right\} = 0. \tag{6.22}$$

The assertion follows from (6.21) and (6.22) if we show that $m^{-1/r}\sum_{i=1}^m (h_m(u_i) - \mu_m(Q))$, $m \in \mathbb{N}$, converges under Q^m stochastically to 0, uniformly on $\mathfrak{Q}$. We have

$$m^{1-2/r}\int (h_m(u) - \mu_m(Q))^2 Q(du)$$
$$\le m^{(r-2)/r}\int (h_m(u))^2 Q(du)$$
$$= m^{(r-2)/r}\int u^2 1_{[0,m^{1/r}\varepsilon_m]}(|u|)Q(du)$$
$$\le \varepsilon_m^{2-r}\int |u|^r Q(du),$$

hence

$$\lim_{m\to\infty} m^{1-2/r}\sup_{Q\in\mathfrak{Q}}\int (h_m(u) - \mu_m(Q))^2 Q(du) = 0.$$

Therefore, $m^{-1/r}\sum_{i=1}^m (h_m(u_i) - \mu_m(Q))$, $m \in \mathbb{N}$, converges to 0 in quadratic mean, hence also stochastically, uniformly on $\mathfrak{Q}$. □

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