Ann. Inst. Statist. Math.
Vol. 51, No. 4, 731–747 (1999)

ERROR BOUNDS FOR ASYMPTOTIC EXPANSION OF THE CONDITIONAL VARIANCE OF THE SCALE MIXTURES OF THE MULTIVARIATE NORMAL DISTRIBUTION

STERGIOS B. FOTOPOULOS AND LIJIAN HE

Department of Management and Systems, and the Program in Statistics, Washington State University, Pullman, WA 99164-4726, U.S.A.

(Received July 7, 1997; revised February 9, 1998)

Abstract. Let $\boldsymbol{X} = A^{1/2}\boldsymbol{G}$ be a scale mixture of a multivariate normal distribution with $\boldsymbol{X}, \boldsymbol{G} \in \mathbb{R}^n$, $\boldsymbol{G}$ is a multivariate normal vector, and A is a positive random variable independent of the multivariate random vector $\boldsymbol{G}$. This study presents asymptotic results of the conditional variance-covariance, $\mathrm{Cov}(\boldsymbol{X}_2 \mid \boldsymbol{X}_1)$, $\boldsymbol{X}_1 \in \mathbb{R}^m$, $m < n$, under some moment expressions. A new representation form is also presented for conditional expectation of the scale variable on the random vector $\boldsymbol{X}_1 \in \mathbb{R}^m$, $m < n$. Both the asymptotic expression and the representation are manageable and in computable form. Finally, an example is presented to illustrate how the computations are carried out.

Key words and phrases: Heteroscedasticity, orthogonal polynomials, Laguerre polynomials, Laplace transform.

1. Introduction

The problem of approximating the scale mixtures of normal distributions has received a lot of interest the last decades. Keilson and Steutel (1974) established moment measures of the distance of mixtures from its parent distribution and showed that Pearson's coefficient of kurtosis plays an important role as a metric. Heyde (1975) and Heyde and Leslie (1976) studied the same properties in greater detail and related the moment measures of distance to more familiar uniform measures. Using a more unified approach, Hall (1979) sharpened Heyde and Leslie's results by reducing a universal constant value. Shimizu (1987, 1995) generalized these results by providing Hermite-type expansion of these mixtures. Within the same framework Fujikoshi and Shimizu (1989) obtained a Hermite-type expansion of multivariate mixture distribution when the scale is distributed in a neighborhood of one.

This article considers expansions of conditional variances for scale mixtures of normal distributions. We accomplish these expansions by adopting ideas found in Shimizu (1987). To state the problem, we assume that $\boldsymbol{X} \in \mathbb{R}^n$, $n \geq 2$ is a (non-degenerate) random vector expressed by the stochastic representation $\boldsymbol{X} = A^{1/2}\boldsymbol{G}$.

It is assumed that A is a positive non-degenerate random variable independent of the n-dimensional Gaussian random (column) vector $\boldsymbol{G}$. The random vector $\boldsymbol{G}$, however, has mean 0 and a positive definite covariance matrix Σ. Cambanis *et al.* (1997) have shown that $\mathrm{Cov}(\boldsymbol{X}_2 \mid \boldsymbol{X}_1 = \boldsymbol{x}_1) = E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1]\ \Sigma_{2|1}$, where $\Sigma_{2|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ with $\boldsymbol{x}_1$ and $\boldsymbol{G}_1$ are m-dimensional ($m < n$) and Σ_{11} is $m \times m$-dimensional. For example, Σ_{11} is the covariance matrix of $\boldsymbol{G}_1$ etc. It is clear that scale mixtures of normal distributions do not have degenerate conditional variances, as in the normal theory, and so provide heteroscedastic examples. Cambanis *et al.* (1997) and Fotopoulos and He (1997) have studied various properties of this conditional variance and obtained several expressions with respect to the moments and/or Laplace transform of A. Their work provided a mathematical development for the behavior of the scale mixtures of multivariate normal distributions under conditioning. In the same spirit, we now continue to investigate the possibility of expanding $E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1]$ in terms of the moments of A and the confluent hyper-geometric functions. We provide explicit ratio expansions and find bounds for the error terms. Even though some of the intermediate calculations become a little tedious, the new expressions derived here are manageable and in a computable form.

Throughout this work we shall use vector notation. The $x \wedge 1 = \min(1, x)$ and $x \vee 1 = \max(1, x)$. The organization of the paper is as follows. The actual expressions of the conditional expectation are introduced in Section 2, where the main results are stated and various conclusions drawn. The proofs of the theorems are deferred to Section 3. In Section 4 we present an example and illustrate how the computations are carried out. Section 5 provides an overview of Laguerre and Hermite polynomials, which are connected with the main results.

2. Background and results

2.1 *Using Laplace expressions*

Cambanis *et al.* (1997) have shown that if the Laplace transform of the scale random variable A satisfies

$$(2.1)\quad \int_{[0,\infty)} u^{m/2-1} E[e^{-uA}]du < \infty \quad \text{and} \quad \int_{[0,\infty)} u^{m/2-1} E[Ae^{-uA}]du < \infty,$$

then for $m = 1$

$$(2.2)\quad E[A \mid X_1 = x_1] = E\left[A \int_0^\infty e^{-t^2A/2} \cos(x_1 t/\sigma_1)dt\right] \Big/ E\left[\int_0^\infty e^{-t^2A/2} \cos(x_1 t/\sigma_1)dt\right],$$

and for $m > 1$

$$(2.3)\quad E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] = E\left[A \int_0^\infty t^{m/2} e^{-t^2A/2} J_{(m-2)/2}(\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}} t)dt\right] \Big/ E\left[\int_0^\infty t^{m/2} e^{-t^2A/2} J_{(m-2)/2}(\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}} t)dt\right].$$

Evaluating (2.2) and/or (2.3) can be a very difficult task. Thus, it is proposed to provide an approximation expression in place of (2.2) and (2.3), which will, of course, be both manageable and in a computable form.

It is clear that $f(A) = e^{-t^2 A/2}$ has absolutely continuous derivatives of any order on any finite segment $[a, b] \subset (0, \infty)$. Based on this information and the assumption that $\frac{A}{E[A]}$ is close to one, (clarification of the closeness to one will be displayed in Theorems 3 and 4) the conditional $E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1]$ is approximated as follows. Conveniently, throughout the results we shall write

$$\bar{a}_i = E[A(A/E[A]-1)^i], \qquad a_i = E[(A/E[A]-1)^i],$$
$$\text{for} \quad i \geq 1, \ \text{and } \bar{a}_0 = a_0 = 1.$$

Then the following theorem is in order.

THEOREM 1. *If $m > 1$ and if the Laplace transform of the scale random variable A satisfies (2.1) and $\Delta_{m,k,j} = E[\frac{A^{2-j}}{(A/E[A]\wedge 1)^{(m+1)/4}}|A/E[A] \vee 1 - 1|^k] < \infty$ for some $k \in \mathbb{N}$ and $j = 1$ or 2, then the following expansion is in order,*

$$\begin{aligned} E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] &= \frac{\sum_{j=0}^{k-1} \bar{a}_j L_j^{(m/2-1)}(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]}{\sum_{j=0}^{k-1} a_j L_j^{(m/2-1)}(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A])} + \epsilon(\boldsymbol{x}_1, A) \\ &= \frac{\bar{L}_k^{(m/2-1)}(\boldsymbol{x}_1, A)}{L_k^{(m/2-1)}(\boldsymbol{x}_1, A)} + \epsilon(\boldsymbol{x}_1, A), \qquad \textit{say}, \end{aligned}$$

where $L_j^{(c)}(x)$, $j \geq 0$, c is independent of j, and $x \geq 0$, are the generalized Laguerre or Sonine polynomials, and

$$|\epsilon(\boldsymbol{x}_1, A)| \leq C(k, \boldsymbol{x}_1, A)\Delta_{m,k,2},$$

where $C(k, \boldsymbol{x}_1, A)$ is a positive constant depending only on k, $\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}$, and some moments of A. Specifically,

$$\begin{aligned} &C(k, \boldsymbol{x}_1, A) \\ &= \frac{|\bar{L}_k^{(m/2-1)}(\boldsymbol{x}_1, A)|}{L_k^{(m/2-1)}(\boldsymbol{x}_1, A)^2} \frac{2^{(m-1)/4}\Gamma\left(\frac{3m-2}{4} + k\right) E[A]^{(m-1)/4}}{k!\Gamma\left(\frac{m}{2}\right)\|\boldsymbol{x}_1\|^{(m-1)/4}_{\Sigma_{11}^{-1}}} \binom{\frac{m}{2}+k}{k} \\ &\quad \cdot \exp(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]). \end{aligned}$$

The next theorem copes with the situation where $m = 1$.

THEOREM 2. *If $m = 1$ and if the Laplace transform of the scale random variable A satisfies (2.1) and $\Delta_{k,j} = E[\frac{A^{2-j}}{(A/E[A]\wedge 1)^{1/2}}|A/E[A] \vee 1 - 1|^k] < \infty$ for* *some $k \in \mathbb{N}$ and $j = 1$ or 2, then the following expansion is in order,*

$$E[A \mid X_1 = x_1] = \frac{\sum_{j=0}^{k-1} \frac{\bar{a}_j}{j!2^j} H_{2j}(x_1/\sigma_1(2E[A])^{1/2})}{\sum_{j=0}^{k-1} \frac{a_j}{j!2^j} H_{2j}(x_1/\sigma_1(2E[A])^{1/2})} + \epsilon(x_1, A)$$
$$= \frac{\bar{H}_k(x_1, A)}{H_k(x_1, A)} + \epsilon(x_1, A), \qquad \text{say,}$$

where $H_i(x)$, $i \geq 0$, are the Hermite polynomials of degree i and $x > 0$, and

$$|\epsilon(x_1, A)| \leq C(k, x_1, A)\Delta_{k,2},$$

where $C(k, x_1, A)$ is a positive constant depending only on k, x_1, and some moments of A. Specifically

$$C(k, x_1, A) = \frac{|\bar{H}_k(x_1, A)}{H_k(x_1, A)^2} \frac{\sqrt{\pi}}{(2E[A])^{1/2}} \frac{1}{2^k} \binom{2k}{k} \exp(x_1^2/2\sigma_1^2 E[A]).$$

Remarks. 1. Note that if $\frac{A}{E[A]}$ becomes close to one, then $A_{m,k,2}$ $(m \geq 1)$ becomes small as $k \in \mathbb{N}$ increases. We may omit the error term presented in Theorems 1 and 2 in calculating the conditional expectation.

2. Observe that, for $x > 0$,

$$L_j^{(m/2-1)}(x) = \binom{\frac{m}{2} + j}{j} M\left(-j, \frac{m}{2}; x\right) \tag{2.4}$$
$$= \binom{\frac{m}{2} + j}{j} {}_1F_1\left(-j, \frac{m}{2}; x\right), \qquad \text{and}$$
$$L_j^{(m/2-1)}(x) = \sum_{i=0}^{j} \binom{j + \frac{m}{2} - 1}{j - i} \frac{(-x)^i}{i!},$$

where ${}_1F_1(a, b, x)$ is the confluent hyper-geometric function and $M(a, b, x)$ is the Kummer's function (see e.g. Rainville (1960), p. 203). Thus, the expression in Theorem 1 may be presented either with respect to the confluent hyper-geometric function or in terms of the Kummer's function.

3. Computing the expressions of the main terms and computing the bounds of the error term of the conditional expectation, $E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1]$, is now a routine work. The example in the next section indicates various steps needed to compute both $\bar{a}_i$ and a_i, for $i \geq 1$. Moreover, it can be found that both the Hermite and Laguerre functions are part of most popular mathematical packages. In light of this, we may now claim that these new results add more insight about the behavior of the equations (2.2) and (2.3).

Based on the knowledge presented in Section 5, it is now clear why the quantity $\frac{A}{E[A]}$ needs to be close to one. Furthermore, with the background developed in the same section, we alternatively furnish a new representation formula for the conditional expectation of A given $\boldsymbol{X}_1 = \boldsymbol{x}_1$ for both $m \geq 2$ and in Theorem 4 for $m = 1$.

THEOREM 3. *If $m > 1$ and if the Laplace transform of the scale random variable A satisfies* (2.1), *and if the sequences $\{\bar{a}_n\}_{n=0}^{\infty} = \{E[A(A/E(A)-1)^n]\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty} = \{E[(A/E[A]-1)^n]\}_{n=0}^{\infty}$ satisfy*

$$\lambda_0 = \max\left\{0, -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\max\{\log|\bar{a}_n|, \log|a_n|\}\right\} < \infty,$$

then

$$E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] = \sum_{j=0}^{\infty}\bar{a}_j L_j^{(m/2-1)}(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A])/\sum_{j=0}^{\infty}a_j L_j^{(m/2-1)}(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]),$$

on every compact subset $\Delta(\lambda_0)$ of $\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]$.

THEOREM 4. *If $m = 1$ and if the Laplace transform of the scale random variable A satisfies* (2.1), *and if the sequences $\{\bar{a}_n\}_{n=0}^{\infty} = \{E[A(A/E(A)-1)^n]\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty} = \{E[(A/E[A]-1)^n]\}_{n=0}^{\infty}$ satisfy*

$$\tau_0 := \max\left\{0, -\limsup_{n\to\infty}(2n+1)^{-1/2}\max\{\log|(2n/e)^{n/2}a_n|, \log|(2n/e)^{n/2}\bar{a}_n|\}\right\} < \infty,$$

then

$$E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] = \sum_{j=0}^{\infty}\frac{\bar{a}_j}{j!2^j}H_{2j}(x_1/\sigma_1(2E[A])^{1/2})/\sum_{j=0}^{\infty}\frac{a_j}{j!2^j}H_{2j}(x_1/\sigma_1(2E[A])^{1/2})$$

on every compact subset $S(\tau_0)$ of $x_1/\sigma - 1(2E[A])^{1/2}$.

Remark. In view of Theorem 2, the moments $E[A(A/E[A]-1)^j]$, and $E[(A/E[A]-1)^j]$ for any $j \in \mathbb{N}$, are then uniquely determined by the expansions presented in the numerator and the denominator, respectively, of both Theorems 3 and 4.

2.2 *Using Moments Expression*

Cambanis *et al.* (1997) have also shown that if $m \geq 2$, or if $m = 1$ and $E[A^{1/2}] < \infty$, then

$$(2.5)\quad E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] = E[A^{-m/2+1}\exp(-\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2A)]/E[A^{-m/2}\exp(-\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2A)].$$

Here again, we are concerned with an asymptotic expansion of this conditional expectation, under the assumption that the unconditional moments of the scale variable exist. Conveniently, for the remaining results we shall write

$$\bar{\beta}_i = E[A(A-E[A])^i], \qquad \beta_i = E[(A-E[A])^i], \qquad \text{for} \quad i \geq 1, \ \text{and} \ \bar{\beta}_0 = \beta_0 = 1.$$

The proposed result is then formulated as follows:

THEOREM 5. *If $m > 1$, or if $m = 1$ and $E[A^{1/2}] < \infty$, and if the scale random variable A satisfies $\Delta_{k,r,m,j} = E[\frac{A(A/E[A]\vee 1)^{r-1}}{(A/E[A]\wedge 1)^{m/2-1}}|A-E[A]|^k] < \infty$ for some $k \in \mathbb{N}$, $r = 1, \ldots, k$, and $j = 1$ or 2, then the following ratio expansion is in order,*

$$\begin{aligned} E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] &= \frac{1 + (\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2)^{-m/2+1} \sum_{j=1}^{k-1} \frac{\bar{\beta}_j}{j!} \lambda_{j,m/2}(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}, A)}{1 + (\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2)^{-m/2+1} \sum_{j=2}^{k-1} \frac{\beta_j}{j!} \lambda_{j,m/2}(\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}, A)} + \epsilon(\boldsymbol{x}_1, A) \\ &= \frac{\bar{\Lambda}_{m/2}(\boldsymbol{x}_1, A)}{\Lambda_{m/2}(\boldsymbol{x}_1, A)} + \epsilon(\boldsymbol{x}_1, A), \qquad \textit{say,} \end{aligned}$$

where, for $(s)_i = \Gamma(s+i)/\Gamma(s)$,

$$\lambda_{j,s}(y,a) = \sum_{i=0}^{j-1} \binom{j-1}{i} (-1)^{i+1} i! L_{j-1,s}\left(\frac{y}{2a}\right) a^{-i-1}, \qquad \textit{and}$$

$$L_{j,s}(y) = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} (s)_i y^{-i}, \qquad y > 0.$$

Moreover, the error term may be bounded by

$$|\epsilon(\boldsymbol{x}_1, A)| \leq C(k, \boldsymbol{x}_1, A) \sum_{s=0}^{k-1} \sum_{r=0}^{k-s} \binom{k-1}{s} s! \binom{k-s}{r} \left(\frac{m}{2}\right)_r \left(\frac{t^2}{2}\right)^{-m/2-r+1} \Delta_{k,r,m,2}$$

with

$$C(k, \boldsymbol{x}_1, A) = \exp\left(\frac{\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}}{2E[A]}\right) E[A]^{m/2-1} \frac{|\bar{\Lambda}_{m/2}(\boldsymbol{x}_1, A)|}{\Lambda_{m/2}(\boldsymbol{x}_1, A)^2}.$$

In obtaining Theorem 5, we introduce the polynomials $L_{j,s}(y)$, $y > 0$. These polynomials are Laguerre-type. However, if the discussed polynomial is expressed with respect to the Laguerre polynomial (L.P.), then it will be noticed that the corresponding L.P. has an upper index depending on the lower one. It is well known that for the L.P. we insist that the upper index be independent of the lower one because many properties which are valid for the independent case, fail to be valid for the dependent one.

Note that for sufficiently large $k \in \mathbb{N}$, the conditional expectation presented in Theorem 5, when the scale variable is concentrated to its expected value, may be simplified by just omitting the error term in the same way as for Theorems 1 and 2.

3. Example

In this section, we present a specific example. We illustrate in an algorithmic procedure how to evaluate the performance of the approximations given in Theorem 2. The algorithm consists of three major steps. The steps provide an outline of the order in which the quantities involve may be computed.

Without loss of generality, we assume that $\boldsymbol{G}$ is a bivariate normal distribution with mean zero and a fixed variance-covariance matrix Σ. The scalar random variable A is assumed to be a functional of a chi-square variate with n degree of freedom.

Set $A^{1/2} = \sqrt{\frac{n}{\chi_n^2}}$ and $\boldsymbol{X} = A^{1/2}\boldsymbol{G}$. Thus, the joint density function of $\boldsymbol{X}$ is given by $f_{\boldsymbol{x}}(\boldsymbol{x}) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{n\pi}}(1 + \frac{\|x\|^2_{\Sigma^{-1}}}{n})^{-(n/2-1)}$, which is the bivariate student t-density. We demonstrate how easy the approximation in Theorem 2 may be achieved and we determine the rate of convergence when the scalar A is specified. In view of Theorem 2, it is easy to see the expression of the c-th moment of A. This is accomplished by first noticing that $\chi_n^2/2$ has a probability density given by

$$f_{\chi_n^2/2}(x) = x^{n/2-1}e^{-x}/\Gamma(n/2), \qquad \text{for} \quad x > 0. \tag{3.1}$$

Step 1: For any $c < n/2$,

$$E[A^c] = \frac{1}{\Gamma(n/2)}\int_0^\infty \left(\frac{n}{2x}\right)^c x^{n/2-1}e^{-x}dx = (n/2)^c \frac{\Gamma\left(\frac{n}{2} - c\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{3.2}$$

Since the expressions in Theorem 2 are in terms of $\bar{a}_i$ and a_i, for $i = 1, \ldots, k-1$, it is necessary to relate them in terms of (3.2).

Step 2: If i is an integer, then the following formulae can be used

$$\bar{a}_i = \sum_{p=0}^{i}(-1)^{i-p}\binom{i}{p}\frac{E[A^{p+1}]}{E[A]^p} \quad \text{and} \quad a_i = \sum_{p=0}^{i}(-1)^{i-p}\binom{i}{p}\frac{E[A^p]}{E[A]^p}, \tag{3.3}$$

to identify the main term in the theorem.

This completes the computation of the main term.

To determine the error term, we proceed as follows:

Step 3: It is clear that, $|E[A \mid X_1 = x_1] - \bar{H}_k(x_1, E[A])/H_k(x_1, E[A])| \leq C(k, x_1, E[A])\Delta_{k,2}$. Hence, the expression that judges the quality of the rate of convergence in the above inequality is $\Delta_{k,2}$. To evaluate $\Delta_{k,2}$, we argue as follows:

As in (3.3), the following equality is in order,

$$\begin{aligned}\Delta_{k,2} &= E\left[\frac{1}{(A/E[A] \wedge 1)^{1/2}}|A/E[A] \vee 1 - 1|^k\right] \\ &= E[(A/E[A] - 1)^k I(A > E[A])] \\ &= \sum_{p=0}^{k}(-1)^{k-p}\binom{k}{p}\frac{E[A^p I(A > E[A])]}{E[A]^p}.\end{aligned} \tag{3.4}$$

Lastly, to obtain a manageable expression of $E[A^p I(A > E[A])]$, for $p = 1, \ldots, k$, we argue as in (3.2) and derive the following:

$$E[A^p I(A > E[A])] = \frac{1}{\Gamma(n/2)} \int_{E[A]}^{\infty} \left(\frac{n}{x}\right)^p x^{n/2-1} e^{-x} dx = (n/2)^p \frac{\Gamma\left(\frac{n}{2} - p, E[A]\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{3.5}$$

where $\Gamma(a, x)$ is the incomplete gamma function.

Upon completing Steps 1–3, one may now substitute equations (3.2)–(3.5) into Theorem 2 and find an expression for the approximation of the conditional expectation and an expression of the bound of the error term. The approximation and the bound of the error term are in computable form and are very easy to be implemented.

4. Proofs

PROOF OF THEOREM 1. In solving Theorem 1, we express the approximations in terms of the Kummer's function instead of Laguerre polynomials. This was chosen for uniformity purposes. It is, however, clear from (2.4) that we may interchange these functions without causing any difficulty.

By the Taylor expansion formula, for any $A > 0$, we have that if A is fixed, then

$$\begin{aligned} e^{-t^2 A/2} &= e^{-t^2 E[A]/2} \sum_{j=0}^{k-1} (-1)^j \frac{t^{2j}(A - E[A])^j}{j! 2^j} \\ &\quad + e^{-t^2(E[A]+\theta(A-E[A]))/2} \frac{(-1)^k t^{2k}(A - E[A])^k}{k! 2^k} \\ &= G_k(t, A) + \Delta_k(t, A), \qquad \text{for} \quad \theta \in (0, 1). \end{aligned} \tag{4.1}$$

In view of Fubini's theorem of successive integration, Lemma 3, and the fact that $M(a, b, z) = e^z M(b - a, b, -z)$ (see e.g. Abramowitz and Stegun (1970), Eq. 13.1.27), it follows that,

$$\begin{aligned} &\int_0^{\infty} t^{m/2} E[G_k(t, A)] J_{(m-2)/2}(\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}} t) dt \\ &= \frac{2^{m/2} \|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}^{(m-2)/2}}{E[A]^{m/2}} \sum_{j=0}^{k-1} \binom{\frac{m}{2} + j}{j} E[A(A/E[A] - 1)^j] \\ &\quad \cdot M\left(\frac{m}{2} + j, \frac{m}{2}; -\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}^2 / 2E[A]\right) \\ &= \frac{2^{m/2} \|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}^{(m-2)/2} \exp(-\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}^2 / 2E[A])}{E[A]^{m/2}} \end{aligned} \tag{4.2}$$

$$\cdot \sum_{j=0}^{k-1} \binom{\frac{m}{2}+j}{j} E[A(A/E[A]-1)^j] M\left(-j, \frac{m}{2}; \|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]\right)$$
$$= 2^{m/2}\|\boldsymbol{x}_1\|^{(m-2)/2}_{\Sigma_{11}^{-1}}/E[A]^{m/2}P_{2,k}(\boldsymbol{x}_1, A), \qquad \text{say.}$$

Similarly, for $\theta \in (0,1)$,

$$\int_0^\infty t^{m/2}E[\Delta_k(t,A)]J_{(m-2)/2}(\|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}t)dt \tag{4.3}$$
$$= \frac{2^{m/2}\|\boldsymbol{x}_1\|^{(m-2)/2}_{\Sigma_{11}^{-1}} \exp(-\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2(E[A]+\theta(A-E[A])))}{E[A]^{m/2}} \cdot \binom{\frac{m}{2}+k}{k}$$

and

$$E\left[\frac{1}{\{1+\theta(A/E[A]-1)\}^{m/2}}\left(\frac{A/E[A]-1}{1+\theta(A/E[A]-1)}\right)^k \cdot M\left(-k, \frac{m}{2}; \|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2(E[A]+\theta(A-E[A]))\right)\right]$$
$$= 2^{m/2}\|\boldsymbol{x}_1\|^{(m-2)/2}_{\Sigma_{11}^{-1}}/E[A]^{m/2}\epsilon_{2,k}(\boldsymbol{x}_1, A), \qquad \text{say.}$$

Using the same arguments for the numerator, we can reveal an expression of the form:

$$E[A \mid \boldsymbol{X}_1 = \boldsymbol{x}_1] \tag{4.4}$$
$$= \frac{\exp(-\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A])\sum_{j=0}^{k-1}\bar{a}_j\binom{\frac{m}{2}+j}{j}M\left(-j, \frac{m}{2}; \|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]\right)+\epsilon_{1,k}(\boldsymbol{x}_1, A)}{\exp(-\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A])\sum_{j=0}^{k-1}a_j\binom{\frac{m}{2}+j}{j}M\left(-j, \frac{m}{2}; \|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2E[A]\right)+\epsilon_{2,k}(\boldsymbol{x}_1, A)}.$$

After some manipulation and (2.4) the expression of the approximation in Theorem 1 is in order.

To understand the usefulness and powerfulness of Theorem 1, we also need to know how small the quantities $\epsilon_{i,k}(\cdot,\cdot)$, for $i = 1,2$, are. To answer this, two preliminary results are required.

Note that

$$M\left(-j, \frac{m}{2}, z\right) = \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+j\right)}e^z z^{-m/2+1}\int_0^\infty e^{-t}t^{m/2+j-1}J_{m/2-1}(2\sqrt{zt})dt \tag{4.5}$$

(see, e.g. Gradshteyn and Ryzhik (1980), Eq. 9.211.3) and

$$J_\nu(z) \le \frac{\left|\frac{1}{2}z\right|^\nu e^{|\operatorname{Im}(z)|}}{\Gamma(\nu+1)}, \qquad \text{for} \quad \nu \ge -\frac{1}{2}. \tag{4.6}$$

Combining (4.5) and (4.6), it follows that for $\mathrm{Im}(z) = 0$ and $z > 0$

$$\left|M\left(-j, \frac{m}{2}, z\right)\right| \le \frac{1}{\Gamma\left(\frac{m}{2}+j\right)} e^z z^{-m/2+1} (z)^{(m-2)/4} \cdot \int_0^\infty e^{-t} t^{m/2+j-1} t^{(m-2)/4} dt = \frac{\Gamma\left(\frac{3m-2}{4}+j\right)}{\Gamma\left(\frac{m}{2}+j\right)} e^z z^{-(m-1)/4}. \tag{4.7}$$

Thus, substituting z with $\|\boldsymbol{x}_1\|^2_{\Sigma_{11}^{-1}}/2(E[A]+\theta(A-E[A]))$ in (4.7), the two error terms in (4.4) can be bounded as follows

$$|\epsilon_{2,k}(\boldsymbol{x}_1, A)| \le \binom{\frac{m}{2}+k}{k} \frac{2^{(m-1)/4}\Gamma\left(\frac{3m-2}{4}+k\right) E[A]^{(m-1)/4}}{\Gamma\left(\frac{m}{2}+k\right) \|\boldsymbol{x}_1\|^{(m-1)/4}_{\Sigma_{11}^{-1}}} \cdot E\left[\frac{1}{\{1+\theta(A/E[A]-1)\}^{(m+1)/4}} \left|\frac{A/E[A]-1}{1+\theta(A/E[A]-1)}\right|^k\right], \tag{4.8}$$

and

$$|\epsilon_{1,k}(\boldsymbol{x}_1, A)| \le \binom{\frac{m}{2}+k}{k} \frac{2^{(m-1)/4}\Gamma\left(\frac{3m-2}{4}+k\right) E[A]^{(m-1)/4}}{\Gamma\left(\frac{m}{2}+k\right) \|\boldsymbol{x}_1\|^{(m-1)/4}_{\Sigma_{11}^{-1}}} \cdot E\left[\frac{A}{\{1+\theta(A/E[A]-1)\}^{(m+1)/4}} \left|\frac{A/E[A]-1}{1+\theta(A/E[A]-1)}\right|^k\right]. \tag{4.9}$$

Note that for $\theta \in (0,1)$

$$1+\theta(A/E[A]-1) \ge \begin{cases} \dfrac{A}{E[A]} & \text{for } \dfrac{A}{E[A]} \ge 1 \\ 1 & \text{for } \dfrac{A}{E[A]} > 1. \end{cases} \tag{4.10}$$

Elaborating (4.10) the proof of Theorem 1 is completed.

PROOF OF THEOREM 2. From (4.1) and Eq. 3.952.9 in Gradshteyn and Ryzhik (1980), it follows that for $m = 1$

$$\int_0^\infty E[G_k(t, A)] \cos\left(\frac{x_1}{\sigma_1} t\right) dt \tag{4.11}$$

$$= \sum_{j=0}^{k-1} (-1)^j \frac{E[(A - E[A])^j]}{j!2^j} \int_0^\infty t^{2j} \cos\left(\frac{x_1}{\sigma_1} t\right) e^{-t^2 E[A]/2} dt$$

$$= \frac{\sqrt{\pi}}{(2E[A])^{1/2}} \sum_{j=0}^{k-1} \frac{E[(A/E[A] - 1)^j]}{j!2^j} \cdot \exp(-x_1^2/2\sigma_1^2 E[A]) H_{2j}(x_1/\sigma_1(2E[A])^{1/2}),$$

where $H_j(\cdot)$ is the Hermite polynomial. Similarly, it can be seen that

$$\begin{aligned} (4.12) \quad & \int_0^\infty E[\Delta_k(t, A)] \cos\left(\frac{x_1}{\sigma_1} t\right) dt \\ &= \frac{\sqrt{\pi}}{(2E[A])^{1/2}} \\ &\quad \cdot E\Bigg[1/\{1 + \theta(A/E[A] - 1)\}^{1/2} \left(\frac{A/E[A] - 1}{1 + \theta(A/E[A] - 1)} \right)^k \Big/ k!2^k \\ &\quad \cdot \exp(-x_1^2/2\sigma_1^2(E[A] + \theta(A - E[A]))) H_{2k} \\ &\quad \cdot (x_1/(2\sigma_1^2(E[A] + \theta(A - E[A])))^{1/2}) \Bigg] \\ &= \epsilon_{2k}(x_1, A), \qquad \text{say.} \end{aligned}$$

Since $\int_0^\infty t^{2k} e^{-p^2t^2} dt = \frac{(2k)!}{2^{k+1} k!(2p)^k} \sqrt{\frac{\pi}{p}}$, then $\epsilon_{2k}(x_1, A)$ can be easily bounded as follows

$$\begin{aligned} (4.13) \quad |\epsilon_{2k}(x_1, A)| &\le \int_0^\infty E[|\Delta_k(t, A)|] dt \\ &= \frac{\sqrt{\pi}}{(2E[A])^{1/2}} \frac{1}{2^k} \binom{2k}{k} \\ &\quad \cdot E\left[\frac{1}{\{1 + \theta(A/E[A] - 1)\}^{1/2}} \left| \frac{A/E[A] - 1}{1 + \theta(A/E[A] - 1)} \right|^k \right]. \end{aligned}$$

Finally, substituting relation (4.10) into (4.13) the proof of Theorem 2 is now completed.

PROOF OF THEOREM 5. Note that if α, β, and γ are continuous maps from $(0, \infty)$ to any real subset and, β and γ are differentiable with respect to α and β respectively of order k, $k \in \mathbb{N}$, then by the Leibnitz' rule for the k-th derivative of product, we have that

$$(4.14) \quad \frac{d\gamma}{d\alpha} = \frac{d\gamma}{d\beta} \frac{d\beta}{d\alpha}, \quad \text{and} \quad \frac{d^k\gamma}{d\alpha^k} = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{d^{k-j}\gamma}{d\beta^{k-j}} \frac{d^j\beta}{d\alpha^j} \quad \text{for} \quad k \ge 1.$$

Similarly, by the Leibnitz' rule, it can be also seen that for $n \in \mathbb{N}$ and $a > 0$

$$\begin{aligned}
(4.15)\qquad D^l(x^{n+a}e^{-x}) &= \sum_{k=0}^{l} \binom{l}{k} D^{l-k}(e^{-x}) D^k(x^{n+a}) \\
&= e^{-x} \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} (n+a)_k x^{n+a-k} \\
&= x^{n+a} e^{-x} L_{l,n+a}(x), \qquad \text{for} \quad l \in \mathbb{N},
\end{aligned}$$

where D is the differential operator $\frac{d}{dx}$.

Note that $L_{l,n+a}(x)$ is a polynomial of order l of x^{-1}. However, if $a = 0$, then

$$(4.15')\qquad D^l(x^n e^{-x}) = x^n e^{-x} \sum_{k=0}^{d} \binom{l}{k} (-1)^{l-k} (n)_k x^{-k} = x^n e^{-x} L_{l,n}(x), \qquad \text{for} \quad l \in \mathbb{N},$$

where $d = n \min(1, \frac{l}{n})$, $l \in \mathbb{N}$, i.e., if $l > n$ then $L_{l,n+a}(x)$ is just a polynomial of order n of x^{-1}. Set $\gamma(a,t) = a^{-m/2} e^{-t^2/2a}$ and $\beta(a,t) = \frac{t^2}{2a}$. Note that $\frac{m}{2}$ may be written either as $\nu + \frac{1}{2}$ or ν, for $\nu \in \mathbb{N}$. Thus, combining (4.14) and (4.15), the following result is in order

$$\begin{aligned}
(4.16)\qquad \frac{d^l}{da^l}(a^{-m/2} e^{-t^2/2a}) &= \left(\frac{t^2}{2}\right)^{-m/2+1} \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{d^{l-j}}{d\left(\frac{t^2}{2a}\right)^{l-j}} \\
&\quad \cdot \left(\left(\frac{t^2}{2a}\right)^{m/2} e^{-t^2/2a} \right) \frac{d^j}{da^j}(a^{-j}) \\
&= \left(\frac{t^2}{2}\right)^{-m/2+1} a^{-m/2} e^{-t^2/2a} \\
&\quad \cdot \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^{j+1} j! L_{l-j,m/2}\left(\frac{t^2}{2a}\right) a^{-j-1} \\
&= \left(\frac{t^2}{2}\right)^{-m/2+1} a^{-m/2} e^{-t^2/2a} \lambda_{l,m/2}(t^2, a).
\end{aligned}$$

By Taylor's expansion series around $E[A]$, it follows that for fixed $A > 0$

$$\begin{aligned}
(4.17)\qquad A^{-m/2} e^{-t^2/2A} &= \sum_{j=0}^{k-1} \frac{(A - E[A])^j}{j!} D^j(E[A]^{-m/2} e^{-t^2/2E[A]}) + \Delta_k(A^*, t) \\
&= S_k(A,t) + \Delta_k(A^*, t),
\end{aligned}$$

where $A^* = E[A] + \theta(A - E[A])$ and $\theta \in (0,1)$, and $t = \|\boldsymbol{x}_1\|_{\Sigma_{11}^{-1}}$.

In connection with (4.16), we now present an explicit form of the $E[S_k(A,t)]$ as follows.

$$(4.18)\quad E[S_k(A,t)] = E[A]^{-m/2}e^{-t^2/2E[A]} \cdot \left\{1+\left(\frac{t^2}{2}\right)^{-m/2+1}\sum_{j=2}^{k-1}\frac{E[(A-E[A])^j]}{j!}\lambda_{j,m/2}\left(\frac{t^2}{2},E[A]\right)\right\}.$$

For the residual term, we proceed as follows. Observe that $A^* \geq A$ for $A \leq E[A]$, $A^* \geq E[A]$ for $A \geq E[A]$. Thus, $E[\Delta_k(A^*,t)]$ can be bounded as

$$(4.19)\quad E[\Delta_k(A^*,t)] \leq \left(\frac{t^2}{2}\right)^{-m/2+1}\sum_{j=0}^{k-1}\sum_{r=0}^{k-j}\binom{k-1}{j}\binom{k-j}{r}\frac{j!\left(\frac{m}{2}\right)_r}{k!}\left(\frac{t^2}{2}\right)^{-r} \cdot E\left[\frac{(A\vee E[A])^r}{(A\wedge E[A])^{m/2-1}}|A-E[A]|^k\right].$$

This completes the proof of Theorem 5.

5. Laguerre and Hermite polynomials and series

In this section we borrow a few standard ideas and definitions from the theory of the classical orthogonal polynomials in order to make our results more revealing and easy to be extrapolated. Rusev (1984) is considered a standard reference book.

5.1 *Definitions*

It is known that every system of orthogonal polynomials $\{P_n(z)\}_{n=0}^{\infty}$ is linearly independent. In particular, for every integer $\nu \geq 0$, $\{P_n(z)\}_{n=0}^{\infty}$ its basis is in the space of all polynomials with degree not greater than ν. This property, together with the orthogonality, leads to the important statement that every system of orthogonal polynomials is the solution of a linear recurrence equation of the kind

$$(5.1)\qquad \alpha_n y_{n+1} + (z-\beta_n)y_n + \gamma_n y_{n-1} = 0,$$

where α_n, and $\gamma_n \geq 0$ for $n \in \mathbb{N}-\{0\}$.

In other words, for every $z \in \mathbb{C}$ and $n \in \mathbb{N}-\{0\}$

$$(5.2)\qquad \alpha_n P_{n+1}(z) + (z-\beta_n)P_n(z) + \gamma_n P_{n-1}(z) = 0.$$

Now, if $\alpha_n = n+1$, $\beta_n = 2n+\alpha+1$, $\gamma_n = n+\alpha$, and $\alpha \in \mathbb{R}-\{-1,-2,\ldots\}$, then $P_n(z) = L_n^{(\alpha)}(z)$, i.e., they are the Laguerre polynomials.

Let $\{P_n(z)\}_{n=0}^{\infty}$ be a system of polynomials orthogonal in the interval $[a,b]$ with respect to the weight function $w(\cdot)$. This system is a solution of the recurrence equation of the kind (4.1). However, it can be shown that the system of functions

$$(5.3)\qquad Q_n(z) = -\int_a^b \frac{w(t)P_n(t)}{t-z}dt, \qquad n \in \mathbb{N},$$

holomorphic in the open set $\mathbb{C} - [a,b]$, is also a solution of (4.1). The functions $Q_n(z)$, $n \in \mathbb{N}$, are called functions of second kind. In fact, it can be shown that the system $\{Q_n(z)\}_{n=0}^{\infty}$ is a second solution of the equation (4.10) in the open set $\mathbb{C} - [a,b]$, i.e., $\forall z \in \mathbb{C} - [a,b]$ the systems $\{P_n(z)\}_{n=0}^{\infty}$ and $\{Q_n(z)\}_{n=0}^{\infty}$ are linearly independent.

Therefore, the Laguerre functions of second kind are given by

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{\alpha}\exp(-t)L_n^{(\alpha)}(t)}{t-z}dt, \qquad n \in \mathbb{N}, \tag{5.4}$$

where $\alpha > -1$, and $z \in \mathbb{C} - [a,b]$.

5.2 *Asymptotic formulas*

If $\alpha \in \mathbb{R} - \{-1,-2,\dots\}$, the asymptotic behavior of the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ on the ray $(0,\infty)$ is given by Fejer's formulas

$$\begin{aligned} L_n^{(\alpha)}(x) &= \pi^{-1/2}\exp(x/2)x^{-\alpha/2-1/4}n^{\alpha/2-1/4} \\ &\cdot \left\{\cos\left((2\pi x)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + l_n^{(\alpha)}(x)\right\}, \end{aligned} \tag{5.5}$$

where $l_n^{(\alpha)}(x) = O(n^{-1/2})$ on $x \in (\epsilon,\omega)$, $0 < \epsilon < \omega < \infty$, for sufficiently large n.

If we are interested only in the growth of $L_n^{(\alpha)}(x)$ as a function of n, we can use the following formula

$$L_n^{(\alpha)}(x) = O(n^{\beta}), \qquad \beta = \max\left\{\frac{\alpha}{2} - \frac{1}{4}, \alpha\right\}, \tag{5.6}$$

which is valid uniformly on every interval $[0,\omega]$, $0 < \omega < \infty$, provided that $\alpha \neq \{-1,-2,\dots\}$ and are real.

In view of the rate of convergence, we shall present the asymptotic behavior of Laguerre polynomials if n and z (independently) tend to infinity.

First, we define the following. If $0 < \lambda < \infty$, $p(\lambda)$ denotes the image of the straight line $\mathrm{Im}(\omega) = \lambda$ under the transformation $z = \omega^2$. This means that $p(\lambda)$ is the curve that can be described by the equality $\mathrm{Re}(-z)^{1/2} = \lambda$, i.e., it is the parabola with focus at the origin and having the real line as its axis. Let $\Delta(\lambda) := \mathrm{Int}\{p(\lambda) : \mathrm{Re}(-z)^{1/2} = \lambda\}$.

If $0 < \lambda < \infty$, $\rho = \max\{1, 2\lambda^2\}$ and $\alpha \in \mathbb{R} - \{-1,-2,\dots\}$, then $\exists$ a constant $A = A(\lambda,\rho,\alpha)$: $\forall n \in \mathbb{N} - \{0\}$ and $z = x+iy \in \Delta^*(\lambda,\rho) := \bar{\Delta}(\lambda) \cap \{z \in \mathbb{C} : |z| \geq \rho\}$ holds the inequality

$$|L_n^{(\alpha)}(z)| \leq A|z|^{-a/2-1/4}n^{\alpha/2-1/4}\exp(-z - 2\lambda\sqrt{n}). \tag{5.7}$$

5.3 *Convergence of series in Laguerre polynomials*

It will be seen that with series in Laguerre polynomials

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z), \qquad \alpha \in \mathbb{R} - \{-1,-2,\ldots\} \tag{5.8}$$

we have to be careful because their regions of convergence are unbounded and this causes some difficulties. For example, by using only the asymptotic formulas (4.5) and (4.6) one can not prove a statement like Abel's Lemma for power series.

As before, if $0 < \lambda < \infty$, by $\Delta(\lambda) = \mathrm{Int}\{p(\lambda) : \mathrm{Re}(-z)^{1/2} = \lambda\}$ and by $\Delta^*(\lambda)$, its exterior. By definition $\Delta(0) := \emptyset$ and $\Delta(\infty) := \mathbb{C}$, respectively $\Delta^*(0) := \mathbb{C} - [0,\infty)$ and $\Delta^*(\infty) := \emptyset$. Further, if $\rho > \max\{1, 2\lambda^2\}$, we define $\Delta(\lambda,\rho) := \Delta(\lambda) \cap \{z \in \mathbb{C} : |z| < \rho\}$.

PROPOSITION 1. *If*

$$\lambda_0 = \max\left\{0, -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n|\right\},$$

then the (4.8) *is absolutely uniformly convergent on every compact subset of* $\Delta(\lambda_0)$ *and divergent in* $\Delta^*(\lambda_0)$.

To see the absolute convergence of (4.8), inequality (4.7) is utilized, namely if $\alpha \in \mathbb{R} - \{-1,-2,\ldots\}$ and $-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n| \geq \lambda_0$, then, $\forall\lambda \in (0,\lambda_0)$ and $\rho > \max\{1, 2\lambda^2\}$, the series

$$\sum_{n=1}^{\infty} a_n z^{a/2+1/4}\exp(-z)L_n^{(a)}(z) \tag{5.9}$$

is absolutely uniformly convergent on the region $\Delta^*(\lambda,\rho)$. Indeed, if $0 < \tau < \lambda_0 - \lambda$, then $|a_n| = O(\exp(-(2\lambda+\tau)\sqrt{n}))$ and (4.7) gives that $|a_n z^{a/2+1/4}\exp(-z)L_n^{(a)}(z)| = O(n^{\alpha/2-1/4}\exp(-2\lambda\sqrt{n}))$, i.e., the series (4.9) is majorized in $\Delta^*(\lambda,\rho)$ by

$$\sum_{n=1}^{\infty} n^{a/2-1/4}\exp(-\tau\sqrt{n}) < \infty. \tag{5.10}$$

5.4 *Uniqueness of the expansions*

A well known fact is that the orthogonal polynomials expansions have the property (usually called uniqueness) that if $\sum_{n=0}^{\infty} a_n P_n(z) \equiv 0$, then $a_n \equiv 0$ $\forall n \in \mathbb{N}$. In other words, the coefficients of an orthogonal expansion are uniquely determined by its sum. For example, in the case of a system of orthogonal $\{P_n(z)\}_{n=0}^{\infty}$ polynomials on a finite interval $[a,b]$ with respect to weight $w(\cdot)$ the coefficients of a series of the kind $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ are given by the equality

$$a_n = \frac{1}{A_n}\int_a^b w(t)P_n(t)f(t)dt, \ \forall n \in \mathbb{N}, \qquad \text{and} \qquad A_n = \int_a^b w(t)[P_n(t)]^2 dt, \tag{5.11}$$

provided that $f(z)$ is uniformly convergent in $[a, b]$.

In the case of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ $(\alpha > -1)$ the interval is infinite and we must be careful when applying representation (4.11). Rusev (1984) has shown that

PROPOSITION 2. *Let $0 < \lambda < \infty$ and $\alpha > -1$. If the complex function $f(\cdot)$ has a representation*

$$f(z) = \sum_{n=0}^{\infty} a_n L_n^{(a)}(z), \qquad z \in \Delta(\lambda_0),$$

then $f(\cdot)$ is holomorphic in $\Delta(\lambda_0)$ and $\forall n \in \mathbb{N}$ holds the equality

$$a_n = \frac{1}{I_n^{(\alpha)}} \int_0^{\infty} t^{\alpha} \exp(-t) L_n^{(\alpha)}(t) f(t) dt, \ \forall n \in \mathbb{N}, \qquad \text{and} \qquad I_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}.$$

In particular, if $f(z) \equiv 0$, then $a_n \equiv 0$ $\forall n \in \mathbb{N}$.

5.5 *Hermite polynomials*

It can be seen (see e.g. Rusey (1984)) that

$$\begin{aligned} &H_{2n}(z) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(z^2), && \text{and} \\ &H_{2n+1}(z) = (-1)^n 2^{2n+1} n! L_n^{(1/2)}(z^2), && n \in \mathbb{N}. \end{aligned} \tag{5.12}$$

Thus, the statements presented for Laguerre polynomials could also be referred to Hermite polynomials. For the sake of convenience, we shall illustrate the following.

We define that if, $S(\tau) := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \tau\}$. By definition $S(0) := \emptyset$ and $S(\infty) := \mathbb{C}$. Similarly, $S^*(\tau) := \{z \in \mathbb{C} : |\operatorname{Im}(z)| > \tau\}$, if $0 < \tau < \infty$ and $S^*(0) := \mathbb{C} - (-\infty, \infty)$, and $S^*(\infty) := \emptyset$. Then, the following Abel's Lemma is in order.

PROPOSITION 3. a. *If*

$$\tau_0 := \max\left\{0, -\limsup_{n\to\infty} (2n+1)^{-1/2} \log |(2n/e)^{n/2} a_n|\right\},$$

then the series $\sum_{n=0}^{\infty} a_n H_n(z)$ is absolutely uniformly convergent on every compact subset of $S(\tau_0)$ and diverges in $S^*(\tau_0)$. And

b. *If a complex function $f(\cdot)$ has in the strip $S(\tau_0)$ $(0 < \tau_0 \le \infty)$ a representation by a series of Hermite polynomials, i.e., $f(z) = \sum_{n=0}^{\infty} a_n H_n(z)$, then $f(\cdot)$ is holomorphic in $S(\tau_0)$ and $\forall n \in \mathbb{N}$*

$$a_n = \frac{1}{I_n} \int_{-\infty}^{\infty} \exp(-t^2) H_n(t) f(t) dt, \qquad \text{and} \qquad I_n = \sqrt{\pi} 2^n n!.$$

In particular, if $f(z) \equiv 0$, then $a_n \equiv 0$ $\forall n \in \mathbb{N}$.

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions.

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