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POINT AND INTERVAL ESTIMATION OF $P(X < Y)$: THE NORMAL CASE WITH COMMON COEFFICIENT OF VARIATION

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Abstract. The problem of estimating $R = P(X < Y)$ originated in the context of reliability where Y represents the strength subjected to a stress X. In this paper we consider the problem of estimating R when X and Y have independent normal distributions with equal coefficient of variation. The maximum likelihood estimation of R when the coefficient of variation is known and when it is unknown is studied. The asymptotic variance of the estimators are obtained and asymptotic confidence intervals are provided. An example is presented to illustrate the procedure. Finally some simulation studies are carried out to study the coverage probability and the lengths of the confidence interval. In particular, lengths of the confidence intervals are compared with and without the assumption of common coefficient of variation. It is observed that the assumption of common coefficient of variation results in considerably tighter intervals.

Key words and phrases: Normal distributions, coefficients of variations, delta method, confidence intervals, coverage probability.

1. Introduction

The problem of estimating $R = P(X < Y)$ has been considered in the literature in both distribution free and parametric frameworks. The problem originated in the context of reliability of a component of strength Y subjected to a stress X. The component fails if at anytime the applied stress is greater than its strength and there is no failure when $Y > X$. Thus $P(X < Y)$ is a measure of the reliability of the component. It may be mentioned that R is of greater interest than just in reliability since it provides a general measure of the difference between two populations and has applications in many areas. For instance, if Y is the response for a control group, and X refers to a treatment group, R is a measure of the effect of the treatment. As Wolfe and Hogg (1971) point out, this measure is one that incorporates similar information to $(\mu_X - \mu_Y)/\sigma$, but is easier to interpret. The function $P(X < Y) - P(Y < X)$ is of practical importance in many situations, including clinical trials and genetics. For more applications of R, see Halperin *et al.* (1987), Simonoff *et al.* (1986), Reiser and Farragi (1994) and Bamber (1975). In fact Bamber (1975) gives a geometrical interpretation of $A(X,Y) = P(X < Y) + \frac{1}{2}P(X = Y)$ and demonstrates that $A(X,Y)$ is a useful measure of the size of the difference between two populations. The problem is to find an estimator of this probability when X and Y have specified probability distributions. Birnbaum (1956), Church and Harris (1970), and others have given practical uses of the results.

Church and Harris (1970) derived the maximum likelihood estimator (MLE) of R assuming that X and Y are independent normals and that the distribution of X is completely known. They also obtained a confidence bound for R and compared with those of Govindaranjulu (1967). Mazumdar (1970) derived the minimum variance unbiased estimator (MVUE) of R when the stress distribution is known and the variance of the strength Y is either known or unknown. Downton (1973) obtained the MVUE of R in the case of independent normals with the parameters of X also unknown. Reiser and Guttman (1987) obtained a predictive estimator of R and compared it with the MLE and MVUE through a simulation study. In a separate study Reiser and Guttman (1986) presented two approximate methods for obtaining confidence intervals and an approximate Bayesian probability interval without the assumption of common coefficient of variation. In addition to the normal case, the problem has been extensively studied for many other models including exponential, gamma, Weibull and Burr distributions; see Constantine *et al.* (1986) for the gamma case, McCool (1991) for the Weibull case and Awad and Gharraf (1986) for the Burr case.

Owen *et al.* (1964) discussed the normal case for equal standard deviations and presented non-parametric confidence limits for this problem. In this paper, instead of assuming equal standard deviations, we assume equal coefficients of variation. This is because the coefficient of variation represents a measure of relative variability and groups can have the same relative variability even if the means and variances of the variable of interest are different. The assumption of homogeneous coefficients of variation is a valid assumption in many types of agricultural, biological and psychological experimentation, because many times the treatment that yields a larger mean also has a larger standard deviation; see Lohrding (1969*a*). Our assumption of equal coefficient of variation will imply that the two means are of equal sign. Thus, in the context of our problem, we can, without loss of generality, assume that the two means are positive. Moreover, since the coefficient of variation is, in general, assumed to be positive, it is desirable to assume that the means are positive, see Sinha *et al.* (1978). More specifically, we assume that X and Y are independent with normal distributions having equal coefficients of variation and study the problem of estimating $P(X < Y)$, which is given by

$$R = P(X < Y) = \Phi\left(\frac{\mu_2 - \mu_1}{\gamma\sqrt{\mu_1^2 + \mu_2^2}}\right)$$

where μ_1, μ_2 (assumed positive) are the means of X and Y, respectively, and γ is the common coefficient of variation. The assumption of equal coefficients of variation can be tested by the tests described by Gupta and Ma (1996).

In Section 2, we derive the maximum likelihood estimators (MLE) of the parameters μ_1, μ_2 and γ under various conditions on the data and obtain the information matrix. These estimators and the information matrix are then used to derive the MLE of R and its asymptotic variance. In Section 3, we assume that γ is known and obtain MLE of R and its variance and in Section 4, we present Reiser and Guttman's (1986) method of constructing a confidence interval of R without the assumption of equal coefficient of variation. An example is presented, in Section 5, to illustrate the procedure. Finally some simulation studies are carried out to study the coverage probability and the lengths of the confidence intervals. In particular lengths of the confidence intervals are compared with and without the assumption of common coefficients of variation. It is observed that our assumption, when valid, results in considerably tighter intervals.

2. Maximum likelihood estimation

Let $X_1, X_2, X_3, \ldots, X_{n_1}$ be a random sample of size n_1 from a normal population with mean μ_1 and variance $\mu_1^2\gamma^2$ and let $Y_1, Y_2, Y_3, \ldots, Y_{n_2}$ be a random sample of size n_2 from a normal population with mean μ_2 and variance $\mu_2^2\gamma^2$ so that the two populations have the common coefficient of variation γ. We wish to estimate the parameters μ_1, μ_2 and γ by means of maximum likelihood assuming the two samples to be independent.

The log likelihood function of the strength stress model is given by $\ln L = -(n_1 + n_2)\ln(\sqrt{2\pi}\gamma) - n_1 \ln \mu_1 - n_2 \ln \mu_2$

$$-\sum_{i=1}^{n_1}(x_i - \mu_1)^2/2\gamma^2\mu_1^2 - \sum_{i=1}^{n_2}(y_i - \mu_2)^2/2\gamma^2\mu_2^2. \tag{2.1}$$

The likelihood equation for μ_1 reduces to

$$\mu_1^2\gamma^2 + \bar{x}\mu_1 - (s_1^2 + \bar{x}^2) = 0. \tag{2.2}$$

Similarly the likelihood equation for μ_2 is

$$\mu_2^2\gamma^2 + \bar{y}\mu_2 - (s_2^2 + \bar{y}^2) = 0 \tag{2.3}$$

where $\bar{x}$ and $\bar{y}$ are the sample means from the first and the second populations and s_1^2 and s_2^2 are the sample variances and are defined by

$$s_1^2 = \sum_{i=1}^{n_1} \frac{(x_i - \bar{x})^2}{n_1} \quad \text{and} \quad s_2^2 = \sum_{i=1}^{n_2} \frac{(y_i - \bar{y})^2}{n_2}.$$

The likelihood equation for γ is given by

$$\frac{\partial}{\partial\gamma}\ln L = -\left(\frac{n_1 + n_2}{\gamma}\right) + \sum_{i=1}^{n_1}\left(\frac{x_i}{\hat{\mu}_1} - 1\right)^2/\gamma^3 + \sum_{i=1}^{n_2}\left(\frac{y_i}{\hat{\mu}_2} - 1\right)^2/\gamma^3 = 0.$$

In order to solve these equations, we proceed as in Gerig and Sen (1980) and Lohrding (1969*a*) and obtain

$$\hat{\gamma}^2 = \frac{1}{n}[\hat{\mu}_1^{-2} n_1 (\hat{\mu}_1^2(1+\hat{\gamma}^2) - \hat{\mu}_1 \bar{x}) + \hat{\mu}_2^{-2} n_2 (\hat{\mu}_2^2(1+\hat{\gamma}^2) - \hat{\mu}_2 \bar{y})].$$

The above equation reduces to

$$n = \frac{n_1 \bar{x}}{\hat{\mu}_1} + \frac{n_2 \bar{y}}{\hat{\mu}_2} \tag{2.4}$$

where $n = n_1 + n_2$. From this and equation (2.3) we get $\hat{\mu}_1$ and $\hat{\gamma}^2$ in terms of $\hat{\mu}_2$ as

$$\hat{\mu}_1 = n_1 \bar{x} \hat{\mu}_2 (n\hat{\mu}_2 - n_2 \bar{y})^{-1} \tag{2.5}$$

$$\hat{\gamma}^2 = \hat{\mu}_2^{-2}(s_2^2 + \bar{y}^2 - \bar{y}\hat{\mu}_2). \tag{2.6}$$

Now from (2.2) and (2.3) we get

$$\hat{\gamma}^2 = (s_1^2 + \bar{x}^2 - \bar{x}\hat{\mu}_1)/\hat{\mu}_1^2 = (s_2^2 + \bar{y}^2 - \bar{y}\hat{\mu}_2)/\hat{\mu}_2^2, \tag{2.7}$$

which enables us to get a quadratic in $\hat{\mu}_2$ as

$$(nC_1^2 + n_2)\hat{\mu}_2^2 - [2n_2 C_1^2 + (2n_2 - n_1)]\bar{y}\hat{\mu}_2 + [n_2^2(C_1^2+1) - n_1^2(C_2^2+1)]\bar{y}^2 n^{-1} = 0, \tag{2.8}$$

where $C_1^2 = s_1^2/\bar{x}^2$ and $C_2^2 = s_2^2/\bar{y}^2$ are the squares of the sample coefficients of variation of the first and second random sample.

Thus the maximum likelihood estimate of the parameter μ_2 can be obtained by solving (2.8). This estimate, in turn will yield $\hat{\mu}_1$ and $\hat{\gamma}$. This set of estimators indeed solves the set of likelihood equations. To check that the joint estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\gamma}$ define a maximum, we refer to the Appendix.

Case 1. Equal sample size $C_1 \neq C_2$. In this case the non-negative root of (2.8) is given by

$$\hat{\mu}_2 = \frac{\bar{y}[1 + \sqrt{(1+2C_2^2)/(1+2C_1^2)}]}{2}. \tag{2.9}$$

Remark. As noted in Lohrding (1969*a*), using the root with minus sign (of the quadratic in (2.8)), one gets the value of the likelihood function as zero under our assumption. The root using the plus sign may give a negative estimate of μ_2, although under our assumptions, the probability of $\hat{\mu}_2 < 0$ tends to zero as the sample size tends to infinity.

It is easy to determine $\hat{\mu}_1$ and $\hat{\gamma}$ since they are expressed in terms of $\hat{\mu}_2$ (see equations (2.5) and (2.6)).

Case 2. Now consider the case when the sample sizes are different i.e. $n_1 \neq n_2$ and $C_1 \neq C_2$. In this case equation (2.8) is

$$(nC_1^2 + n_2)\hat{\mu}_2^2 - (2n_2 C_1^2 + (2n_2 - n_1))\bar{y}\hat{\mu}_2 + (n_2^2(C_1^2+1) - n_1^2(C_2^2+1))\bar{y}^2 n^{-1} = 0 \tag{2.10}$$

whose nonnegative root is given by

$$\hat{\mu}_2 = \frac{a+b}{c}$$

where

$$\begin{aligned}
a &= \{(n_1n_2 + 2n_2^2 - n_1^2)\bar{x}^2 + 2n_2(n_1+n_2)s_1^2\}\bar{y} \\
b &= n_1\sqrt{n_1+n_2}\,\bar{x} \\
&\quad \cdot \sqrt{(n_1+n_2)\bar{x}^2\bar{y}^2 + 4n_1\bar{y}^2s_1^2 + 4n_2\bar{x}^2s_2^2 + 4n_2\bar{x}^2s_2^2 + 4(n_1+n_2)s_1^2s_2^2} \\
c &= [2(n_1+n_2)\{n_2\bar{x}^2 + (n_1+n_2)s_1^2\}].
\end{aligned}$$

Note that we take only the positive root in front of the radical sign in the numerator because of the assumption that the population mean is positive. Once $\hat{\mu}_2$ is obtained, it is easy to determine $\hat{\mu}_1$ and $\hat{\gamma}$ as they are functions of $\hat{\mu}_2$, $\bar{x}$, $\bar{y}$, C_1 and C_2. That this solution defines a maximum of the likelihood follows from the negative definiteness of the matrix $A = [\frac{\partial^2 \ln L}{\partial\theta_r\partial\theta_s}]$ evaluated at the above solutions, where $\theta_1 = \mu_1$, $\theta_2 = \mu_2$, $\theta_3 = \gamma$; see Kendall and Stuart (1967). The negative definiteness of the matrix A is investigated in the Appendix.

The estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\gamma}$ will be used in estimating R of our model.

Asymptotic variances and covariances of the estimators

The Fisher information matrix denoted by I is given by

$$I = -E\begin{bmatrix} \frac{\partial^2 L}{\partial\mu_1^2} & \frac{\partial^2 L}{\partial\mu_1\partial\mu_2} & \frac{\partial^2 L}{\partial\mu_1\partial\gamma} \\ \frac{\partial^2 L}{\partial\mu_2\partial\mu_1} & \frac{\partial^2 L}{\partial\mu_2^2} & \frac{\partial^2 L}{\partial\mu_2\partial\gamma} \\ \frac{\partial^2 L}{\partial\gamma\partial\mu_1} & \frac{\partial^2 L}{\partial\gamma\partial\mu_2} & \frac{\partial^2 L}{\partial\gamma^2} \end{bmatrix} = \begin{bmatrix} \frac{n_1(1+2\gamma^2)}{\gamma^2\mu_1^2} & 0 & \frac{2n_1}{\gamma\mu_1} \\ 0 & \frac{n_2(1+2\gamma^2)}{\gamma^2\mu_2^2} & \frac{2n_2}{\gamma\mu_2} \\ \frac{2n_1}{\gamma\mu_1} & \frac{2n_2}{\gamma\mu_2} & \frac{2n}{\gamma^2} \end{bmatrix}.$$

The asymptotic variances and covariances of the estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\gamma}$ are given by $A = I^{-1}$ where

$$A = \begin{pmatrix} \frac{\gamma^2\mu_1^2(n_1(1+2\gamma^2)+n_2)}{n_1(1+2\gamma^2)(n_1+n_2)} & \frac{2\gamma^4\mu_1\mu_2}{(1+2\gamma^2)(n_1+n_2)} & \frac{-\gamma^3\mu_1}{(n_1+n_2)} \\ \frac{2\gamma^4\mu_1\mu_2}{(1+2\gamma^2)(n_1+n_2)} & \frac{\gamma^2\mu_2^2(n_2(1+2\gamma^2)+n_1)}{n_2(1+2\gamma^2)(n_1+n_2)} & \frac{-\gamma^3\mu_2}{(n_1+n_2)} \\ \frac{-\gamma^3\mu_1}{(n_1+n_2)} & \frac{-\gamma^3\mu_2}{(n_1+n_2)} & \frac{\gamma^2(1+2\gamma^2)}{2(n_1+n_2)} \end{pmatrix}.$$

In the above matrix, the diagonal entries give the asymptotic variances of $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\gamma}$ and the off diagonal entries give the covariances of $(\hat{\mu}_1, \hat{\mu}_2)$, $(\hat{\mu}_2, \hat{\gamma})$ and $(\hat{\mu}_1, \hat{\gamma})$.

The asymptotic variances and covariances of the estimators $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\gamma}$ will be needed to find the variance of the estimator of the reliability function $\hat{R}$. The variance of $\hat{R}$ will be obtained by using the delta method, see Cassela and Berger (1990).

The MLE of the reliability function is given by $\hat{R} = \Phi(\frac{\hat{\mu}_2-\hat{\mu}_1}{\hat{\gamma}\sqrt{(\hat{\mu}_1^2+\hat{\mu}_2^2)}})$. In order to construct a confidence interval for R, let us find confidence interval for $S = \frac{\mu_2-\mu_1}{\gamma\sqrt{\mu_1^2+\mu_2^2}}$. The MLE of S is given by $\hat{S} = \frac{\hat{\mu}_2-\hat{\mu}_1}{\hat{\gamma}\sqrt{\hat{\mu}_1^2+\hat{\mu}_2^2}}$.

The Variance of $\hat{S}$ is obtained as follows:

For convenience let $S_1 = \frac{\partial \hat{s}}{\partial \hat{\mu}_1}$, $S_2 = \frac{\partial \hat{s}}{\partial \hat{\mu}_2}$ and $S_3 = \frac{\partial \hat{s}}{\partial \hat{\gamma}}$ evaluated at (μ_1, μ_2, γ).

$$\begin{aligned}\operatorname{Var}(\hat{S}) &= S_1^2 \operatorname{Var}(\hat{\mu}_1) + S_2^2 \operatorname{Var}(\hat{\mu}_2) + S_3^2 \operatorname{Var}(\hat{\gamma}) \\ &\quad + 2S_1S_2 \operatorname{Cov}(\hat{\mu}_1, \hat{\mu}_2) + 2S_2S_3 \operatorname{Cov}(\hat{\mu}_2, \hat{\gamma}) + 2S_1S_3 \operatorname{Cov}(\hat{\mu}_1, \hat{\gamma}).\end{aligned}$$

Let us now derive the expression for $\operatorname{Var}(\hat{S})$. It can be verified that

$$S_1 = -\frac{(\mu_1+\mu_2)\mu_2}{\gamma(\mu_1^2+\mu_2^2)^{3/2}}, \quad S_2 = \frac{\mu_1(\mu_1+\mu_2)}{\gamma(\mu_1^2+\mu_2^2)^{3/2}} \quad \text{and} \quad S_3 = \frac{\mu_1-\mu_2}{\gamma^2(\mu_1^2+\mu_2^2)^{1/2}}.$$

Thus the variance of $\hat{S}$ is given by

$$\begin{aligned}\operatorname{Var}(\hat{S}) &= \frac{S_1^2\gamma^2\mu_1^2(n_1(1+2\gamma^2)+n_2)}{n_1(1+2\gamma^2)(n_1+n_2)} + \frac{S_2^2\gamma^2\mu_2^2(n_2(1+2\gamma^2)+n_1)}{n_2(1+2\gamma^2)(n_1+n_2)} \\ &\quad + S_3^2\frac{\gamma^2(1+2\gamma^2)}{2(n_1+n_2)} + 2S_1S_2\frac{2\gamma^4\mu_1\mu_2}{(n_1+n_2)(1+2\gamma^2)} \\ &\quad + 2S_2S_3\frac{-\gamma^3\mu_2}{(n_1+n_2)} + 2S_1S_3\frac{-\gamma^3\mu_1}{(n_1+n_2)}.\end{aligned}$$

An estimate of $\operatorname{Var}(\hat{S})$ can be obtained by using the MLE's $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\gamma}$. Once an estimate of the variance is obtained, it is easy to set up a $100(1-\alpha)\%$ confidence interval for S by assuming asymptotic normally of $\hat{S}$. Since $\Phi(\cdot)$ is an increasing function of S, one can construct a confidence interval for R.

3. Estimation of reliability function when γ is known

In the previous section we estimated the parameters μ_1, μ_2 and γ. Now we consider the case where γ is known. Here we have to estimate only the parameters μ_1 and μ_2.

In this case the log likelihood function is given by

$$\begin{aligned}L = &-(n_1+n_2)\ln(\sqrt{2\pi}\gamma) - n_1\ln(\mu_1) - n_2\ln(\mu_2) \\ &-\left[\sum_{i=1}^{n_1}\frac{(x_i-\mu_1)^2}{2\gamma^2\mu_1^2} + \sum_{i=1}^{n_2}\frac{(y_i-\mu_2)^2}{2\gamma^2\mu_2^2}\right].\end{aligned} \tag{3.1}$$

It can be verified that the MLE's $\hat{\mu}_1$ and $\hat{\mu}_2$ are given by

$$\hat{\mu}_1 = \frac{-\bar{x} + \sqrt{\bar{x}^2 + 4\gamma^2(s_1^2 + \bar{x}^2)}}{2\gamma^2} \quad \text{and}$$

(3.2)

$$\hat{\mu}_2 = \frac{-\bar{y} + \sqrt{\bar{y}^2 + 4\gamma^2(s_2^2 + \bar{y}^2)}}{2\gamma^2}.$$

Proceeding as before the information matrix is given by

$$I = \begin{bmatrix} \frac{n_1(1+2\gamma^2)}{\mu_1^2\gamma^2} & 0 \\ 0 & \frac{n_2(1+2\gamma^2)}{\mu_2^2\gamma^2} \end{bmatrix}$$

whose inverse is given by

$$A = \begin{bmatrix} \frac{\mu_1^2\gamma^2}{n_1(1+2\gamma^2)} & 0 \\ 0 & \frac{\mu_2^2\gamma^2}{n_2(1+2\gamma^2)} \end{bmatrix}.$$

The diagonal entries give the asymptotic variances of $\hat{\mu}_1$ and $\hat{\mu}_2$ and the covariances are given by the off diagonal entries which turn out to be zero. So we can use the same method as described in the last section to determine the variance of $\hat{S}$ by the delta method.

4. Estimation of reliability function without the assumption of common coefficient of variation

Reiser and Guttman (1986) presented confidence intervals for $P(X < Y)$ without the assumption of a common coefficient of variation. Their method is presented below for comparison purposes.

$$R = P(X < Y) = \Phi(\delta),$$

where $\delta = (\mu_2 - \mu_1)/\sqrt{\sigma_1^2 + \sigma_2^2}$.

Let $\hat{\delta} = \frac{\bar{y}-\bar{x}}{s}$ where

$$s = \sqrt{s_1^2 + s_2^2}, \quad s_1^2 = \sum(x_1 - \bar{x})^2/(n_1 - 1) \quad \text{and}$$
$$s_2^2 = \sum(y_i - \bar{y})^2/(n_2 - 1).$$

Let

$$M = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2/n_1 + \sigma_2^2/n_2}, \quad \text{with} \quad \hat{M} = \frac{s_1^2 + s_2^2}{s_1^2/n_1 + s_2^2/n_2}.$$

Again let

$$f = \frac{(\sigma_1^2 + \sigma_2^2)^2}{\sigma_1^4/(n_1 - 1) + \sigma_2^4/(n_2 - 1)}, \quad \text{with} \quad \hat{f} = \frac{(s_1^2 + s_2^2)^2}{s_1^4/(n_1 - 1) + s_2^4/(n_2 - 1)}.$$

Then the two sided $100(1-\alpha)\%$ confidence interval for δ is given by

$$\hat{\delta} \mp a$$

and the confidence interval for R is

$$(\Phi(\hat{\delta}-a), \Phi(\hat{\delta}+a))$$

where

$$a = \left(\frac{1}{\hat{M}} + \frac{\hat{\delta}^2}{2\hat{f}}\right)^{1/2} Z_{\alpha/2}$$

and $Z_{\alpha/2}$ is the upper $\alpha/2$-th percentile of the standard normal distribution.

5. An example

The data in Table 1 are taken from Nelson ((1990), p. 115) and represent the hours to failure of 20 motorettes with a new class-H insulation run at 240° and 220°C. It has been observed by Nelson (1990) that lognormal distribution adequately fits at the two temperatures.

Table 1.

	X(240°C)*	Y(220°C)*
1	7.0690	7.4753
2	7.0690	7.7981
3	7.3271	7.7981
4	7.3582	7.7981
5	7.3883	7.7981
6	7.4176	7.7981
7	7.4176	8.0417
8	7.4460	8.0417
9	7.4736	8.0417
10	7.5771	8.0417

*Note that X and Y denote the natural logarithm of the failure times. We thus assume that the above data come from independent normal distributions.

We now briefly outline the score test developed by Gupta and Ma (1996) to test the equality of the coefficients of variation.

We want to test the hypothesis

$$H_0 : \sigma_1/\mu_1 = \sigma_2/\mu_2 = \gamma \quad \text{(unknown)}$$

against

$$H_1 : \sigma_1/\mu_1 \neq \sigma_2/\mu_2,$$

where σ_1^2 and σ_2^2 are the population variances. Then the test statistic is given by

$$T = \frac{\hat{\gamma}^2(1+2\hat{\gamma}^2)}{2}\left(\frac{a_1^2}{n_1}+\frac{a_2^2}{n_2}\right)$$

where

$$a_1 = \sum_{i=1}^{n_1}\frac{(x_i-\hat{\mu}_1)}{\hat{\mu}_1^2\hat{\gamma}^3} - \frac{n_1}{\hat{\gamma}}$$

and

$$a_2 = \sum_{i=1}^{n_2}\frac{(y_i-\hat{\mu}_2)^2}{\hat{\mu}_2^2\hat{\gamma}^3} - \frac{n_2}{\hat{\gamma}}.$$

Under H_0, T has a chi-square distribution with 1 degree of freedom. In our example, using the results of Section 2, the maximum likelihood estimates are given by $\hat{\mu}_1 = 7.354238$, $\hat{\mu}_2 = 7.863381$, and $\hat{\gamma} = .021644$. The computed values of a_1 and a_2 are

$$a_1 = -16.04, \qquad a_2 = 15.06.$$

The observed value of the test statistic $T = .0113$. Hence the null hypothesis is not rejected and the p value of the test is almost one.

Remark. It is well known that not rejecting H_0 does not necessarily mean that the two coefficients of variation are equal. However, in this case such a high p value would indicate that the null hypothesis is true and, therefore, the procedure of this paper would be applicable.

Using the estimates obtained before, the MLE of R is given by $\hat{R} = .9854$. The variance-covariance matrix of the estimators is given by

$$\begin{bmatrix} 2532 & 1.268 & -3.728 \\ 1.268 & 2895 & -3.986 \\ -3.728 & -3.986 & 11.72 \end{bmatrix}.$$

Note that the above entries should be multiplied by 10^{-6}. These give $\mathrm{Var}(\hat{S}) = .2188$ and the 95% asymptotic confidence interval for R as $(.898, .999)$. The length of the interval is .101. The length of the interval, as presented in Section 4, without the assumption of equal coefficients of variation is .124. Thus the assumption of equal coefficients of variation results in a gain of 23%.

6. Simulation studies

In this section simulation studies are carried out to study the distribution of $\hat{R}$, the coverage probability and the mean length of the confidence intervals (C.I.) assuming equal coefficients of variation and without assuming equal c.v.s. 10,000 observations were generated for parameter values $\mu_1 = 0.5$, $\mu_2 = 0.8$ and for various values of γ between .2 and 2 for each of the four cases (i) $n_1 = 30$,

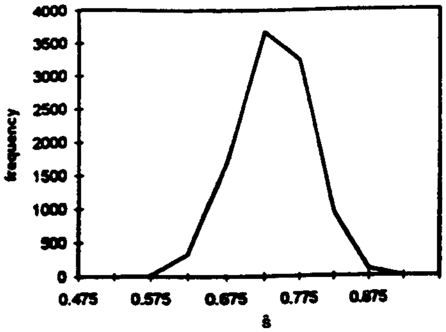

Fig. 1. Distribution of $\hat{S}$. $\mu_1 = 0.5$, $\mu_2 = 0.8$, $\gamma = 0.5$.

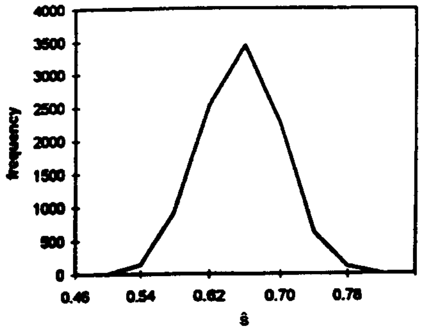

Fig. 2. Distribution of $\hat{S}$. $\mu_1 = 0.5$, $\mu_2 = 0.8$, $\gamma = 0.8$.

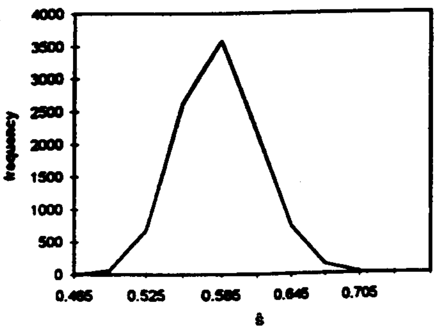

Fig. 3. Distribution of $\hat{S}$. $\mu_1 = 0.5$, $\mu_2 = 0.8$, $\gamma = 1.5$.

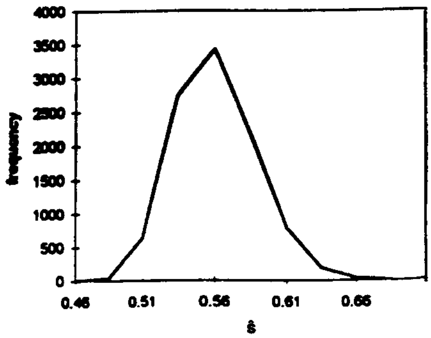

Fig. 4. Distribution of $\hat{S}$. $\mu_1 = 0.5$, $\mu_2 = 0.8$, $\gamma = 2.0$.

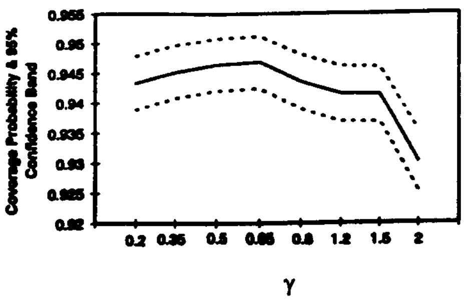

Fig. 5. Coverage probability. $\mu_1 = 0.5$, $\mu_2 = 0.8$.

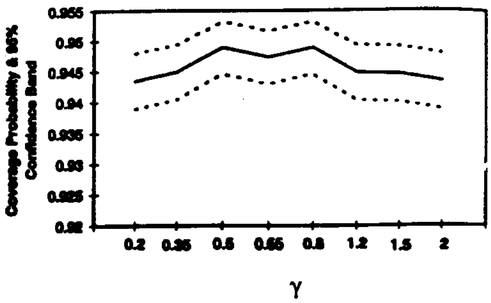

Fig. 6. Coverage probability. $\mu_1 = 0.5$, $\mu_2 = 1.5$.

$n_2 = 32$, $\alpha = .05$, (ii) $n_1 = 30$, $n_2 = 32$, $\alpha = .10$, (iii) $n_1 = 20$, $n_2 = 25$, $\alpha = .05$ and (iv) $n_1 = 20$, $n_2 = 25$, $\alpha = .10$. The results are presented only for case (i) for the sake of brevity because the others are similar.

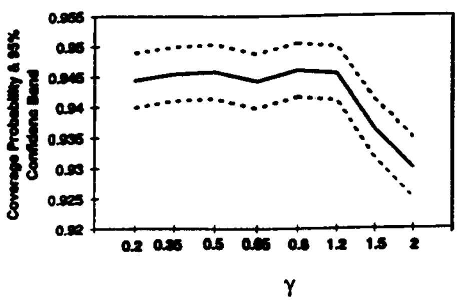

Fig. 7. Coverage probability. $\mu_1 = 1.5$, $\mu_2 = 1.0$.

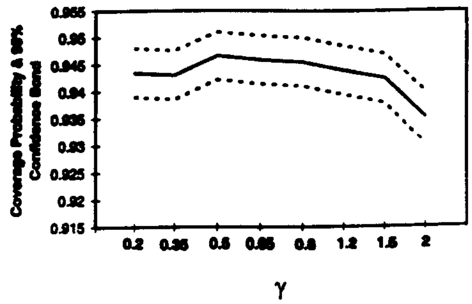

Fig. 8. Coverage probability. $\mu_1 = 2.0$, $\mu_2 = 1.5$.

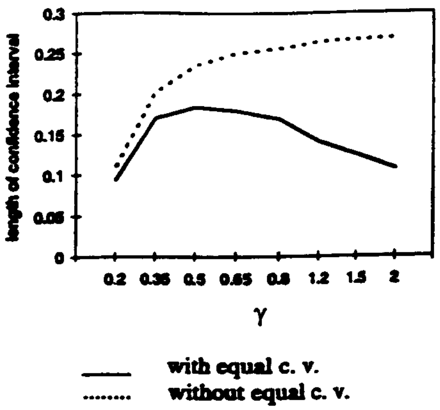

Fig. 9. Length of the confidence interval for R. $\mu_1 = 0.5$, $\mu_2 = 0.8$.

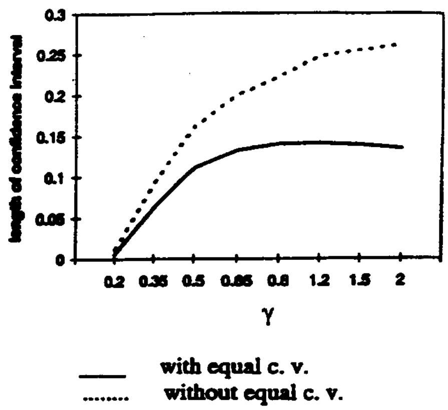

Fig. 10. Length of the confidence interval for R. $\mu_1 = 0.5$, $\mu_2 = 1.5$.

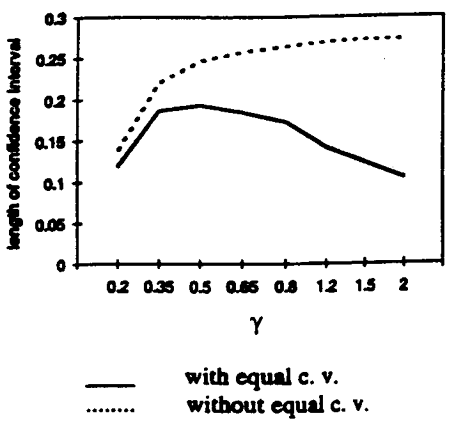

Fig. 11. Length of the confidence interval for R. $\mu_1 = 1.5$, $\mu_2 = 1.0$.

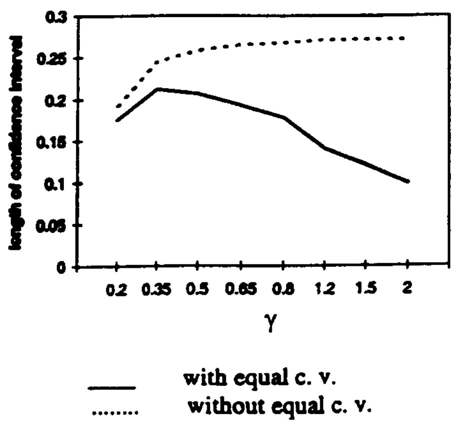

Fig. 12. Length of the confidence interval for R. $\mu_1 = 2.0$, $\mu_2 = 1.5$.

Distribution of $\hat{S}$. The distribution of $\hat{S}$, for values of $\gamma = 0.5$, .8, 1.5 and 2.0, is shown, in Figs. 1–4. It is observed that the distribution of $\hat{S}$ is approximately normal. In fact it has been noticed that the distribution of $\hat{R}$ is also approximately normal.

Coverage probabilities. In all cases, the coverage probabilities are close to the nominal value (.95). The length of the C.I. is less for small and large values of γ. For $\mu_1 = 0.5$ and $\mu_2 = 0.8$, the length is decreasing for $\gamma > .5$ and for $\mu_1 = 0.5$ and $\mu_2 = 1.5$, the length starts decreasing for values of $\gamma > 1$ (approximately). Overall both the coverage probability and the length of the C.I. are good.

Figures 5–8 also contain confidence bands for the coverage probabilities. These confidence bands have been obtained, to give a measure of error to the estimated coverage probabilities, by computing 95% C.I.'s based on the binomial.

Comparison with Reiser and Guttman. The lengths of the C.I. were compared by using our assumption of equal C.V. and by not using this assumption. The results are shown in Figs. 9–12. In all the cases, the lengths are much larger by Reiser and Guttman's method. In fact, in some cases, the length of the C.I. is more than double (by using Reiser and Guttman) the length obtained by using the results of this paper.

7. Concluding remarks

The simulation results, in this paper, indicate that assuming homogeneous coefficients of variation the new C.I.'s are better than the Reiser and Guttman (1986) results. One way to test the homogenity of coefficients of variation is to use Gupta and Ma's (1996) procedure. However, not rejecting the null hypothesis by Gupta and Ma test does not guarantee that the properties shown for the new procedure holds. A better comparison would be to compare the two methods when the coefficients are close but different. It would be useful to know how different the coefficients can be before the proposed method becomes suitable. This robustness issue is quite involved and requires further research.

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Appendix

If the matrix

$$A = \left[\frac{\partial \ln L}{\partial \theta_r \theta_s}\right]_{\hat{\theta}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where $\theta = \mu_1$, $\theta_2 = \mu_2$ and $\theta_3 = \gamma$, is negative definite, then the point estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\gamma}$ define a maximum, see Kendall and Stuart (1967).

In our case A is of the form

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Using a result of Rao (1965), p. 34, it can be verified that the above matrix is negative definite if (1) $a_{11} < 0$, (2) $a_{22} < 0$ and (3) $a_{33} < \frac{a_{13}^2}{a_{11}} + \frac{a_{23}^2}{a_{22}}$.

In our case

$$A = \begin{bmatrix} \frac{n_1\gamma^2\mu_1^2 + 2n_1\bar{x} - 3\sum x_i^2}{\gamma^2\mu_1^4} & 0 & \frac{2\mu_1 n_1\bar{x} - 2\sum x_1^2}{\gamma^3\mu_1^3} \\ 0 & \frac{n_2\gamma^2\mu_2^2 + 2n_2\bar{y} - 3\sum y_j^2}{\gamma^2\mu_2^4} & \frac{2\mu_2 n_2\bar{y} - 2\sum y_j^2}{r^3\mu_2^3} \\ \frac{2\mu_1 n_1\bar{x} - 2\sum x_i^2}{\gamma^3\mu_1^3} & \frac{2\mu_2 n_2\bar{y} - 2\sum y_j^2}{\gamma^3\mu_2^3} & \frac{(n_1+n_2)\gamma^2 - 3\sum(x_i-\mu_1)^2 - 3\sum(y_j-\mu_2)^2}{\gamma^4} \end{bmatrix}$$

where the matrix A is evaluated at $(\hat{\mu}_1, \hat{\mu}_2, \hat{\gamma})$.

Proceeding as in Lohrding (1969*b*), after considerable manipulation of inequalities, one can show that the matrix A is negative definite. The details are omitted.

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