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APPROXIMATION OF THE POSTERIOR DISTRIBUTION IN A CHANGE-POINT PROBLEM

SUBHASHIS GHOSAL[1]*, JAYANTA K. GHOSH[1] AND TAPAS SAMANTA[2]**

[1] *Division of Theoretical Statistics and Mathematics, Indian Statistical Institute, 203 B. T. Road, Calcutta-700035, India*
[2] *Computer Science Unit, Indian Statistical Institute, 203 B. T. Road, Calcutta-700035, India*

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Abstract. We consider a family of models that arise in connection with sharp change in hazard rate corresponding to high initial hazard rate dropping to a more stable or slowly changing rate at an unknown change-point θ. Although the Bayes estimates are well behaved and are asymptotically efficient, it is difficult to compute them as the posterior distributions are generally very complicated. We obtain a simple first order asymptotic approximation to the posterior distribution of θ. The accuracy of the approximation is judged through simulation. The approximation performs quite well. Our method is also applied to analyze a real data set.

Key words and phrases: Change-point, Gibbs sampling, hazard rate, posterior distribution.

1. Introduction

We consider a change-point model $\{f(x;\theta) : x \geq 0\}$ for life-time data where the density $f(x;\theta)$ is decreasing in x for every θ and is continuous everywhere except at $x = \theta$ where the form of the density changes. The parameter θ is called the change-point for such a model. Such models are of significant importance in reliability theory. For equipments with high infant mortality rate up to a change-point θ, it is often customary to "burn in" equipments up to θ or an estimated value of θ and only sell survivors. Such models have been considered by Nguyen *et al.* (1984), Basu *et al.* (1988), Ghosh and Joshi (1992) and Ghosh *et al.* (1993, 1996). For a different application to leukemia patients, see Matthews and Farewell (1982) and Achcar and Bolfarine (1989). An important example of such a model

* Now at Department of Mathematics, Free University, 1081 HV Amsterdam, The Netherlands. Research supported by the National Board of Higher Mathematics, Department of Atomic Energy, Bombay, India.

** Now at Applied Statistics Unit, Indian Statistical Institute, 203 B. T. Road, Calcutta-700035, India.

is given by the density

$$f(x;\theta) = \begin{cases} ae^{-ax}, & 0 \le x \le \theta, \\ b\exp[-(a-b)\theta]e^{-bx}, & \theta < x < \infty, \end{cases} \tag{1.1}$$

$0 < b < a < \infty$. More generally, we can consider

$$f(x;\theta) = \begin{cases} \lambda_1(x+\mu)^{\alpha-1}e^{-ax}, & 0 \le x \le \theta, \\ \lambda_2(\theta)(x+\mu)^{\alpha-1}e^{-bx}, & \theta < x < \infty, \end{cases} \tag{1.2}$$

where $\lambda_1^{-1} = \int_0^\infty (x+\mu)^{\alpha-1}e^{-ax}dx$ and

$$\lambda_2(\theta) = \lambda_1 \frac{\int_\theta^\infty (x+\mu)^{\alpha-1}e^{-ax}dx}{\int_\theta^\infty (x+\mu)^{\alpha-1}e^{-bx}dx};$$

here a, b, α and μ are fixed positive constants satisfying $0 < b < a < \infty$, $\mu \ge (\alpha-1)/b$ if $1 \le \alpha < \infty$ and $\mu > 0$ if $0 < \alpha < 1$. The condition on μ is imposed here so as to make the density decreasing and bounded (in x) as required for the application of our results.

The class of densities described above does not satisfy the "usual regularity conditions". In fact, θ is a point of discontinuity. This kind of nonregular cases were first studied by Chernoff and Rubin (1956). Ibragimov and Has'minskii (1981) (henceforth abbreviated as IH) studied a wide variety of nonregular cases (that covers the change-point models also, if the change-point parameter is bounded) and obtained asymptotic properties of the maximum likelihood estimate (MLE) and the Bayes estimates. It follows that the MLE and the Bayes estimates are consistent and converge at a rate n^{-1}. Further, the Bayes estimates are asymptotically more efficient than the MLE. Also, by Proposition 1 of Ghosal *et al.* (1995) and in view of the results of Chapter V of IH, the posterior is consistent with probability one and concentrates in $O(n^{-1})$-neighbourhoods of the true parameter with probability tending to one for priors which are positive and continuous. However, the form of the actual posterior distribution may be awkward which prevents one from various Bayesian computations. If the posterior distribution (of a suitably centered and normalized parameter) approaches a simple form, as the sample size increases indefinitely, one can easily execute approximate (up to the first order) Bayesian computations based on the limiting form. Unfortunately, as typical in most of the nonregular cases, Theorem 2.4 of Ghosh *et al.* (1994) implies that the posterior distribution in the change-point problem does not approach a limit (details are shown in pp. 32–35 of Ghosal (1994)). In this paper, we first find a useful approximation to the posterior distribution which depends on the sample size, assuming θ is the only unknown parameter. In presence of additional parameters (for example, if a and b in (1.1) are unknown), a similar approximation to the marginal posterior of θ is also obtained. This is established by using an approximation similar to (2.8) below for the corresponding likelihood ratio process $Z_n(\cdot)$ obtained in Ghosal and Samanta (1995). Also, this suggests an approximation of the form (2.11) to the joint posterior of all the parameters. To justify this, one has to prove an analogous version of Lemma 2.2. Nevertheless, we show that our approximations coupled with the Gibbs sampling technique enable us to approximately compute the posterior distribution in the multiparameter case even if some observations are censored.

The paper is organized as follows. We deal with the one-parameter case in Section 2 and the multiparameter case is considered in Section 3. The proofs of the auxiliary lemmas are given in the Appendix. In Section 4, we present the results of a simulation study to judge the accuracy of the approximations obtained in the paper and illustrate our method with a real data set.

2. One-parameter case

Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) observations with a common density

$$f(x;\theta) = \begin{cases} f_1(x), & 0 \le x \le \theta, \\ (\bar{F}_1(\theta)/\bar{F}_2(\theta))f_2(x), & \theta < x < \infty, \end{cases} \tag{2.1}$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are (completely specified) nonincreasing, continuous densities with survival functions $\bar{F}_1(\cdot)$ and $\bar{F}_2(\cdot)$ respectively, the hazard rate of $f_1(\cdot)$ is greater than that of $f_2(\cdot)$, i.e.,

$$r_1(x) := \frac{f_1(x)}{\bar{F}_1(x)} > r_2(x) := \frac{f_2(x)}{\bar{F}_2(x)}, \qquad x > 0, \tag{2.2}$$

and $0 < \theta < K$, K being a known bound.

The point of discontinuity θ is often referred to as a change-point or a point where a new density takes over. It is easily observed that the conditions stated in IH (p. 242) are satisfied. At the change-point θ, the left and right hand limits of $f(x;\theta)$ are

$$\begin{aligned} q(\theta) &:= \lim_{x \uparrow \theta} f(x;\theta) = f_1(\theta), \\ p(\theta) &:= \lim_{x \downarrow \theta} f(x;\theta) = (\bar{F}_1(\theta)/\bar{F}_2(\theta))f_2(\theta). \end{aligned} \tag{2.3}$$

It follows from (2.2) that $q(\theta) > p(\theta)$.

Note that the assumption of nonincreasing density is satisfied if the hazard rates $r_1(x)$ and $r_2(x)$ are nonincreasing. The assumption that θ is bounded is a crucial one. If all the parameters are unbounded both the MLE or the Bayes estimates may misbehave (see Pham and Nguyen (1990)) and so one has to look for alternative estimators. Pham and Nguyen (1990) considered an estimator which maximizes likelihood over random compact sets and obtained strong consistency and asymptotic distribution of the resulting estimator. As the asymptotic distribution is complicated, Pham and Nguyen (1993) also considered an useful bootstrap approximation. Since the posterior distribution may be inconsistent when θ is unbounded and the objective of the present paper is to approximate the posterior, we restrict our attention only to the case when θ is bounded. From a practical point of view, however, this assumption is a mild requirement since failure before the change-point may be viewed as infant mortality.

We fix $0 < \theta_0 < K$ and consider the likelihood ratio process

$$Z_n(u) = \prod_{i=1}^{n} \frac{f(X_i; \theta_0 + u/n)}{f(X_i; \theta_0)}, \qquad u \in U_n := (-n\theta_0, n(K - \theta_0)), \tag{2.4}$$

and set $Z_n(u) = 0$ otherwise.

Suppose we have a prior density $\pi(\theta)$ which is positive and continuous at θ_0 and bounded on $(0, K)$. The posterior density of the normalized parameter $u = n(\theta - \theta_0)$ given the observations $X_1, \ldots, X_n$ is given by

$$\pi_n(u) = \frac{Z_n(u)\pi(\theta_0 + u/n)}{\int Z_n(w)\pi(\theta_0 + w/n)dw}. \tag{2.5}$$

We are interested in obtaining an approximate expression (correct up to the first order) for $\pi_n(u)$ as $n \to \infty$. In view of Theorem 2.1 of Ghosh *et al.* (1994), we may assume, without loss of generality, that $\pi(\theta) \equiv 1$.

Let

$$\tilde{Z}_n(u) = \exp\left[c(\theta_0)u + \delta(\theta_0)\,\mathrm{sign}(u) \sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right] \tag{2.6}$$

for $u > -n\theta_0$ and set it to be zero otherwise where $c(\theta_0) = p(\theta_0) - q(\theta_0) < 0$ and $\delta(\theta_0) = \log(q(\theta_0)/p(\theta_0)) > 0$. We shall drop θ_0 whenever there is no source of confusion.

Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics based on $X_1, \ldots, X_n$ and let $X_{0:n} = 0$ and $X_{n+1:n} = \infty$ as convention. We set $U_{i:n} = n(X_{i:n} - \theta_0)$, $i = 0, 1, \ldots, n, n+1$. Then $\tilde{Z}_n(u)$ can be expressed as

$$\tilde{Z}_n(u) = \begin{cases} \exp[cu + \delta(r - k)], & U_{r:n} < u \leq U_{r+1:n},\ r = 0, \ldots, k-1, \\ \exp[cu], & U_{k:n} < u < U_{k+1:n}, \\ \exp[cu + \delta(r - k)], & U_{r:n} \leq u < U_{r+1:n},\ r = k+1, \ldots, n, \end{cases} \tag{2.7}$$

where k is such that $U_{k:n} < 0 < U_{k+1:n}$.

It has been shown in IH (Chapter V) that

$$Z_n(u) = \tilde{Z}_n(u) + o_p(1) \tag{2.8}$$

and the convergence is uniform in u belonging to compacts.

It has also been shown in IH (Chapter V) that the finite dimensional distributions of the processes $Z_n(\cdot)$ and $\tilde{Z}_n(\cdot)$ converge to the process $Z(\cdot)$ defined by

$$Z(u) = \begin{cases} \exp[cu + \delta\nu(u)], & u \geq 0, \\ \exp[cu + \delta\tilde{\nu}(-u)], & u < 0, \end{cases} \tag{2.9}$$

where $\nu(\cdot)$ and $\tilde{\nu}(\cdot)$ are independent homogeneous Poisson processes with rates p and q respectively. It is straightforward to check that

$$EZ(u) = 1, \qquad u \in \mathbb{R}. \tag{2.10}$$

From (2.8), it is natural to expect that the posterior density is approximated by $\tilde{\pi}_n(\cdot)$, where

$$\tilde{\pi}_n(u) = \frac{\tilde{Z}_n(u)}{\int \tilde{Z}_n(w)dw}. \tag{2.11}$$

Indeed we have the following result:

THEOREM 2.1. *Under the above set up,*

$$\int |\pi_n(u) - \tilde{\pi}_n(u)|du \to_p 0. \tag{2.12}$$

To prove Theorem 2.1, we shall use the following lemmas whose proofs are deferred to the Appendix.

LEMMA 2.1. *For any $u \in \mathbb{R}$,*

$$\lim_{n\to\infty} E|Z_n(u) - \tilde{Z}_n(u)| = 0. \tag{2.13}$$

LEMMA 2.2. *The following assertions hold:*
(i) $E\tilde{Z}_n(u) \leq 1$, $u \in \mathbb{R}$.
(ii) $E\tilde{Z}_n^{1/2}(u) \leq \exp[-\lambda|u|]$, $u \in \mathbb{R}$, *where* $\lambda = (p^{1/2} - q^{1/2})^2/2$.
(iii) *Given any $\varepsilon > 0$, there exist $N \geq 1$, $B_0 > 0$ such that for all $n \geq N$,*

$$E|\tilde{Z}_n^{1/2}(u_1) - \tilde{Z}_n^{1/2}(u_2)|^2 \leq B_0 \exp[\varepsilon R]|u_1 - u_2|$$

whenever $u_1, u_2 \in \mathbb{R}$ with $|u_1| \leq R$, $|u_2| \leq R$, $R \geq 0$.

LEMMA 2.3. *There exist $N \geq 1$, $B, b > 0$ such that for all $n \geq N$,*

$$E\left(\int_{|u|>H} \tilde{\pi}_n(u)du\right) \leq B\exp[-bH], \qquad H > 0. \tag{2.14}$$

Remark 2.1. It can be noted from the proof of Lemma 2.3 in the Appendix that (2.14) can be strengthened to:

There exist $\varepsilon_0 > 0$, $N \geq 1$, $B, b > 0$, such that for all $n \geq N$,

$$E\left(\int_{|u|>H} \exp[\varepsilon_0|u|]\tilde{\pi}_n(u)du\right) \leq B\exp[-bH], \qquad H > 0. \tag{2.15}$$

In particular, $\int \exp[\varepsilon_0|u|]\tilde{\pi}_n(u)du$ is stochastically bounded.

PROOF OF THEOREM 2.1. For any $H > 0$, we have

$$\int |\pi_n(u) - \tilde{\pi}_n(u)|du \le \int_{|u|\le H} |\pi_n(u) - \tilde{\pi}_n(u)|du + \int_{|u|>H} \pi_n(u)du + \int_{|u|>H} \tilde{\pi}_n(u)du. \tag{2.16}$$

Fix $\eta > 0$ and choose $H > 0$ and $N \ge 1$ sufficiently large such that

$$E\left(\int_{|u|>H} \pi_n(u)du\right) < \eta, \qquad n \ge N, \tag{2.17}$$

and

$$E\left(\int_{|u|>H} \tilde{\pi}_n(u)du\right) < \eta, \qquad n \ge N. \tag{2.18}$$

Relation (2.17) follows from Lemma I.5.2 and Theorem V.2.5 of IH while (2.18) follows from Lemma 2.3 above. For this chosen H, set $I_{1n} = \int_{|u|>H} \pi_n(u)du$ and $I_{2n} = \int_{|u|>H} \tilde{\pi}_n(u)du$. Now

$$\begin{aligned}
&\int_{|u|\le H} |\pi_n(u) - \tilde{\pi}_n(u)|du \\
&\quad \le \int_{|u|\le H} \left| \frac{Z_n(u)}{\int Z_n(w)dw} - \frac{\tilde{Z}_n(u)}{\int Z_n(w)dw} \right| du \\
&\qquad + \int_{|u|\le H} \left| \frac{\tilde{Z}_n(u)}{\int Z_n(w)dw} - \frac{\tilde{Z}_n(u)}{\int \tilde{Z}_n(w)dw} \right| du \\
&\quad = \left(\int Z_n(w)dw\right)^{-1} \int_{|u|\le H} |Z_n(u) - \tilde{Z}_n(u)|du \\
&\qquad + \left| \left(\int Z_n(w)dw\right)^{-1} - \left(\int \tilde{Z}_n(w)dw\right)^{-1} \right| \int_{|u|\le H} \tilde{Z}_n(u)du.
\end{aligned} \tag{2.19}$$

The second term on the RHS of (2.19) is bounded by

$$\begin{aligned}
&\left(\int Z_n(w)dw\right)^{-1} \left(\int \tilde{Z}_n(w)dw\right)^{-1} \int_{|u|\le H} \tilde{Z}_n(u)du \\
&\quad \times \left(\int_{|w|>H} Z_n(w)dw + \int_{|w|>H} \tilde{Z}_n(w)dw \right. \\
&\qquad \left. + \int_{|w|\le H} |Z_n(w) - \tilde{Z}_n(w)|dw \right)
\end{aligned} \tag{2.20}$$

$$\le I_{1n} + I_{2n}\left(\int Z_n(w)dw\right)^{-1}$$
$$\times\left(\int_{|u|\le H} Z_n(u)du + \int_{|u|\le H}|Z_n(u) - \tilde{Z}_n(u)|du\right)$$
$$+\left(\int Z_n(w)dw\right)^{-1}\int_{|w|\le H}|Z_n(w) - \tilde{Z}_n(w)|dw$$
$$\le I_{1n} + I_{2n} + 2\left(\int Z_n(u)du\right)^{-1}\int_{|u|\le H}|Z_n(u) - \tilde{Z}_n(u)|du.$$

In view of (2.17) and (2.18), it is thus enough to show that

$$\left(\int Z_n(u)du\right)^{-1}\int_{|u|\le H}|Z_n(u) - \tilde{Z}_n(u)|du \to_p 0. \tag{2.21}$$

Since for any $M > 0$,

$$\left(\int Z_n(u)du\right)^{-1} \le \left(\int_{|u|\le M} Z_n(u)du\right)^{-1} \tag{2.22}$$

and $\int_{|u|\le M} Z_n(u)du$ converges to the positive random variable $\int_{|u|\le M} Z(u)du$ by virtue of Theorems V.2.1, V.2.5 and 1.A.22 of IH (here $Z(u)$ is as defined in (2.9)), it follows that

$$\left(\int Z_n(u)du\right)^{-1} = O_p(1). \tag{2.23}$$

Alternatively, (2.23) can be deduced from Lemma I.5.1 of IH. Hence it remains to show that

$$\int_{|u|\le H}|Z_n(u) - \tilde{Z}_n(u)|du \to_p 0. \tag{2.24}$$

To this end, observe that

$$E\left(\int_{|u|\le H}|Z_n(u) - \tilde{Z}_n(u)|du\right) = \int_{|u|\le H} E|Z_n(u) - \tilde{Z}_n(u)|du \to 0 \tag{2.25}$$

by Lemma 2.1 and the dominated convergence theorem, since

$$E|Z_n(u) - \tilde{Z}_n(u)| \le EZ_n(u) + E\tilde{Z}_n(u) \le 2. \tag{2.26}$$

□

Remark 2.2. In a manner similar to the proof of Theorem 2.1, one can also show that for any $k \ge 0$,

$$\int |u|^k|\pi_n(u) - \tilde{\pi}_n(u)|du \to_p 0. \tag{2.27}$$

This yields approximations for the posterior moments.

Theorem 2.1 provides an approximation to the posterior distribution of $u = n(\theta - \theta_0)$ (and hence of θ); the approximation, however, involves the unknown parameter point θ_0. We therefore consider another version of Theorem 2.1 where θ_0 is replaced by a "good" estimate $\hat{\theta}$.

Let $\hat{\theta}$ be an n-consistent estimate of the parameter, i.e., $n(\hat{\theta} - \theta_0) = O_p(1)$. Existence of such estimators is ensured as the MLE or Bayes estimates are n-consistent (see Section V.4 of IH). The Bayes estimate, however, cannot be evaluated without computing the posterior which we are trying to approximate. The MLE or the approximate MLE (as suggested by Chernoff and Rubin (1956)) can, however, be used for this purpose.

Set $w = n(\theta - \hat{\theta})$. The posterior density of w is given by

$$\pi_n^*(w) = \pi_n(w + n(\hat{\theta} - \theta_0)). \tag{2.28}$$

From Theorem 2.1, we immediately have

$$\int |\pi_n^*(w) - \tilde{\pi}_n^*(w)| dw \to_p 0, \tag{2.29}$$

where

$$\begin{aligned}\tilde{\pi}_n^*(w) &= \tilde{\pi}_n(w + n(\hat{\theta} - \theta_0)) \\ &= \frac{|c| \exp[cw + \delta r]}{\sum_{s=0}^{n} (\exp[cW_{s:n}] - \exp[cW_{s+1:n}]) e^{\delta s}}, \\ &\qquad\qquad w \in (W_{r:n}, W_{r+1:n}],\end{aligned} \tag{2.30}$$

and $W_{r:n} = U_{r:n} - n(\hat{\theta} - \theta_0) = n(X_{r:n} - \hat{\theta})$, $r = 0, \ldots, n+1$.

The approximation $\tilde{\pi}_n^*(w)$ still involves θ_0 through the constants $c = c(\theta_0)$ and $\delta = \delta(\theta_0)$. Set

$$\begin{aligned}\hat{\pi}_n(w) &= \frac{|\hat{c}| \exp[\hat{c}w + \hat{\delta} r]}{\sum_{s=0}^{n} (\exp[\hat{c}W_{s:n}] - \exp[\hat{c}W_{s+1:n}]) e^{\hat{\delta} s}}, \\ &\qquad\qquad W_{r:n} < w \le W_{r+1:n},\ r = 0, \ldots, n,\end{aligned} \tag{2.31}$$

where $\hat{c} = c(\hat{\theta})$ and $\hat{\delta} = \delta(\hat{\theta})$. Clearly $c(\cdot)$ and $\delta(\cdot)$ are continuously differentiable and hence $\hat{c}$ and $\hat{\delta}$ are n-consistent estimates of c and δ respectively.

THEOREM 2.2. *Under the above set up,*

$$\int |\pi_n^*(w) - \hat{\pi}_n(w)| dw \to_p 0. \tag{2.32}$$

PROOF. Note that

$$\int |\tilde{\pi}_n^*(w) - \tilde{\pi}_n(w)|dw \le 2\left(\int \tilde{Z}_n(u)du\right)^{-1} \int |\tilde{Z}_n(u) - \hat{Z}_n(u)|du, \tag{2.33}$$

where $\hat{Z}_n(u)$ is obtained from $\tilde{Z}_n(u)$ by replacing c and δ by $\hat{c}$ and $\hat{\delta}$ respectively, i.e.,

$$\hat{Z}_n(u) = \exp\left[\hat{c}u + \hat{\delta}\,\mathrm{sign}(u)\sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right], \quad \text{if} \quad \theta_0 + u/n > 0,$$

and equals to zero otherwise. Now

$$\begin{aligned}
&\left(\int \tilde{Z}_n(u)du\right)^{-1} \int |\tilde{Z}_n(u) - \hat{Z}_n(u)|du \\
&\le \left(\int \tilde{Z}_n(u)du\right)^{-1} \int |\exp[cu] - \exp[\hat{c}u]| \\
&\quad \times \exp\left[\hat{\delta}\,\mathrm{sign}(u)\sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right] du \\
&\quad + \left(\int \tilde{Z}_n(u)du\right)^{-1} \int \exp[\hat{c}u] \\
&\quad \times \left|\exp\left[\hat{\delta}\,\mathrm{sign}(u)\sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right]\right. \\
&\qquad \left. - \exp\left[\delta\,\mathrm{sign}(u)\sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right]\right| du \\
&= J_{1n} + J_{2n} \qquad \text{(say)}.
\end{aligned} \tag{2.34}$$

Using the inequality

$$|e^x - e^y| \le |x - y| e^x \exp[|x - y|], \tag{2.35}$$

we have

$$J_{1n} \le |\hat{c} - c| \int |u| \exp[|\hat{c} - c||u|]\tilde{\pi}_n(u)du, \tag{2.36}$$

which converges to zero in probability by consistency of $\hat{c}$ and Remark 2.1.

By another application of (2.35),

$$\begin{aligned}
J_{2n} \le \int &|\delta - \hat{\delta}| \sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\} \exp[|c - \hat{c}||u|] \\
&\times \exp\left[|\delta - \hat{\delta}| \sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right] \tilde{\pi}_n(u)du.
\end{aligned} \tag{2.37}$$

We split the integral in (2.37) over the regions $|u| \leq H$ and $|u| > H$, where H is to be chosen later. Since $\sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\} \leq n$ and $\hat{\delta}$ is n-consistent for δ, it follows from (2.15) that the integral over $|u| > H$ can be made arbitrarily small with probability arbitrarily close to one by choosing H sufficiently large. For this chosen H, the integral over $|u| \leq H$ is dominated by

$$2H|\delta - \hat{\delta}| \sum_{i=1}^{n} I\{X_i \in [\theta_0 - H/n, \theta_0 + H/n]\} \exp[|c - \hat{c}|H] \times \exp\left[|\delta - \hat{\delta}| \sum_{i=1}^{n} I\{X_i \in [\theta_0 - H/n, \theta_0 + H/n]\}\right], \tag{2.38}$$

which converges to zero in probability since $\hat{c} \rightarrow_p c$, $\hat{\delta} \rightarrow_p \delta$ and $\sum_{i=1}^{n} I\{X_i \in [\theta_0 - H/n, \theta_0 + H/n]\}$ converges in law (to a Poisson$((p+q)H)$ random variable). Combining (2.29) and (2.33)–(2.38), we obtain (2.32). □

Remark 2.3. As in Remark 2.2, one can also show that for any $k \geq 0$,

$$\int |w|^k |\pi_n^*(w) - \tilde{\pi}_n(w)| dw \rightarrow_p 0. \tag{2.39}$$

Remark 2.4. In some cases (as in the model (1.1)), the parametric function $\delta(\cdot)$ may be free of θ. In such cases, one can use any consistent estimate $\hat{\theta}$ (such as estimates obtained by the method of moments) to obtain the convergence in (2.32). However, the approximation is likely to perform better if one uses superior estimates.

Remark 2.5. Theorems 2.1 and 2.2 are first order asymptotic results where the approximation does not depend on the prior provided it is continuous and positive. In practice, the approximations suggested by (2.12) and (2.32) are likely to perform better for relatively flat priors.

Remark 2.6. The approximations to the actual posterior given by (2.12) and (2.32) do not involve the bound K of θ. This enables one to use these results even if no precise knowledge of K is available.

3. Multiparameter case

We now consider the case with a density of the form given in (2.1) where $f_1(\cdot)$ and $f_2(\cdot)$ involve unknown parameters $\boldsymbol{\varphi}$. Thus we consider iid observations $X_1, \ldots, X_n$ with a common density

$$f(x; \theta, \boldsymbol{\varphi}) = \begin{cases} f_1(x; \boldsymbol{\varphi}), & 0 \leq x \leq \theta, \\ (\bar{F}_1(\theta; \boldsymbol{\varphi})/\bar{F}_2(\theta; \boldsymbol{\varphi})) f_2(x; \boldsymbol{\varphi}), & \theta < x < \infty, \end{cases} \tag{3.1}$$

$0 < \theta < K$ and $\boldsymbol{\varphi} \in \Phi$, an open subset of $\mathbb{R}^d$, $d \geq 1$.

The assumptions and notations are analogous to those in Section 2 with obvious modifications in presence of the additional parameter $\boldsymbol{\varphi}$. For example, the quantities p, q, c and δ involve the parameter $\boldsymbol{\varphi}$ and the prior density π is defined on $(0, K) \times \Phi$.

We fix $0 < \theta_0 < K$, $\boldsymbol{\varphi}_0 \in \Phi$ and consider the likelihood ratio process

$$Z_n(u, \boldsymbol{v}) = \prod_{i=1}^{n} \frac{f(X_i; \theta_0 + u/n, \boldsymbol{\varphi}_0 + n^{-1/2}\boldsymbol{v})}{f(X_i; \theta_0, \boldsymbol{\varphi}_0)}. \tag{3.2}$$

By convention, we set $Z_n(u, \boldsymbol{v}) = 0$ if $(\theta_0 + u/n, \boldsymbol{\varphi}_0 + n^{-1/2}\boldsymbol{v})$ does not belong to the parameter space.

For a prior density $\pi(\theta, \boldsymbol{\varphi})$, the posterior density of the normalized parameter $u = n(\theta - \theta_0)$ and $\boldsymbol{v} = n^{1/2}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_0)$ is given by

$$\pi_n(u, \boldsymbol{v}) = \frac{Z_n(u, \boldsymbol{v})\pi(\theta_0 + u/n, \boldsymbol{\varphi}_0 + n^{-1/2}\boldsymbol{v})}{\int Z_n(t, \boldsymbol{s})\pi(\theta_0 + t/n, \boldsymbol{\varphi}_0 + n^{-1/2}\boldsymbol{s})dtd\boldsymbol{s}} \tag{3.3}$$

and the marginal posterior for u has density

$$\pi_n(u) = \int \pi_n(u, \boldsymbol{v})d\boldsymbol{v}. \tag{3.4}$$

An approximation of $Z_n(u, \boldsymbol{v})$ obtained in Ghosal and Samanta (1995) is given by

$$\tilde{Z}_n(u, \boldsymbol{v}) = \tilde{Z}_n^{(1)}(u)\tilde{Z}_n^{(2)}(\boldsymbol{v}), \tag{3.5}$$

where

$$\tilde{Z}_n^{(1)}(u) = \exp\left[cu + \delta\,\mathrm{sign}(u)\sum_{i=1}^{n} I\{X_i \in [\theta_0, \theta_0 + u/n]\}\right],$$
$$\tilde{Z}_n^{(2)}(\boldsymbol{v}) = \exp\left[\boldsymbol{v}^T\boldsymbol{\Delta}_n - \frac{1}{2}\boldsymbol{v}^T\boldsymbol{F}\boldsymbol{v}\right],$$
$$\boldsymbol{\Delta}_n \to_d \boldsymbol{\Delta} \sim N_d(\boldsymbol{0}, \boldsymbol{F})$$

and $\boldsymbol{F}$ is the Fisher information matrix for $\boldsymbol{\varphi}$.

It is shown in Ghosal and Samanta (1995) that

$$Z_n(u, \boldsymbol{v}) = \tilde{Z}_n(u, \boldsymbol{v}) + o_p(1) \tag{3.6}$$

and the convergence is uniform in $(u, \boldsymbol{v})$ belonging to compacts. It is also shown in Ghosal and Samanta (1995) that the finite dimensional distributions of the processes $Z_n(\cdot)$ and $\tilde{Z}_n(\cdot)$ converge to those of the process $Z(\cdot)$ defined by

$$Z(u, \boldsymbol{v}) = Z^{(1)}(u)Z^{(2)}(\boldsymbol{v}), \tag{3.7}$$

where $Z^{(1)}(u)$ is given by (2.9) and

$$Z^{(2)}(\boldsymbol{v}) = \exp\left[\boldsymbol{v}^T\boldsymbol{\Delta} - \frac{1}{2}\boldsymbol{v}^T\boldsymbol{F}\boldsymbol{v}\right] \tag{3.8}$$

with $\boldsymbol{\Delta}$ independent of $Z^{(1)}(\cdot)$ and distributed as $N_d(\mathbf{0}, \boldsymbol{F})$.

The following theorem shows that even if there is an additional "regular" parameter $\boldsymbol{\varphi}$, the marginal posterior of $u = n(\theta - \theta_0)$ has the same approximation as that obtained in Theorem 2.1.

THEOREM 3.1. *Under the above set up,*

$$\int |\pi_n(u) - \tilde{\pi}_n(u)| du \to_p 0, \tag{3.9}$$

where $\pi_n(u)$ *is the marginal posterior density of* u *defined in* (3.4) *and*

$$\tilde{\pi}_n(u) = \frac{\int \tilde{Z}_n(u, \boldsymbol{v}) d\boldsymbol{v}}{\iint \tilde{Z}_n(t, \boldsymbol{s}) dt d\boldsymbol{s}} = \frac{\tilde{Z}_n^{(1)}(u)}{\int \tilde{Z}_n^{(1)}(t) dt}. \tag{3.10}$$

PROOF. We proceed along the same line as in the proof of Theorem 2.1. For notational simplicity, below we shall omit dummy variables of integration. As in (2.16), we bound the LHS of (3.9) by

$$\int_{|u| \le H} |\pi_n - \tilde{\pi}_n| + \int_{|u| > H} \pi_n + \int_{|u| > H} \tilde{\pi}_n.$$

In view of Lemma I.5.2 of IH, the second term can be made small by choosing H large. The third term is also small by Lemma 2.3. Now

$$\begin{aligned} \int_{|u| \le H} |\pi_n - \tilde{\pi}_n| &\le \left(\iint Z_n\right)^{-1} \int_{|u| \le H} \int |Z_n - \tilde{Z}_n| \\ &\quad + \frac{|\iint Z_n - \iint \tilde{Z}_n|}{\iint Z_n \iint \tilde{Z}_n} \int_{|u| \le H} \int \tilde{Z}_n \\ &= I_1 + I_2 \quad \text{(say)}. \end{aligned} \tag{3.11}$$

We have $I_1 \le I_{11} + I_{12} + I_{13}$, where

$$\begin{aligned} I_{11} &= \left(\iint Z_n\right)^{-1} \int_{|u| \le H} \int_{\|\boldsymbol{v}\| \le H} |Z_n - \tilde{Z}_n| \\ I_{12} &= \left(\iint Z_n\right)^{-1} \int_{|u| \le H} \int_{\|\boldsymbol{v}\| > H} Z_n \\ I_{13} &= \left(\iint Z_n\right)^{-1} \int_{|u| \le H} \int_{\|\boldsymbol{v}\| > H} \tilde{Z}_n. \end{aligned}$$

By the arguments used to prove (2.23), $(\iint Z_n)^{-1}$ is stochastically bounded. Thus by (3.6), I_{11} tends to zero in probability. For large enough H, I_{12} is small by Lemma I.5.2 of IH. To handle the term I_{13}, it suffices to prove that

$$\lim_{H \to \infty} \sup_{n \ge 1} P\left\{ \int_{|u| \le H} \tilde{Z}_n^{(1)} > H^2 \right\} = 0, \tag{3.12}$$

$$\lim_{H \to \infty} \sup_{n \ge 1} P\left\{ \int_{\|\boldsymbol{v}\| > H} \tilde{Z}_n^{(2)} > H^{-4} \right\} = 0. \tag{3.13}$$

Indeed, we then can choose H large enough to make the product $\int_{|u|\leq H} \tilde{Z}_n^{(1)} \int_{\|v\|>H} \tilde{Z}_n^{(2)}$ sufficiently small (uniformly in n) with a preassigned probability. By the stochastic boundedness of $(\iint Z_n)^{-1}$, it will then follow that I_{13} can be made small.

To prove (3.12), simply observe that

$$P\left\{\int_{|u|\leq H} \tilde{Z}_n^{(1)} > H^2\right\} \leq H^{-2}\int_{|u|\leq H} E\tilde{Z}_n^{(1)} \leq 2H^{-1}.$$

Relation (3.13) follows from the stochastic boundedness of $\mathbf{\Delta}_n$ and the nature of the tails of a normal distribution.

We now note that $I_2 \leq I_{21} + I_{22} + I_{23}$, where

$$I_{21} = \left|\int_{|u|\leq H}\int Z_n - \int_{|u|\leq H}\int \tilde{Z}_n\right| \left(\iint Z_n\right)^{-1}\left(\iint \tilde{Z}_n\right)^{-1}\int_{|u|\leq H}\int \tilde{Z}_n$$

$$I_{22} = \left(\int_{|u|>H}\int Z_n\right)\left(\iint Z_n\right)^{-1}\left(\iint \tilde{Z}_n\right)^{-1}\int_{|u|\leq H}\int \tilde{Z}_n$$

$$I_{23} = \left(\int_{|u|>H}\int \tilde{Z}_n\right)\left(\iint Z_n\right)^{-1}\left(\iint \tilde{Z}_n\right)^{-1}\int_{|u|\leq H}\int \tilde{Z}_n.$$

The term $I_{21} \leq I_1$ whereas I_{22} is bounded above by $(\iint Z_n)^{-1}\int_{|u|>H}\int Z_n$, which can be made small by Lemma I.5.2 of IH. Now

$$\begin{aligned} I_{23} &= \frac{\int_{|u|>H}\tilde{Z}_n}{(\iint Z_n)(\iint \tilde{Z}_n)}\left(\int_{|u|\leq H}\int Z_n + \int_{|u|\leq H}\int |Z_n - \tilde{Z}_n|\right) \\ &\leq \left(\int \tilde{Z}_n^{(1)}\right)^{-1}\int_{|u|>H}\tilde{Z}_n^{(1)} + I_1. \end{aligned}$$

The first term above can be made small by Lemma 2.3. This completes the proof. □

As mentioned in the introduction, an analytic approximation for the joint posterior distribution of (θ, φ) has not been obtained. We, however, describe below a procedure of obtaining an approximation to the joint posterior distribution using the one-dimensional approximation obtained in Section 2, the well-known normal approximation to the posterior distribution of φ and the Gibbs sampling procedure. It is interesting to observe that here asymptotic approximation and Markov chain Monte-Carlo methods are not competitors, but complementary to each other.

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be observations from a density of the form (3.1), where the possibility of some observations being censored (right censoring) is not ruled out. We proceed in the following steps.

Step 1. Compute an initial estimate $(\hat{\theta}, \hat{\varphi})$, which could be the MLE.

Step 2. Fix an interval of reasonable values of (θ, φ) around the initial estimate and generate an initial sample (θ_0, φ_0) at random from this interval.

Step 3. Impute the censored observations to obtain an artificial sample z assuming (θ_0, φ_0) to be the actual value of the parameters. For example, for Model (1.1), if the i-th observation is censored at x_i, then the corresponding imputed value is $z_i = x_i + V$, where V is a random outcome from the density $f(\cdot; \theta_0 - x_i, a_0, b_0)$ if $x_i \le \theta_0$ and from exponential with mean b_0^{-1} otherwise.

Step 4. With $\varphi = \varphi_0$ compute the MLE $\hat{\theta}$ based on the (complete) data z.

Step 5. Generate w from the density $\tilde{\pi}_n(\cdot)$ defined in (2.31). Generating a sample from $\tilde{\pi}_n(\cdot)$ is not difficult as $\tilde{\pi}_n(\cdot)$ is a piecewise exponential density. Put $\theta_1 = \hat{\theta} + w/n$.

Step 6. With $\theta = \theta_1$ and z as the observations, compute the MLE $\hat{\varphi}$ of φ. For Model (1.1) we compute the MLE $(\hat{a}, \hat{b})$ of (a, b) subject to the restriction $a > b$. Letting $J = \{i : z_i \le \theta_1\}$, m = number of elements in J, $T_1 = \sum_{i \in J} z_i$, $T_2 = \sum_{i \notin J} z_i$ and $S = \sum z_i$, it turns out that $\hat{a} = m/(T_1 + (n-m)\theta_1)$, $\hat{b} = (n-m)/(T_2 - (n-m)\theta_1)$ provided $1 \le m < n$ and $m/(T_1 + (n-m)\theta_1) > (n-m)/(T_2 - (n-m)\theta_1)$ and $\hat{a} = \hat{b} = n/S$ otherwise.

Step 7. We generate a sample φ_1 from the posterior distribution of φ using its normal approximation (if the exact distribution is non-standard). We may use reparametrization of φ for more accurate approximation to a normal distribution. For example, for Model (1.1) we consider $\alpha = \log a$ and $\beta = \log b$ and generate $\alpha \sim N(\hat{\alpha}, (1 - \exp[-\hat{a}\theta_1])^{-1}/n)$ and $\beta \sim N(\hat{\beta}, \exp[\hat{a}\theta_1]/n)$ until $\alpha > \beta$, where $\hat{\alpha} = \log \hat{a}$, $\hat{\beta} = \log \hat{b}$ and $\hat{a}$, $\hat{b}$ are as in Step 6. Put $a_1 = e^{\alpha}$ and $b_1 = e^{\beta}$.

Step 8. Replace (θ_0, φ_0) by (θ_1, φ_1) and perform Steps 3–7. Repeat the operation a large number of times to obtain final samples from the joint posterior distribution. These may now be used to (approximately) compute the posterior means, medians, standard deviations etc.

Our method is illustrated with a real data set in Section 4.

After completing the revision, we have become aware of a recent work by Ebrahimi *et al.* (1997) where they have proposed to compute the exact posterior by a combination of Gibbs sampling with other computation based methods. The difference between our approaches is that we approximate complicated conditionals by asymptotics and use only Gibbs whereas they use computation based methods such as Metropolis or certain tailored strategies to draw samples from the complicated conditionals. Our method is computationally less intensive than theirs but, on the other hand, is not exact. Also we have obtained a simple approximation in the general case so that we need not really treat the different particular cases separately (except for calculating quantities like $p(\hat{\theta})$ and $q(\hat{\theta})$). For large samples one can just use our simpler approximation that performs quite well.

4. A simulation study and an application to real data

We first present the results of a simulation study to judge the accuracy of the approximations obtained in Section 2. For this we consider iid observations $X_1, X_2, \ldots, X_n$, each having distribution possessing a density given in (1.1) with $0 < \theta < K$, K being a known bound. We consider a uniform prior on $(0, K)$ and obtain the expression for the exact posterior density $\pi_n^*(w)$ of the normalized (and centered) parameter $w = n(\theta - \hat{\theta})$, where $\hat{\theta}$ is the MLE of θ. Equation (2.31)

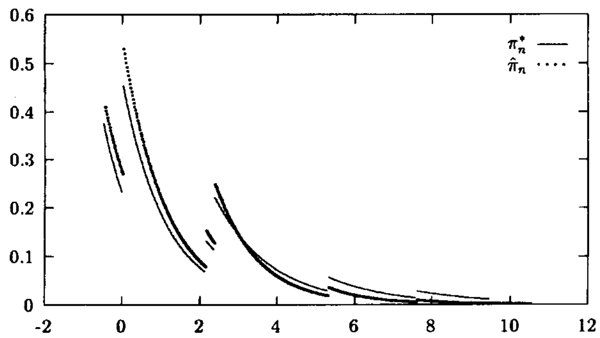

Fig. 1. Comparison of exact and approximate posterior of $w = n(\theta - \hat{\theta})$ in the one-parameter case.

gives an expression for the approximate posterior $\tilde{\pi}_n$ with $\hat{\delta} = (\delta =) \log(a/b)$ and $\hat{c} = (b-a)\exp(-a\hat{\theta})$. In our experiment, iid observations are generated from a distribution (1.1) with $a = 2$, $b = 1$, $\theta = 0.1$ and K is taken to be 1. For different choices of the sample size n we calculate the exact posterior $\pi_n^*(w)$ and its approximation $\tilde{\pi}_n(w)$. For comparison of the approximation $\tilde{\pi}_n$ with the actual posterior π_n^*, we plot them together. It is observed that the approximations are good for moderate sample sizes. For a sample of size 10, the plots are shown in Fig. 1.

In our experiment we have chosen the model (1.1) so that we are able to compute the exact posterior easily and compare it with its approximation. For more complicated models (e.g., (1.2)), the actual posterior may be awfully complicated while the approximations can be computed easily.

We now illustrate the method we suggested in Section 3 (pp. 20–21) for dealing with the multiparameter case both with a set of simulated data and a set of real data. For illustration we again consider Model (1.1). We take as the prior distribution a product of a uniform distribution over (δ_1, K) for θ and a noninformative prior $\pi(a,b)$ for (a,b) given by $d\pi(a,b) = \frac{1}{ab} da db$ on $\delta_2 < b < a < \infty$, δ_1, δ_2 being very small positive numbers. For a random sample of size 100 from (1.1) with $\theta = 1$, $a = 1$ and $b = 0.2$, censored by an exponential variable, we compare the characteristics of the exact and approximate posterior distributions for θ in Table 1 (we take $K = 4$, $\delta_1 = 0.1$, $\delta_2 = 0.01$). We also consider a data on remission duration for 84 patients with acute non-lymphoblastic leukemia (Glucksberg *et al.* (1981)) which was used in Matthews and Farewell ((1982), Section 4). As in Matthews and Farewell (1982), the data is modelled as independently censored observations from the density (1.1) with all the parameters (θ, a, b) unknown. For certain advantages in computation and putting it more or less in the same scale as that of the simulation experiment we have divided the original data by 500. Matthews and Farewell (1982) computed the MLE as (1.394, 1.02, 0.215) which we take as the initial estimate mentioned in Step 1. We take $K = 3.5$, $\delta_1 = 0.05$, $\delta_2 = 0.01$. The values of various characteristics of the exact and approximate posterior distributions of θ are shown in Table 2.

Table 1. Values of various characteristics of the exact and approximate posterior of θ for a simulated data.

	Mean	Median	S.D.
Exact	1.05	1.02	0.07
Approximate	1.07	1.07	0.06

Table 2. Values of various characteristics of the exact and approximate posterior of θ for the data on remission duration.

	Mean	Median	S.D.
Exact	1.83	1.60	0.50
Approximate	1.79	1.51	0.61

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Appendix

PROOF OF LEMMA 2.1. The result directly follows from Theorem V.2.4 of IH. However, the full strength of that theorem is not needed and we propose an alternative simpler proof. In view of (2.8), it suffices to show that $Z_n(u)$ and $\tilde{Z}_n(u)$ are uniformly integrable. Since both $Z_n(u)$ and $\tilde{Z}_n(u)$ converge in law to the random variable $Z(u)$ defined in (2.9), we only have to show that

$$\lim_{n\to\infty} EZ_n(u) = EZ(u), \qquad \lim_{n\to\infty} E\tilde{Z}_n(u) = EZ(u). \tag{A.1}$$

Since $EZ_n(u) \le 1$ (by definition) and $E\tilde{Z}_n(u) \le 1$ (by Lemma 2.2(i)), Fatou's lemma and (2.10) imply (A.1). □

PROOF OF LEMMA 2.2. Throughout the proof, we use the fact that $f(x;\theta)$ is nonincreasing in x and the inequality $e^x \ge 1+x$, $x \in \mathbb{R}$, without mention.

(i) We show the derivation for the case $u \ge 0$; the case $u < 0$ is similar.

$$\begin{aligned}
E\tilde{Z}_n(u) &= \exp(cu)(1+(e^\delta-1)P\{X_1 \in [\theta_0, \theta_0+u/n]\})^n \\
&= \exp(cu)\left(1+(e^\delta-1)\int_{\theta_0}^{\theta_0+u/n}(\bar{F}_1(\theta_0)/\bar{F}_2(\theta_0))f_2(x)dx\right)^n \\
&\le \exp(cu)(1+(e^\delta-1)pu/n)^n \\
&\le \exp[(p-q)u + n(q/p-1)pu/n] \\
&= 1.
\end{aligned}$$

(ii) We do for $u < 0$ only.

$$\begin{aligned}E\tilde{Z}_n^{1/2}(u) &= \exp(cu/2)(1-(1-e^{-\delta/2})P\{X_1 \in [\theta_0 + u/n, \theta_0]\})^n \\ &\le \exp(cu/2)(1-(1-e^{-\delta/2})q(-u)/n)^n \\ &\le \exp[(p-q)u/2 + n(1-(p/q)^{1/2})qu/n] \\ &= \exp[(p^{1/2}-q^{1/2})^2 u/2].\end{aligned}$$

(iii) Below, for a random variable X, $E^n(X)$ will stand for $(E(X))^n$. There are three different cases—$0 \le u_1 \le u_2 \le R$, $-R \le u_1 \le u_2 \le 0$ and $-R \le u_1 \le 0 \le u_2 \le R$. We show calculations for the last case only.

$$\begin{aligned}&E|\tilde{Z}_n^{1/2}(u_1) - \tilde{Z}_n^{1/2}(u_2)|^2 \\ &\quad = E^n(\exp[cu_2/n + \delta I\{X_1 \in [\theta_0, \theta_0 + u_2/n]\}]) \\ &\qquad + E^n(\exp[cu_1/n - \delta I\{X_1 \in [\theta_0 + u_1/n, \theta_0]\}]) \\ &\qquad - 2E^n\Big(\exp\Big[\frac{c}{2n}(u_1+u_2) - (\delta/2)\Big(I\Big\{X_1 \in \Big[\theta_0, \theta_0 + \frac{u_2}{n}\Big]\Big\} \\ &\qquad\qquad + I\Big\{X_1 \in \Big[\theta_0 + \frac{u_1}{n}, \theta_0\Big]\Big\}\Big)\Big]\Big) \\ &\quad \le 2nE(|c||u_2-u_1|/(2n) + (\delta/2)I\{X_1 \in (\theta_0 + u_1/n, \theta_0 + u_2/n]\}) \\ &\qquad \cdot \exp[R|c|/n + \delta] \\ &\quad = [|c||u_2-u_1| + \delta(pu_2 + f_1(0)(-u_1))]\exp[R|c|/n+\delta] \\ &\quad \le (|c| + f_1(0)\delta)\exp[R|c|/n+\delta]|u_2-u_1|.\end{aligned}$$

Given any $\varepsilon > 0$, choose $N \ge 1$ such that $|c|/N < \varepsilon$. Then (iii) follows from the above calculations. □

PROOF OF LEMMA 2.3. The proof is somewhat similar to that of Lemma I.5.2 of IH, and therefore is briefly sketched. Fix $M > 0$ and set $I = \int_\Gamma \tilde{Z}_n(u)du$, $Q = I/\int \tilde{Z}_n(u)du$, where $\Gamma = \{u : M \le |u| < M+1\}$. Divide Γ into L subintervals $\Delta_1, \ldots, \Delta_L$ of equal length, where L is to be chosen later. Choose $u_i \in \Delta_i$, $i = 1, \ldots, L$, and set $S = \sum_{i=1}^L \tilde{Z}_n(u_i)\,\mathrm{mes}(\Delta_i) = \sum_{i=1}^L \int_{\Delta_i} \tilde{Z}_n(u_i)du$. Let λ be as in Lemma 2.2(ii). Then,

$$\begin{aligned}\text{(A.2)}\quad P\Big\{S > \frac{1}{2}\exp[-\lambda M/8]\Big\} &\le P\Big\{\Big(\max_{1\le i\le L} \tilde{Z}_n^{1/2}(u_i)\Big) > \frac{1}{2}\exp[-\lambda M/16]\Big\} \\ &\le 2\sum_{i=1}^L E(\tilde{Z}_n^{1/2}(u_i))\exp[\lambda M/16] \\ &\le 2L\exp[-15\lambda M/16];\end{aligned}$$

here we have used part (ii) of Lemma 2.2.

Using Cauchy-Schwarz inequality and Lemma 2.2 (iii) (with $\varepsilon = \lambda/8$), we have for some $B_1 > 0$,

$$\text{(A.3)}\qquad E|S - I| \le \sum_{i=1}^L \int_{\Delta_i} E|\tilde{Z}_n(u) - \tilde{Z}_n(u_i)|du$$

$$\begin{aligned}
&\leq \sum_{i=1}^{L} \int_{\Delta_i} (E(\tilde{Z}_n^{1/2}(u) + \tilde{Z}_n^{1/2}(u_i))^2)^{1/2} \\
&\qquad \cdot (E(\tilde{Z}_n^{1/2}(u) - \tilde{Z}_n^{1/2}(u_i))^2)^{1/2} du \\
&\leq 2B^{1/2} \exp[\lambda(M+1)/16] \sum_{i=1}^{L} \int_{\Delta_i} |u - u_i|^{1/2} du \\
&\leq B_1 \exp[\lambda M/16] L^{-1/2}.
\end{aligned}$$

Hence by (A.2) and (A.3),

$$\text{(A.4)} \quad \begin{aligned}
P\{I > \exp[-\lambda M/8]\} &\leq P\left\{S > \frac{1}{2}\exp[-\lambda M/8]\right\} \\
&\quad + P\left\{|S - I| > \frac{1}{2}\exp[-\lambda M/8]\right\} \\
&\leq 2L\exp[-15\lambda M/16] + 2B_1 L^{-1/2}\exp[3\lambda M/16].
\end{aligned}$$

We now choose L such that $\exp[3\lambda M/4] \leq L \leq 2\exp[3\lambda M/4]$. Then (A.4) yields that for some $B_2 > 0$,

$$\text{(A.5)} \qquad P\{I > \exp[-\lambda M/8]\} \leq B_2 \exp[-\lambda M/8].$$

From Lemma I.5.1 of IH, for sufficiently small $\eta > 0$,

$$\text{(A.6)} \qquad P\left\{\int \tilde{Z}_n(u)du < \eta/4\right\} \leq 4B_0^{1/2}\exp[\lambda/16]\eta^{1/2}.$$

Thus from (A.5) and (A.6), we obtain

$$\text{(A.7)} \quad \begin{aligned}
EQ &\leq P\left\{\int \tilde{Z}_n(u)du < \eta/4\right\} + P\{I > \exp[-\lambda M/8]\} \\
&\quad + \frac{4}{\eta}\exp[-\lambda M/8] \\
&\leq 4B_0^{1/2}\exp[\lambda/16]\eta^{1/2} + B_2\exp[-\lambda M/8] + \frac{4}{\eta}\exp[-\lambda M/8].
\end{aligned}$$

Putting $\eta = \exp[-\lambda M/12]$ in (A.7), we have for some $B_3 > 0$,

$$\text{(A.8)} \qquad EQ \leq B_3 \exp[-\lambda M/24].$$

Hence for all $n \geq N$ (where N corresponds to $\varepsilon = \lambda/8$ in Lemma 2.2 (iii)),

$$\text{(A.9)} \quad \begin{aligned}
E\left(\frac{\int_{|u|>H} \tilde{Z}_n(u)du}{\int \tilde{Z}_n(u)du}\right) &\leq \sum_{r=0}^{\infty} B_3 \exp[-\lambda(H+r)/24] \\
&\leq B\exp[-\lambda H/24]. \qquad \square
\end{aligned}$$

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