Ann. Inst. Statist. Math.
Vol. 51, No. 3, 449–462 (1999)

# CERTAIN PROBABILISTIC ASPECTS OF SEMISTABLE LAWS

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(Received January 22, 1998; revised April 30, 1998)

**Abstract.** We study the extent to which the property of semistability of a random vector in $\Re^d$ is determined by semistability of its marginals, and the place of semistable laws within the family of type $G$ and sub-stable laws. Similarities and differences between stable and semistable laws are discussed.

*Key words and phrases*: Semistable distribution, type $G$ distribution, infinitely divisible distribution, stable distribution.

## 1. Introduction

The law $\mu$ of a non-Gaussian random vector $\boldsymbol{X}$ in $\Re^d$ (or even in a more general space) is called semistable if it is infinitely divisible and there exist $r, b \in (0,1) \cup (1,\infty)$ and $\boldsymbol{c} \in \Re^d$ such that

$$\mu^{*r} = \mu(b\cdot) * \delta_{\boldsymbol{c}}, \tag{1.1}$$

where for positive $r$, $\mu^{*r}$ stands for the $r$-th convolution power of $\mu$, $*$ means the convolution of two measures, and $\delta_{\boldsymbol{c}}$ is the point mass at $\boldsymbol{c}$. If $\mu$ is semistable, then $\boldsymbol{X}$ is also called semistable, and we often say that $\boldsymbol{X}$ satisfies (1.1). One can alternatively define the semistability of $\boldsymbol{X}$ by requiring existence of i.i.d. random vectors $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \dots$ in $\Re^d$, vectors $\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}, \dots$ in $\Re^d$ and positive numbers $a_1, a_2, \dots$ such that

$$a_k(\boldsymbol{Z}^{(1)} + \cdots + \boldsymbol{Z}^{(n_k)}) + \boldsymbol{c}^{(k)} \Rightarrow \boldsymbol{X} \tag{1.2}$$

as $k \to \infty$ for a sequence $\{n_k, k \ge 1\}$ such that $n_k \to \infty$ and $n_{k+1}/n_k \to 1/r$ or $r$ as $k \to \infty$, according as $0 < r < 1$ or $r > 1$, where $\Rightarrow$ means the convergence

* Samorodnitsky's research was partially supported by NSF Grant DMS-97-04982 at Cornell University. This work was done when Gennady Samorodnitsky visited Department of Mathematics at Keio University in August 1997. The hospitality and support provided during that visit is gratefully acknowledged.

in law. It is also known that there exists an $\alpha \in (0,2)$ depending only on $\boldsymbol{X}$ such that $b$ in (1.1) is expressed as $b = r^{-1/\alpha}$, namely

$$\mu^{*r} = \mu(r^{-1/\alpha}\cdot) * \delta_c. \tag{1.3}$$

Thus $\alpha$ is a characteristic of $\boldsymbol{X}$ and is called the index of $\boldsymbol{X}$.

Suppose $\boldsymbol{X}$ is semistable with index $\alpha$ ($SS(\alpha)$ for short). Let $\Gamma$ be the collection of all $r \in (0,\infty)$ for which $\boldsymbol{X}$ satisfies (1.3) for some $\boldsymbol{c}$ in $\Re^d$. Following Rajput and Rama-Murthy (1987), for a fixed $r \in \Gamma \setminus \{1\}$, we say that $\boldsymbol{X}$ is $r$-semistable index $\alpha$ ($r$-$SS(\alpha)$ for short). It follows directly from (1.1) that if $\boldsymbol{X}$ is $r_i$-$SS(\alpha)$ for some $r_i \in \Re \setminus \{1\}$, $i = 1,2$, then $\boldsymbol{X}$ is $r_1r_2$-$SS(\alpha)$, and hence $\Gamma$ is a closed multiplicative subgroup of $(0,\infty)$. We refer the reader to Chung *et al.* (1982) and Rajput and Rama-Murthy (1987) for these and other facts on semistable laws. We will be using a somewhat unorthodox terminology that $\boldsymbol{X}$ is $\Gamma$-semistable index $\alpha$, whose meaning is, however, obvious.

A $\Gamma$-semistable index $\alpha$ random vector $\boldsymbol{X}$ for which $\Gamma = (0,\infty)$ (this follows automatically if $\Gamma$ contains a sequence of $r$'s approaching 1) is $\alpha$-stable. Since $\alpha$-stable random vectors, $0 < \alpha < 2$, are often viewed as heavy tailed analogs of Gaussian random vectors, the dependence structure of $\alpha$-stable random vectors and processes has been extensively studied. See, for instance, Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994) for two recent books on the subject. The tails of semistable random variables are similar (even though not necessarily strictly comparable) to those of stable random variables, and since the family of $\alpha$-stable laws is, from many points of view, a small subset of the family of all $SS(\alpha)$ laws, the latter offer higher flexibility in stochastic modeling than the former. This fact points to potential uses of semistable laws in applied probability. The first step in realizing such potential is to understand the probabilistic structures of semistable laws. This step has not been, so far, taken. Rather, much of the works on semistable laws have been concentrated on more abstract properties of the latter, like the structure of their support (Rajput *et al.* (1994)) or the tail properties of the norm (Louie and Rajput (1979), Rajput (1997)).

In this paper we study certain probabilistic aspects of semistable laws. We concentrate on two issues, that are of interest in clarifying the place of semistable laws among all the infinitely divisible laws. In the next section we study the extent to which the property of semistability of a random vector in $\Re^d$ is determined by the property of semistability of its marginals, and in Section 3 we clarify which of the semistable random variables in $\Re$ are of type $G$, and which of them are, in fact, sub-stable.

We finish this introductory section by recalling a few basic facts about semistable random vectors. Let $\boldsymbol{X}$ be an $r$-$SS(\alpha)$ random vector in $\Re^d$, $0 < r < 1$, $0 < \alpha < 2$. The Lévy measure $\nu$ of $\boldsymbol{X}$ has then the scaling property

$$r^n \nu = \nu(r^{-n/\alpha}\cdot), \qquad n = \pm 1, \pm 2, \dots. \tag{1.4}$$

Conversely, any infinitely divisible random vector $\boldsymbol{X}$ whose Lévy measure has the scaling property (1.4) is $r$-$SS(\alpha)$. An immediate conclusion from (1.4) is the well known fact that a non-degenerate $\Gamma$-semistable index $\alpha$ real-valued random variable $X$ with $\Gamma \neq \{1\}$ has a finite $p$-th moment, $p > 0$, if and only if $p < \alpha$.

An $r$-$SS(\alpha)$ random vector $\boldsymbol{X}$ is called *strictly* $r$-$SS(\alpha)$ if $\mathbf{c} = \mathbf{0}$ in (1.3). It is obvious that a symmetric $r$-$SS(\alpha)$ random vector $\boldsymbol{X}$ (i.e. $\boldsymbol{X} \stackrel{\mathrm{d}}{=} -\boldsymbol{X}$) is strictly $r$-$SS(\alpha)$. Furthermore, the notion of strictness applies equally well to the notion of $\Gamma$-semistability, in the sense that if $r_i \in \Gamma$ for $i = 1, 2$ are numbers different from 1, and $\boldsymbol{X}$ is strictly $r_1$-semistable index $\alpha$, then it is also strictly $r_2$-semistable index $\alpha$. As before, we refer the reader to, for instance, Rajput and Rama-Murthy (1987).

## 2. What can we say about a random vector whose marginals are all semistable?

Let $\boldsymbol{X}$ be a random vector in $\Re^d$. If it is $r$-$SS(\alpha)$ (that is, if it satisfies (1.3)) then it is easy to see that for every $\boldsymbol{\gamma} \in \Re^d$ the real-valued random variable (a marginal of $\boldsymbol{X}$) $Y_{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, \boldsymbol{X})$ (where $(\,,\,)$ is the inner product in $\Re^d$) satisfies (1.3) as well, and so is $r$-$SS(\alpha)$ (in $\Re^1$). If the converse is true, then one can use it as an alternative definition of semistability in a multidimensional space. This approach (through one-dimensional projections) is a well known way to define a multivariate Gaussian vector, and it was stated by Dudley and Kanter (1974) that the same was true for $\alpha$-stable random vectors. However, their argument turned out to be valid only for $\alpha \in (1, 2)$, or, alternatively, under the assumption of *strict stability*. Indeed, Marcus (1983) gave an example of a non-stable random vector in $\Re^2$ whose marginals were all $\alpha$-stable, with an $\alpha \in (0, 1)$. It follows from the result of Giné and Hahn (1983) that the random vector constructed in Marcus (1983) is not even infinitely divisible and, hence, not $SS(\alpha)$. Therefore, the $SS(\alpha)$ property of the marginals does not imply the $SS(\alpha)$ property of $\boldsymbol{X}$ if $0 < \alpha < 1$, the statement on p. 141 of Rajput and Rama-Murthy (1987) notwithstanding. On the other hand, Samorodnitsky and Taqqu (1991) showed that the statement of Dudley and Kanter (1974) was true for $\alpha = 1$. The main result of this section, Theorem 1 below, shows that the situation in the $SS(\alpha)$ case is similar to that in the $\alpha$-stable case.

THEOREM 1. *Let $\boldsymbol{X}$ be a random vector in $\Re^d$, such that for every $\boldsymbol{\gamma} \in \Re^d$ the marginal $Y_{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, \boldsymbol{X})$ is $\Gamma_{\boldsymbol{\gamma}}$-semistable index $\alpha(\boldsymbol{\gamma})$ for some $\Gamma_{\boldsymbol{\gamma}} \neq \{1\}$ and $0 < \alpha(\boldsymbol{\gamma}) < 2$. Assume that $\Gamma = \bigcap_{\boldsymbol{\gamma} \in \Re^d} \Gamma_{\boldsymbol{\gamma}} \neq \{1\}$. Then the index $\alpha(\boldsymbol{\gamma})$ does not depend on $\boldsymbol{\gamma}$. That is, there is an $\alpha \in (0, 2)$ such that $\alpha(\boldsymbol{\gamma}) = \alpha$ for all $\boldsymbol{\gamma} \in \Re^d$ such that $Y_{\boldsymbol{\gamma}}$ is not degenerate (i.e. not constant). Moreover,*

(i) *If $1 \leq \alpha < 2$, then the vector $\boldsymbol{X}$ is $\Gamma$-semistable index $\alpha$ (with $\Gamma = \bigcap_{\boldsymbol{\gamma} \in \Re^d} \Gamma_{\boldsymbol{\gamma}}$).*

(ii) *In the case $0 < \alpha < 1$ the conclusion of part* (i) *remains true if, additionally, one of the following two conditions holds.*

(a) *For every $\boldsymbol{\gamma} \in \Re^d$, the marginal $Y_{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, \boldsymbol{X})$ is strictly $r$-semistable for some $r \in \Gamma_{\boldsymbol{\gamma}}$.*

(b) *For every $\boldsymbol{\gamma}^{(1)} \in \Re^d$ and $\boldsymbol{\gamma}^{(2)} \in \Re^d$, the random vector $(Y_{\boldsymbol{\gamma}^{(1)}}, Y_{\boldsymbol{\gamma}^{(2)}})$ in $\Re^2$ is infinitely divisible. This is true, in particular, if the random vector $\boldsymbol{X}$ is infinitely divisible.*

PROOF. Suppose, there are $\boldsymbol{\gamma}^{(1)} \in \Re^d$ and $\boldsymbol{\gamma}^{(2)} \in \Re^d$ such that $Y_{\boldsymbol{\gamma}^{(1)}}$ and $Y_{\boldsymbol{\gamma}^{(2)}}$ are non-degenerate, and $0 < \alpha(\boldsymbol{\gamma}^{(1)}) < \alpha(\boldsymbol{\gamma}^{(2)}) < 2$ (say). Let $\rho$ be any nonzero number, and let $\boldsymbol{\gamma}(\rho) = \rho\boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(2)}$. Then $\rho Y_{\boldsymbol{\gamma}^{(1)}} + Y_{\boldsymbol{\gamma}^{(2)}} = Y_{\boldsymbol{\gamma}(\rho)}$ is $\Gamma_{\boldsymbol{\gamma}(\rho)}$-semistable index $\alpha(\boldsymbol{\gamma}(\rho))$. Observe that for every $p \in [\alpha(\boldsymbol{\gamma}^{(1)}), \alpha(\boldsymbol{\gamma}^{(2)}))$, we have

$$E|Y_{\boldsymbol{\gamma}(\rho)}|^p = E|\rho Y_{\boldsymbol{\gamma}^{(1)}} + Y_{\boldsymbol{\gamma}^{(2)}}|^p = \infty,$$

because $E|Y_{\boldsymbol{\gamma}^{(1)}}|^p = \infty$ and $E|Y_{\boldsymbol{\gamma}^{(2)}}|^p < \infty$. Therefore, $Y_{\boldsymbol{\gamma}(\rho)}$ is non-degenerate and, further, $\alpha(\boldsymbol{\gamma}(\rho)) \le \alpha(\boldsymbol{\gamma}^{(1)})$. Let now $\{\rho_n\}$ be a sequence of (say) positive numbers that converge to 0. Let $Y_n = Y_{\boldsymbol{\gamma}(\rho_n)}$ and $\alpha(n) = \alpha(\boldsymbol{\gamma}(\rho_n))$, $n \ge 1$. Observe that $Y_n \Rightarrow Y_{\boldsymbol{\gamma}^{(2)}}$ as $n \to \infty$. Therefore, $\nu_n \Rightarrow \nu$, where $\nu_n$ is the Lévy measure of $Y_n$, $n \ge 1$ and $\nu$ is the Lévy measure of $Y_{\boldsymbol{\gamma}^{(2)}}$. Choose an $r < 1$ in $\Gamma$. Then every $Y_n$ is $r$-semistable index $\alpha(n)$, while $Y_{\boldsymbol{\gamma}^{(2)}}$ is $r$-semistable index $\alpha(\boldsymbol{\gamma}^{(2)})$.

Choose an $a > 0$ such that for all $m \ge 0$ the point $r^{m/\alpha(\boldsymbol{\gamma}^{(2)})}a$ is a continuity point of $\nu$. Choose an $\epsilon > 0$ so small that

$$1 + \epsilon < \frac{1}{r}. \tag{2.1}$$

Then there is an $n_0$ such that for all $n \ge n_0$ we have

$$\nu_n((a, \infty)) \le (1 + \epsilon)\nu((a, \infty)).$$

Choose, further, an $m \ge 0$ so big that $m/\alpha(\boldsymbol{\gamma}^{(1)}) \ge (m+1)/\alpha(\boldsymbol{\gamma}^{(2)})$. We have by (1.4), for all $n \ge n_0$,

$$\begin{aligned}\nu_n((r^{(m+1)/\alpha(\boldsymbol{\gamma}^{(2)})}a, \infty)) &\le \nu_n((r^{m/\alpha(\boldsymbol{\gamma}^{(1)})}a, \infty)) \le \nu_n((r^{m/\alpha(n)}a, \infty)) \\ &= r^{-m}\nu_n((a, \infty)) \le (1 + \epsilon)r^{-m}\nu((a, \infty)).\end{aligned}$$

Letting $n \to \infty$, we conclude that

$$\nu((r^{(m+1)/\alpha(\boldsymbol{\gamma}^{(2)})}a, \infty)) \le (1 + \epsilon)r^{-m}\nu((a, \infty)).$$

On the other hand, once again by (1.4) we have

$$\nu((r^{(m+1)/\alpha(\boldsymbol{\gamma}^{(2)})}a, \infty)) = r^{-(m+1)}\nu((a, \infty)),$$

and so we must have

$$(1 + \epsilon)r^{-m} \ge r^{-(m+1)},$$

which contradicts the choice (2.1) for $\epsilon$. Therefore, the index $\alpha(\boldsymbol{\gamma})$ does not depend on $\boldsymbol{\gamma}$.

To proceed, we need the next lemma which provides a simple estimate we will need in the sequel. Let $X$ be an $r$-$SS(\alpha)$ real-valued random variable. Letting $\nu$, as before, denote its Lévy measure, one has

$$Ee^{i\theta X} = \exp\left\{\int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \le 1))\nu(dx) + i\theta c\right\}, \qquad \theta \in \Re, \tag{2.2}$$

for some $c \in \Re$, where $\mathbf{1}(A)$ is the indicator function of $A$.

LEMMA 1. *Let $\{X(t), t \geq 0\}$ be a Lévy process in the sense that it has independent and stationary increments, and $X(0) = 0$ a.s. Suppose that $X := X(1)$ is $r$-$SS(\alpha)$, and its characteristic function has the form in (2.2). Fix $0 < r < 1$. Then, if $\alpha \neq 1$, for each $k = \pm 1, \pm 2, \dots$ we have*

$$r^{-k/\alpha} X(r^k) + \frac{r^{k(\alpha-1)/\alpha} - 1}{1 - r^{(1-\alpha)/\alpha}}(m_+ + m_-) + (1 - r^{k(\alpha-1)/\alpha})c \stackrel{\mathrm{d}}{=} X, \tag{2.3}$$

*where $\stackrel{\mathrm{d}}{=}$ means the equality in law, and*

$$m_+ = \int_{r^{1/\alpha}}^{1} x\nu(dx), \qquad m_- = \int_{-1}^{-r^{1/\alpha}} x\nu(dx).$$

*If $\alpha = 1$, then*

$$r^{-k} X(r^k) + k(m_+ + m_-) \stackrel{\mathrm{d}}{=} X. \tag{2.4}$$

*Moreover, if $\alpha \neq 1$,*

$$n^{-1/\alpha} X(n) + \frac{r^{k(1-\alpha)/\alpha} - 1}{1 - r^{(1-\alpha)/\alpha}}(m_+ + m_-) + (1 - r^{k(1-\alpha)/\alpha})c \Rightarrow X \tag{2.5}$$

*as $k \to \infty$ along the sequence $n = [r^{-k}]$. If $\alpha = 1$, then along the same sequence we have*

$$n^{-1} X(n) - k(m_+ + m_-) \Rightarrow X. \tag{2.6}$$

PROOF. We have for every $k \geq 1$ (say), using (1.4)

$$\begin{aligned} (2.7)\quad & E\exp\{i\theta r^{-k/\alpha} X(r^k)\} \\ &= \exp\left\{ r^k \int_{-\infty}^{\infty} (e^{i\theta r^{-k/\alpha}x} - 1 - i\theta r^{-k/\alpha} x\mathbf{1}(|x| \leq 1))\nu(dx) \right. \\ &\qquad \left. + i\theta r^{-k(1-\alpha)/\alpha} c \right\} \\ &= \exp\left\{ r^k \int_{-\infty}^{\infty} (e^{i\theta r^{-k/\alpha}x} - 1 - i\theta r^{-k/\alpha} x\mathbf{1}(|r^{-k/\alpha}x| \leq 1))\nu(dx) \right. \\ &\qquad - i\theta r^{-k(1-\alpha)/\alpha} \int_{-\infty}^{\infty} x\mathbf{1}(r^{k/\alpha} < |x| < 1)\nu(dx) \\ &\qquad \left. + i\theta r^{-k(1-\alpha)/\alpha} c \right\} \end{aligned}$$

$$= \exp\left\{\int_{-\infty}^{\infty}(e^{i\theta x}-1-i\theta x\mathbf{1}(|x|\le 1))\nu(dx) - i\theta r^{-k(1-\alpha)/\alpha}\sum_{j=0}^{k-1} r^{-j(\alpha-1)/\alpha}(m_+ + m_-) + i\theta r^{-k(1-\alpha)/\alpha}c\right\}.$$

The statements (2.3) and (2.4) are now obvious, and the case $k \le -1$ is completely analogous.

Similarly, for every $n \ge 1$

$$Ee^{i\theta n^{-1/\alpha}X(n)} = \exp\left\{n\int_{-\infty}^{\infty}(e^{i\theta n^{-1/\alpha}x}-1-i\theta n^{-1/\alpha}x\mathbf{1}(|x|\le 1))\nu(dx) + i\theta n^{1-1/\alpha}c\right\}.$$

Let now $n = [r^{-k}]$ for $k = 1, 2, \ldots$. It is straightforward to check that for any $\theta \in \Re$ we have

$$\left| n\int_{-\infty}^{\infty}(e^{i\theta n^{-1/\alpha}x}-1-i\theta n^{-1/\alpha}x\mathbf{1}(|x|\le 1))\nu(dx) - r^{-k}\int_{-\infty}^{\infty}(e^{i\theta r^{k/\alpha}x}-1-i\theta r^{k/\alpha}x\mathbf{1}(|x|\le 1))\nu(dx)\right| \to 0$$

as $k \to \infty$. Therefore, as $k \to \infty$, we have

$$Ee^{i\theta n^{-1/\alpha}X(n)} \sim \exp\left\{r^{-k}\int_{-\infty}^{\infty}(e^{i\theta r^{k/\alpha}x}-1-i\theta r^{k/\alpha}x\mathbf{1}(|x|\le 1))\nu(dx) + i\theta r^{k(1-\alpha)/\alpha}c\right\},$$

and (2.5) and (2.6) of Lemma 1 now follow from (2.3) and (2.4). □

We now go back to the proof of Theorem 1. Pick and fix any $r \in \Gamma \cap (0,1)$, and recall that for every $\boldsymbol{\gamma} \in \Re^d$, $Y_{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, \boldsymbol{X})$ is $r$-$SS(\alpha)$. Regarding $c$ in (2.2) and $m_+$ and $m_-$ defined in Lemma 1 as parameters of an infinitely divisible random variable, we denote by $c(\boldsymbol{\gamma})$, $m_+(\boldsymbol{\gamma})$ and $m_-(\boldsymbol{\gamma})$ the corresponding parameters of $Y_{\boldsymbol{\gamma}}$, and set

$$\beta(\boldsymbol{\gamma}) = \begin{cases} \dfrac{m_+(\boldsymbol{\gamma}) + m_-(\boldsymbol{\gamma})}{1 - r^{\alpha-1}} + c(\boldsymbol{\gamma}) & \text{if } \alpha \ne 1 \\ m_+(\boldsymbol{\gamma}) + m_-(\boldsymbol{\gamma}) & \text{if } \alpha = 1, \end{cases}$$

$\boldsymbol{\gamma} \in \Re^d$. Further, for $\boldsymbol{\gamma} = \boldsymbol{e}_j$, the $j$-th coordinate vector, we denote $\beta(\boldsymbol{e}_j)$ by $\beta_j$, $j = 1, \ldots, d$. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)$.

With $n = [r^{-k}]$, $k = 1, 2, \ldots$, we consider a subsequence of normalized partial sums

$$\boldsymbol{S}^{(k)} = \frac{1}{n^{1/\alpha}}(\boldsymbol{X}^{(1)} + \cdots + \boldsymbol{X}^{(n)}) - c_k(\alpha)\boldsymbol{\beta}, \tag{2.8}$$

where $\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \ldots$ are i.i.d. copies of $\boldsymbol{X}$, and

$$c_k(\alpha) = \begin{cases} r^{k(1-\alpha)/\alpha} - 1 & \text{if } \alpha \neq 1 \\ k & \text{if } \alpha = 1, \end{cases} \tag{2.9}$$

$k = 0, \pm 1, \pm 2, \ldots$. It follows from Lemma 1 that all of the $d$ coordinates of the sequence $(\boldsymbol{S}^{(k)}, k \geq 1)$ converge weakly as $k \to \infty$, and so the whole sequence is tight. Therefore, for every $\boldsymbol{\gamma} \in \Re^d$, the sequence $((\boldsymbol{\gamma}, \boldsymbol{S}^{(k)}), k \geq 1)$ is tight as well. Here

$$(\boldsymbol{\gamma}, \boldsymbol{S}^{(k)}) = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} Y_{\boldsymbol{\gamma}}^{(j)} - c_k(\alpha)(\boldsymbol{\beta}, \boldsymbol{\gamma}), \tag{2.10}$$

where $Y_{\boldsymbol{\gamma}}^{(1)}, Y_{\boldsymbol{\gamma}}^{(2)}, \ldots$ are i.i.d. copies of $Y_{\boldsymbol{\gamma}}$. However, another application of Lemma 1 shows that the sequence

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} Y_{\boldsymbol{\gamma}}^{(j)} - c_k(\alpha)\beta(\boldsymbol{\gamma}), \qquad k \geq 1, \tag{2.11}$$

converges weakly, and so is tight as well. Now suppose $1 \leq \alpha < 2$. Then, since $c_k(\alpha) \to \infty$ as $k \to \infty$, the only way the sequences (2.10) and (2.11) can be tight simultaneously is

$$\beta(\boldsymbol{\gamma}) = (\boldsymbol{\beta}, \boldsymbol{\gamma}), \tag{2.12}$$

and since the above argument works for every $\boldsymbol{\gamma} \in \Re^d$, the relation (2.12) must hold for all such $\boldsymbol{\gamma}$. However, (2.10), (2.12) and Lemma 1 imply that for every $\boldsymbol{\gamma} \in \Re^d$ the sequence $((\boldsymbol{\gamma}, \boldsymbol{S}^{(k)}), k \geq 1)$ converges weakly to $Y_{\boldsymbol{\gamma}}$, and so

$$\boldsymbol{S}^{(k)} \Rightarrow \boldsymbol{X} \tag{2.13}$$

as $k \to \infty$, which is the same as

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} \boldsymbol{X}^{(j)} - c_k(\alpha)\boldsymbol{\beta} \Rightarrow \boldsymbol{X}.$$

Therefore, the random vector $\boldsymbol{X}$ satisfies (1.2) with $n_k = [r^{-k}]$ and so is $r$-$SS(\alpha)$. Thus the proof of part (i) is complete.

If $0 < \alpha < 1$, then $c_k(\alpha) \to -1$ as $k \to \infty$, and so the above argument does not work. However, to make it work one only needs to establish (2.12). It is easy to check that a one-dimensional $r$-$SS(\alpha)$ random variable $X$ with characteristic function given by (2.2) is strictly $r$-$SS(\alpha)$ if and only if $\beta = 0$, where

$$\beta = \begin{cases} \dfrac{m_+ + m_-}{1 - r^{(\alpha-1)/\alpha}} + c & \text{if } \alpha \neq 1 \\ m_+ + m_- & \text{if } \alpha = 1. \end{cases}$$

Therefore, if for every $\gamma \in \Re^d$ the marginal $Y_\gamma = (\gamma, X)$ is strictly $r$-$SS(\alpha)$ for some $r \in \Gamma \cap (0,1)$, then $\beta(\gamma) = 0$ for all $\gamma \in \Re^d$, which establishes (trivially) (2.12) and, hence, the $r$-semistability index $\alpha$ property of $X$, no matter what $\alpha$ is. This shows part (ii) of the theorem.

Consider now the situation described in part (ii), (b). Fix, once again, an $r \in \Gamma \cap (0,1)$, and fix arbitrary $\gamma^{(1)} \in \Re^d$ and $\gamma^{(2)} \in \Re^d$. Consider the two-dimensional random vector $Y = (Y_{\gamma^{(1)}}, Y_{\gamma^{(2)}})$, which is, by assumption, infinitely divisible. Define

$$Y_k = r^{-k/\alpha} Y(r^k) - c_{-k}(\alpha)\tilde{\beta}, \qquad k = 1, 2, \ldots,$$

with $\tilde{\beta} = (\beta(\gamma^{(1)}), \beta(\gamma^{(2)})) \in \Re^2$. Here $Y(t)$ denotes an $\Re^2$-valued Lévy process with $Y(1) \stackrel{d}{=} Y$. Such a Lévy process exists because $Y$ is infinitely divisible. Observe from (2.3) that the two coordinates of the sequence $Y_1, Y_2, \ldots$ have fixed $r$-$SS(\alpha)$ distributions, and so the whole sequence is tight. Therefore, the same is true for the sequence $\{(Y_k, t), k = 1, 2, \ldots\}$ for every $t = (t_1, t_2) \in \Re^2$. However,

$$(2.14) \qquad (Y_k, t) = r^{-k/\alpha} Y_{t_1\gamma^{(1)} + t_2\gamma^{(2)}}(r^k) - c_{-k}(\alpha)(\tilde{\beta}, t), \qquad k = 1, 2, \ldots,$$

where $\{Y_{t_1\gamma^{(1)} + t_2\gamma^{(2)}}(t)\}$ is again an $\Re^1$-valued Lévy process with $Y_{t_1\gamma^{(1)} + t_2\gamma^{(2)}}(1) \stackrel{d}{=} Y_{t_1\gamma^{(1)} + t_2\gamma^{(2)}}$. On the other hand, once again from (2.3) the sequence

$$(2.15) \qquad r^{-k/\alpha} Y_{t_1\gamma^{(1)} + t_2\gamma^{(2)}}(r^k) - c_{-k}(\alpha)\beta(t_1\gamma^{(1)} + t_2\gamma^{(2)}), \qquad k = 1, 2, \ldots$$

has a fixed $r$-$SS(\alpha)$ distribution, and so is tight as well. However, if $0 < \alpha < 1$, then $c_{-k}(\alpha) \to \infty$ as $k \to \infty$, and so the only way the two sequences, (2.14) and (2.15), can be tight at the same time is when

$$(2.16) \qquad \beta(t_1\gamma^{(1)} + t_2\gamma^{(2)}) = (\tilde{\beta}, t) = t_1\beta(\gamma^{(1)}) + t_2\beta(\gamma^{(2)}).$$

However, this shows that (2.16) must hold for all $\gamma^{(1)}$ and $\gamma^{(2)} \in \Re^d$ and all real $t_1, t_2$. Hence (2.12) holds, and hence $X$ is $r$-$SS(\alpha)$. This completes the proof of the theorem. □

We would like to mention that we do not know whether the assumption $\cap_{\gamma \in \Re^d} \Gamma_\gamma \neq \{1\}$ in Theorem 1 is superfluous or not, that is, if this fact is already implied by the assumption that for every $\gamma \in \Re^d$ the marginal $Y_\gamma = (\gamma, X)$ is $\Gamma_\gamma$-semistable index $\alpha(\gamma)$ for some $\Gamma_\gamma \neq \{1\}$ and $0 < \alpha(\gamma) < 2$.

## 3. Which $r$-$SS(\alpha)$ random variables are of type $G$ and which are sub-stable?

Recall that a real-valued random variable $X$ is said to be of type $G$ if

$$X \stackrel{d}{=} S^{1/2} Z, \tag{3.1}$$

where $Z$ is a standard normal random variable independent of a nonnegative infinitely divisible random variable $S$. A similar definition is used in multidimensional (including infinite dimensional) settings. Here $X$ is a random vector and $Z$ a centered Gaussian random vector in the appropriate space. A type $G$ random variable is itself infinitely divisible, and is a variance mixture of Gaussian random variables. One important example of type $G$ random variables is symmetric $\alpha$-stable (S$\alpha$S for short) random variables. Moreover, the type $G$ property extends to S$\alpha$S processes; see Samorodnitsky and Taqqu (1994). Hence, S$\alpha$S processes are covariance mixtures of (centered) Gaussian processes, and many properties of the former have been understood via (conditional) reduction to the properties of the latter; some examples can be found in Marcus and Pisier (1984), Talagrand (1988), Rosiński *et al.* (1993) and Adler *et al.* (1993). The approach of the latter paper was applied to a larger class of type $G$ infinitely divisible processes in Marcus and Shen (1997). The power of this approach alone makes it important to understand which semistable random variables and vectors are of type $G$. This will also serve to clarify, further, the structure of semistable laws. Type $G$ processes were introduced by Marcus (1987), and we refer the reader to Rosiński (1991) for more information on type $G$ random variables and processes.

A notion parallel to that of a type $G$ random variable is that of a sub-stable random variable. A random variable $X$ is said to be sub-$\beta$-stable, $0 < \beta \leq 2$, if

$$X \stackrel{d}{=} S^{1/\beta} Z_\beta, \tag{3.2}$$

where this time $Z_\beta$ is a standard symmetric $\beta$-stable random variable independent of a nonnegative infinitely divisible random variable $S$. That is,

$$E e^{i\theta Z_\beta} = e^{-|\theta|^\beta}, \qquad \theta \in \Re, \tag{3.3}$$

with a similar definition in a multidimensional case. We mentioned above that symmetric stable random variables are of type $G$. Moreover, in that case the variance mixing random variable in (3.1) is positive stable, and one can use this fact to conclude immediately that any sub-stable random variable is also of type $G$. Let us denote by $\Xi(\beta)$ the collection of all distributions of sub-$\beta$-stable random variables, $0 < \beta \leq 2$. Every symmetric $\alpha$-stable random variable with $0 < \alpha < 2$ is sub-$\beta$-stable for every $\beta \in (\alpha, 2]$. That is, a standard S$\alpha$S random variable $Z_\alpha$, $0 < \alpha < 2$, can be represented in the form

$$Z_\alpha \stackrel{d}{=} (Z^+_{\alpha/\beta})^{1/\beta} Z_\beta \tag{3.4}$$

for every $\beta \in (\alpha, 2]$. Here $Z^+_{\alpha/\beta}$ is a positive strictly $\alpha/\beta$-stable random variable, whose Laplace transform is given by

$$E e^{-u Z^+_{\alpha/\beta}} = e^{-u^{\alpha/\beta}}, \qquad u \geq 0. \tag{3.5}$$

This shows that

$$\Xi(\beta_1) \subset \Xi(\beta_2) \quad \text{for} \quad 0 < \beta_1 < \beta_2 \le 2.$$

Clearly, $\Xi(2)$ is the collection of type $G$ random variables. In this section we study not only the question, whether or not $r$-$SS(\alpha)$ random variables are of type $G$, but also whether or not they belong to other families $\Xi(\beta)$. We refer the reader to Samorodnitsky and Taqqu (1994) for more information on the sub-stability property of symmetric stable random variables.

Let us introduce some notation. Denote by $\mathcal{SS}(\alpha, r)$ the collection of all distributions of symmetric $r$-$SS(\alpha)$ real-valued random variables, $0 < \alpha < 2$, $0 < r < 1$, and by $\mathcal{S}(\alpha) \subset \mathcal{SS}(\alpha, r)$ the collection of all distributions of symmetric $\alpha$-stable real-valued random variables, $0 < \alpha < 2$. It follows then from (3.4) that

$$\mathcal{S}(\alpha) \subset \Xi(\beta), \quad \text{all} \quad \alpha < \beta \le 2. \tag{3.6}$$

We are interested in the common parts of $\Xi(\beta)$ and $\mathcal{SS}(\alpha, r) \setminus \mathcal{S}(\alpha)$, $\alpha < \beta \le 2$.

The following Theorem 2 is the main result of this section. Throughout, $0 < r < 1$ is fixed. This result shows that among non-stable symmetric $r$-$SS(\alpha)$ random variables there are both those inside $\Xi(\beta)$ and outside of it. Furthermore, (3.6) cannot be extended to any part of $\mathcal{SS}(\alpha, r)$ other than $\mathcal{S}(\alpha)$. Part of the information provided in Theorem 2 can also be found in Theorem 2.4 of Rajput and Rama-Murthy (1984).

THEOREM 2. (i) *Let $0 < \alpha < \beta \le 2$, and let $S^+_{\alpha/\beta}$ be a positive strictly $r$-$SS(\alpha/\beta)$ random variable, independent of a symmetric $\beta$-stable random variable $Z_\beta$. Then*

$$X = (S^+_{\alpha/\beta})^{1/\beta} Z_\beta \tag{3.7}$$

*is a symmetric $r$-$SS(\alpha)$ random variable. Moreover, $X$ is not $S\alpha S$ unless $S^+_{\alpha/\beta}$ is a positive strictly $\alpha/\beta$-stable random variable. That is,*

$$\Xi(\beta) \cap (\mathcal{SS}(\alpha, r) \setminus \mathcal{S}(\alpha)) \ne \emptyset. \tag{3.8}$$

(ii) *If $0 < \alpha < \beta \le 2$, and a symmetric $r$-$SS(\alpha)$ random variable $X$ is sub-$\beta$-stable, then the random variable $S$ in* (3.2) *must be a positive strictly $r$-$SS(\alpha/\beta)$ random variable. Furthermore, for every $0 < \alpha < \beta \le 2$ there are $r$-$SS(\alpha)$ random variables that are not sub-$\beta$-stable. That is,*

$$\mathcal{SS}(\alpha, r) \cap (\Xi(\beta))^c \ne \emptyset. \tag{3.9}$$

*Moreover, for any $\alpha < \beta_1 < \beta_2 \le 2$ the inclusion*

$$\Xi(\beta_1) \cap \mathcal{SS}(\alpha, r) \subset \Xi(\beta_2) \cap \mathcal{SS}(\alpha, r) \tag{3.10}$$

*is proper, and*

$$\bigcap_{\alpha < \beta \le 2} (\Xi(\beta) \cap \mathcal{SS}(\alpha, r)) = \mathcal{S}(\alpha). \tag{3.11}$$

PROOF. Let $H$ denote the law of a symmetric $\beta$-stable random variable $Z_\beta$, $0 < \beta \leq 2$, satisfying (3.3) and let $\eta$ denote a multiple of the Lévy measure of $S^+_{\alpha/\beta}$, $0 < \alpha < \beta \leq 2$, such that

$$Ee^{-uS^+_{\alpha/\beta}} = \exp\left\{-\int_0^\infty (1 - e^{-uz})\eta(dz)\right\},$$

$u \geq 0$. Observe that for every $z > 0$ and real $\theta$

$$1 - e^{-|\theta|^\beta z} = 2\int_0^\infty (1 - \cos(\theta z^{1/\beta} y))H(dy).$$

Therefore, for $X$ defined by (3.7) we have

$$\begin{aligned}(3.12)\qquad Ee^{i\theta X} &= Ee^{-|\theta|^\beta S^+_{\alpha/\beta}} = \exp\left\{-\int_0^\infty (1 - e^{-|\theta|^\beta z})\eta(dz)\right\}\\ &= \exp\left\{-2\int_0^\infty \eta(dz)\int_0^\infty (1 - \cos(\theta z^{1/\beta} y))H(dy)\right\}.\end{aligned}$$

We see immediately that $X$ is a symmetric infinitely divisible random variable with Lévy measure $\nu$ given by

$$(3.13)\qquad \nu(A) = (\eta \otimes H)\{(z, y), z > 0, y \in \Re : z^{1/\beta} y \in A\},$$

$A$ a Borel set.

Since $S^+_{\alpha/\beta}$ is an $r$-$SS(\alpha/\beta)$ random variable, its Lévy measure $\eta$ satisfies (1.4). That is,

$$\eta(r^{-\beta/\alpha}A) = r\eta(A)$$

for every Borel set $A$. Therefore, for every such $A$

$$\begin{aligned}\nu(r^{-1/\alpha}A) &= \int_{-\infty}^\infty \eta(y^{-\beta} r^{-\beta/\alpha} A^\beta)H(dy)\\ &= r\int_{-\infty}^\infty \eta(y^{-\beta}A^\beta)H(dy) = r\nu(A).\end{aligned}$$

That is $\nu$ satisfies (1.4) and, hence, $X$ is symmetric $r$-$SS(\alpha)$. The first part of (i) is proved.

In the opposite direction, suppose now that a symmetric $r$-$SS(\alpha)$ random variable $X$ is sub-$\beta$-stable, in the form (3.2), $0 < \alpha < \beta \leq 2$, and let $\nu$ be the Lévy measure of $X$. We have, as in (3.12),

$$(3.14)\qquad Ee^{-uS} = \exp\left\{-2\int_0^\infty (1 - \cos(u^{1/\beta} x))\nu(dx)\right\},$$

$u \geq 0$. Since $X$ is $r$-$SS(\alpha)$, the Lévy measure $\nu$ satisfies (1.4). Therefore, for every $u \geq 0$

$$\begin{aligned} Ee^{-ur^{\beta/\alpha}S} &= \exp\left\{-2\int_0^\infty (1-\cos(u^{1/\beta}r^{1/\alpha}x))\nu(dx)\right\} \\ &= \exp\left\{-2r\int_0^\infty (1-\cos(u^{1/\beta}x))\nu(dx)\right\} = (Ee^{-uS})^r. \end{aligned}$$

That is, $S$ must be a positive strictly $r$-$SS(\alpha/\beta)$ random variable. The first part of (ii) is proved. To show the second part of (i), observe that the above argument also implies that if $X$ is S$\alpha$S, then $S$ must be a positive strictly $r$-$SS(\alpha/\beta)$ random variable for all $0 < r < 1$, and hence it is a positive strictly $\alpha/\beta$-stable random variable. Therefore, if $S^+_{\alpha/\beta}$ in (3.7) is non-stable, then $X$ is not S$\alpha$S.

We have now proved (i) and the first part of (ii) of the theorem. To prove the remaining statements of (ii) of the theorem, it is enough to note that the law $H$ of $S_\beta$ is absolutely continuous with respect to the Lebesgue measure on $\Re$, and we immediately see from (3.13) that the Lévy measure of any symmetric $r$-$SS(\alpha)$ random variable $X$ that is sub-$\beta$-stable for some $\beta > \alpha$ is absolutely continuous as well. Therefore, any symmetric $r$-$SS(\alpha)$ random variable whose Lévy measure is not absolutely continuous, cannot be sub-$\beta$-stable for any $\beta > \alpha$.

Moreover, let us denote by $\mathcal{SS}^+(\alpha, r)$ the collection of (the laws of) all positive strictly $r$-$SS(\alpha)$ random variables, $0 < \alpha < 1$, $0 < r < 1$. It follows from part (i) of the theorem that for every random variable $S^+_\alpha$ whose law is in $\mathcal{SS}^+(\alpha, r)$, independent of a symmetric $\beta$-stable random variable $Z_\beta$, the product

$$X = (S^+_\alpha)^{1/\beta} Z_\beta \tag{3.15}$$

is a symmetric $r$-$SS(\alpha\beta)$ random variable. For a $\beta_1 \in (0, \beta)$ denote by $\mathcal{SS}^+(\alpha; \beta, \beta_1, r)$ the subset of $\mathcal{SS}^+(\alpha, r)$ consisting of the laws of such $S^+_\alpha$ that the product in (3.15) is sub-$\beta_1$-stable. Observe that if the law of $S^+_\alpha$ is in $\mathcal{SS}^+(\alpha; \beta, \beta_1, r)$ then we can alternatively represent $X$ in (3.15) as

$$X \stackrel{d}{=} (S^+_{\alpha\beta/\beta_1})^{1/\beta_1} Z_{\beta_1} \stackrel{d}{=} (S^+_{\alpha\beta/\beta_1})^{1/\beta_1} (Z^+_{\beta_1/\beta})^{1/\beta} Z_\beta, \tag{3.16}$$

where we are using our usual notation, and all random variables in a product are independent. Now, it follows from (3.14) that the law of $X$ in (3.7) uniquely determines the law of the factor $S^+_{\alpha/\beta}$. Hence, (3.15) and (3.16) imply that

$$S^+_\alpha \stackrel{d}{=} (S^+_{\alpha\beta/\beta_1})^{\beta/\beta_1} Z^+_{\beta_1/\beta}. \tag{3.17}$$

We have then

$$\begin{aligned} \exp\left\{-\int_0^\infty (1-e^{-uz})\eta(dz)\right\} &= Ee^{-uS^+_\alpha} = Ee^{-u^{\beta_1/\beta} S^+_{\alpha\beta/\beta_1}} \\ &= \exp\left\{-\int_0^\infty (1-e^{-u^{\beta_1/\beta}z})\eta_1(dz)\right\} \\ &= \exp\left\{\int_0^\infty \eta_1(dz)\int_0^\infty (1-e^{-uz^{\beta/\beta_1}y})Q(dy)\right\}, \end{aligned}$$

where $\eta$ and $\eta_1$ are (multiples of) the Lévy measures of $S^+_\alpha$ and $S^+_{\alpha\beta/\beta_1}$ accordingly, and $Q$ is the law of $Z^+_{\beta_1/\beta}$. Therefore, we conclude that

$$\eta(A) = \eta_1 \otimes Q\{(z,y), z > 0, y > 0 : z^{\beta/\beta_1} y \in A\}. \tag{3.18}$$

This is a description of the Lévy measures of the laws in $\mathcal{SS}^+(\alpha;\beta,\beta_1,r)$. Since it follows as above that, in particular, any law in $\mathcal{SS}^+(\alpha;\beta,\beta_1,r)$ must have an absolutely continuous Lévy measure, we conclude that $\mathcal{SS}^+(\alpha;\beta,\beta_1,r)$ is a proper subset of $\mathcal{SS}^+(\alpha,r)$.

Let $\alpha < \beta_1 < \beta_2 \le 2$, and let the law of $X$ be in $\Xi(\beta_2) \cap \mathcal{SS}(\alpha,r)$. Then $X$ has a representation

$$X = (S^+_{\alpha/\beta_2})^{1/\beta_2} Z_{\beta_2},$$

with $S^+_{\alpha/\beta_2}$ being a positive strictly $r$-$SS(\alpha/\beta_2)$ random variable. Since for $X$ above to be also a sub-$\beta_1$-stable random variable we must have the law of $S^+_{\alpha/\beta_2}$ to be in $\mathcal{SS}^+(\alpha/\beta_2;\beta_2,\beta_1,r)$, which has been proved to be a proper subset of all laws of positive strictly $r$-$SS(\alpha/\beta_2)$ random variables, we conclude that $\Xi(\beta_1)\cap\mathcal{SS}(\alpha,r)$ is a proper subset of $\Xi(\beta_2) \cap \mathcal{SS}(\alpha,r)$.

It remains to prove (3.11). Assume that an $r$-$SS(\alpha)$ random variable $X$ is sub-$\beta$-stable for all $\beta \in (\alpha, 2]$, and consider the family of corresponding positive strictly $r$-$SS(\alpha/\beta)$ random variables that make (3.7) hold (in distribution) for $\beta \in (\alpha,2]$. This family is, obviously, tight. Its all possible limiting points, as $\beta \downarrow \alpha$, have to be nonnegative $r$-$SS(1)$ random variables, hence nonnegative constants. Therefore, taking weak limits along an appropriate subsequence of $\beta \downarrow \alpha$ we conclude that $X$ is equal in distribution to a constant multiple of $Z_\alpha$ and, hence, the law of $X$ is in $\mathcal{S}(\alpha)$.

This completes the proof of the theorem. □

*Remark* 1. From the argument used in the proof of Theorem 2 we see, for example, that an $r$-$SS(\alpha)$ random variable $X$ with Lévy measure given by

$$\nu = \sum_{n=-\infty}^{\infty} r^n (\delta_{r^{n/\alpha}} + \delta_{-r^{n/\alpha}}),$$

is not sub-$\beta$-stable for *any* $\beta > \alpha$.

*Remark* 2. One can give a somewhat more complete description of type $G$, $r$-$SS(\alpha)$ random variables than that given in Theorem 2. It follows from (3.13) that the Lévy measure $\nu$ of such random variables has a derivative of the form

$$\frac{d\nu(x)}{dx} = g(x^2),$$

where $g$ is a completely monotone function that can be represented in the form

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y/(2z)} z^{-1/2} \eta(dz),$$

where $\eta$ is the Lévy measure of a positive strictly $r$-$SS(\alpha/2)$ random variable. See also Rosiński (1991).

*Remark* 3. It follows immediately from Theorems 1 and 2 that for $0 < \alpha < \beta \le 2$, a positive strictly $r$-$SS(\alpha/\beta)$ random variable $S^+_{\alpha/\beta}$ independent of a symmetric $\beta$-stable random vector $\boldsymbol{Z}_\beta$, the random vector

$$\boldsymbol{X} = (S^+_{\alpha/\beta})^{1/\beta}\boldsymbol{Z}_\beta$$

is a symmetric $r$-$SS(\alpha)$ random vector. In a similar way, starting with appropriate centered Gaussian or symmetric stable processes one can construct families of symmetric $r$-$SS(\alpha)$ processes that have features one would like to model, e.g. stationarity, self-similarity, etc.

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