Ann. Inst. Statist. Math.
Vol. 51, No. 2, 359–367 (1999)

ON THE FIRST ENTRY TIME OF A $\mathbb{Z}_+$-VALUED $AR(1)$ PROCESS

EMAD-ELDIN A. A. ALY[1]* AND NADJIB BOUZAR[2]**

[1] *Department of Mathematical Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1*
[2] *Department of Mathematics, University of Indianapolis, 1400 East Hanna Avenue, Indianapolis, IN 46227, U.S.A.*

(Received November 20, 1995; revised February 19, 1997)

Abstract. In this paper we derive an explicit formula for the expected value of the first time a $\mathbb{Z}_+$-valued $AR(1)$ process exceeds a given level. Using martingale theory we obtain a generalized Wald's equation that holds under a simple integrability condition. As an application, we give an asymptotic formula for the expected value of the first exit time of the $AR(1)$ process with a thinned Poisson innovation.

Key words and phrases: Autoregressive, innovation sequence, martingale, first entry time, Poisson distribution.

1. Introduction

Time series models for sequences of $\mathbb{Z}_+$-valued r.v.'s have been the object of several recent papers. McKenzie (1986, 1988) used the concept of binomial thinning to construct $\mathbb{Z}_+$-valued ARMA models. The binomial thinning operator (introduced by Steutel and van Harn (1979)) is defined as follows. For a $\mathbb{Z}_+$-valued r.v. X and $\alpha \in (0,1)$, let

$$\alpha \circ X = \sum_{i=1}^{X} Y_i, \tag{1.1}$$

where $(Y_i,\ i \geq 1)$ is a sequence of iid Bernoulli (α) r.v.'s independent of X. The operator $\circ$ is treated as the analogue of the scalar multiplication and incorporates the discrete nature of the variates. A sequence $(X_n,\ n \geq 0)$ of $\mathbb{Z}_+$-valued r.v.'s is said to be an $AR(1)$ process (cf. McKenzie (1986)) if for every $n \geq 0$

$$X_{n+1} = \alpha \circ X_n + \epsilon_{n+1} = \sum_{i=1}^{X_n} Y_i^{(n)} + \epsilon_{n+1}, \tag{1.2}$$

* Research supported by an NSERC Canada grant at the University of Alberta.
** Research supported by a University of Indianapolis Summer Research Grant.

where $0 < \alpha < 1$, $Y_i^{(n)}$ $(n \geq 0, i \geq 1)$ are iid Bernoulli (α) r.v.'s, $(\epsilon_n,\ n \geq 1)$ is a sequence of iid $\mathbb{Z}_+$-valued r.v.'s with finite first moment, and the sequences $(Y_i^{(n)},\ n \geq 0, i \geq 1)$ and $(\epsilon_n,\ n \geq 1)$ are independent. The sequence $(\epsilon_n,\ n \geq 1)$ is referred to as the innovation sequence. Regression and correlation properties of model (1.2) are similar to those of the Gaussian $AR(1)$ process. Stationary $AR(1)$ processes with given marginals (such as Poisson, negative binomial, etc...) were extensively studied (cf. Aly and Bouzar (1994) for further references as well as for a generalized $AR(1)$ model).

The purpose of this paper is to study the following first exit time (over a given level) of a $\mathbb{Z}_+$-valued $AR(1)$ process $(X_n,\ n \geq 0)$:

$$\tau(A) = \inf\{n \in \mathbb{N} : X_n \geq A\}, \tag{1.3}$$

for some $A \in \mathbb{N}$. This class of stopping times arises naturally both in the theory and applications of stochastic processes as well as in areas of statistics such as sequential analysis. The main goal is to derive an explicit formula for the expected value $E(\tau(A))$ of the first exit time. Using martingale theory we obtain a generalized Wald's equation that holds under a simple integrability condition. Wald's equations for sequences of dependent variables were obtained by Novikov (1990) who studied first exit times of standard $AR(1)$ processes. It is principally Novikov's approach that we adopt here.

The paper is organized as follows. In Section 2 a class of martingales associated with a $\mathbb{Z}_+$-valued $AR(1)$ process is constructed. The main result and some examples are given in Section 3. As an application, an asymptotic formula for $E(\tau(A))$ is obtained in Section 4 in the case where the $AR(1)$ process has a thinned Poisson innovation.

2. Martingales associated with $\mathbb{Z}_+$-valued $AR(1)$ processes

$\mathbb{Z}_+$-valued $AR(1)$ processes are Galton-Watson branching processes with immigration, and as such their existence follows from Harris (1963). We will therefore assume that for a $\mathbb{Z}_+$-valued $AR(1)$ process $(X_n,\ n \geq 0)$ all the r.v.'s in (1.2) are defined on some probability space $(\Omega, \mathcal{F}, P)$. For each $n \in \mathbb{N}$, we denote by $\mathcal{F}_n$ the σ-field generated by $\{(Y_i^{(k)}, i \geq 1), k \leq n-1\}$ and $X_0,\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n$ (with $\mathcal{F}_0$ generated by X_0 only). We further assume that the exponential moments $\phi_\epsilon(z) = E(z^{\epsilon_1})$ and $\phi_{X_0}(z) = E(z^{X_0})$ exist and are finite for every $z \geq 1$. The cumulant exponential moment of ϵ_1 is defined by

$$\psi(z) = \log \phi_\epsilon(z), \qquad z \geq 1. \tag{2.1}$$

It can be easily deduced that for a $\mathbb{Z}_+$-valued $AR(1)$ process $(X_n,\ n \geq 0)$ the following relations hold for each $n \geq 1$ and $z \geq 1$:

$$\begin{aligned} &E(z^{\alpha \circ X_n} \mid \mathcal{F}_n) = E(z^{\alpha \circ X_n} \mid X_n) = (1 - \alpha + \alpha z)^{X_n}, \\ &\phi_{X_n}(z) = E(z^{X_n}) = \phi_{X_0}(1 - \alpha^n + \alpha^n z) \prod_{k=0}^{n-1} \phi_\epsilon(1 - \alpha^k + \alpha^k z). \end{aligned} \tag{2.2}$$

Following Novikov (1990), we define the function Ψ_α as follows:

$$\Psi_\alpha(z) = \sum_{k=0}^{\infty} \psi(1 - \alpha^k + \alpha^k z), \qquad z \geq 1, \tag{2.3}$$

where $0 < \alpha < 1$ and ψ is as in (2.1).

The following lemma is useful.

LEMMA 2.1. *The function $\Psi_\alpha(z)$ is well defined and is finite for all $z \geq 1$. Moreover, it satisfies*

$$\Psi_\alpha(z) = \Psi_\alpha(1 - \alpha + \alpha z) + \psi(z), \qquad z \geq 1. \tag{2.4}$$

PROOF. Since $E(\epsilon_1) < \infty$, we have for z near 1, $|\psi(z)| \leq c|z-1|$, for some constant $c > 0$. This implies that for k large enough $|\psi(1-\alpha^k + \alpha^k z)| \leq c\alpha^k|z-1|$ which in turn implies that the series of (2.3) converges absolutely for $z \geq 1$. Equation (2.4) follows then from the fact that

$$\Psi_\alpha(1 - \alpha + \alpha z) = \sum_{k=0}^{\infty} \psi(1 - \alpha^{k+1} + \alpha^{k+1} z), \qquad z \geq 1. \qquad \square$$

Next, we compute the function Ψ_α for some discrete distributions. If ϵ_1 takes the values $0, 1, \ldots, N$, $N \in \mathbb{N}$, with probabilities $p_0, p_1, \ldots, p_N$ respectively, ($\sum_{i=0}^N p_i = 1$ and $0 \leq p_0 < 1$), then $\psi(z) = \log \sum_{i=0}^N p_i z^i$. It follows easily that

$$\Psi_\alpha(z) = \log \prod_{k=0}^{\infty} (1 + g_k(z)) = \log \left(1 + \sum_{k=1}^{\infty} b_k (z-1)^k\right), \qquad z \geq 1, \tag{2.5}$$

where $g_k(z) = \sum_{i=1}^N \sum_{l=0}^i p_i \alpha^{kl} \binom{i}{l} (z-1)^l$ and $0 \leq b_k < \infty$, $k \geq 1$. Equation (2.5) applies in particular to the binomial and hypergeometric distributions. In the case where ϵ_1 has a Poisson distribution with parameter $\lambda > 0$, $\psi(z) = \lambda(z-1)$ and

$$\Psi_\alpha(z) = \frac{\lambda}{1-\alpha}(z-1), \qquad z \geq 1. \tag{2.6}$$

More generally, if ϵ_1 has a discrete stable distribution (cf. Steutel and van Harn (1979)) with exponent $\gamma \in (0,1]$, then $\psi(z) = \lambda(z-1)^\gamma$, $\lambda > 0$, and

$$\Psi_\alpha(z) = \frac{\lambda}{1-\alpha^\gamma}(z-1)^\gamma, \qquad z \geq 1. \tag{2.7}$$

We now define two processes related to a $\mathbb{Z}_+$-valued $AR(1)$ process (X_n, $n \geq 0$). For $\mu > 0$ and $n \geq 0$, let

$$W_n^{(\mu)} = \alpha^{\mu n} \int_1^\infty (z-1)^{\mu - 1} z^{X_n} \exp(-\Psi_\alpha(z)) dz, \tag{2.8}$$

and

$$Z_n = H_\alpha(X_n) + n \log \alpha, \tag{2.9}$$

where

$$H_\alpha(\cdot) = \int_1^\infty (z-1)^{-1}(z^{(\cdot)} - z^{X_0}) \exp(-\Psi_\alpha(z))dz. \tag{2.10}$$

The processes $(W_n^{(\mu)},\ n \geq 0)$ and $(Z_n,\ n \geq 0)$ are adapted with respect to $(\mathcal{F}_n,\ n \geq 0)$. Moreover, letting

$$I(\mu) = \int_2^\infty (z-1)^{\mu-1} \phi_{X_0}(z) \exp(-\Psi_\alpha(z))dz, \qquad \mu \geq 0, \tag{2.11}$$

we have the following result.

LEMMA 2.2. *Let $n \in \mathbb{N}$.*
i) *For each $\mu > 0$, $E(W_n^{(\mu)}) < \infty$ if and only if $I(\mu) < \infty$.*
ii) *If $I(0) < \infty$, then $E(|Z_n|) < \infty$.*

PROOF. By Fubini's theorem and equations (2.2)–(2.4), we have

$$E(W_n^{(\mu)}) = \alpha^{n\mu} \int_1^\infty (z-1)^{\mu-1} \phi_{X_0}(1 - \alpha^n + \alpha^n z) \exp(-\Psi_\alpha(1 - \alpha^n + \alpha^n z))dz.$$

The change of variable $v = 1 - \alpha^n + \alpha^n z$ implies

$$E(W_n^{(\mu)}) \int_1^\infty (z-1)^{\mu-1} \phi_{X_0}(z) \exp(-\Psi_\alpha(z))dz$$

from which i) follows. Decomposing the integral in (2.10) along the partition $\{(1,2], (2,\infty)\}$ and using the same argument as above, we obtain $E(|Z_n|) \leq 2I(0) + B_0$ for some constant $B_0 \geq 0$, thus proving ii). □

Next, we state and prove the main result of the section.

PROPOSITION 2.3.
i) *For each $\mu > 0$, $I(\mu) < \infty$ implies $(W_n^{(\mu)},\ \mathcal{F}_n,\ n \in \mathbb{N})$ is a martingale.*
ii) *If $I(0) < \infty$, then $(Z_n,\ \mathcal{F}_n,\ n \in \mathbb{N})$ is a martingale.*

PROOF. i) From Lemma 2.2, $E(W_n^{(\mu)}) < \infty$. By the conditional Fubini's theorem, the independence assumption, (2.2) and (2.4), we have for every $n \in \mathbb{N}$,

$$E(W_{n+1}^{(\mu)} \mid \mathcal{F}_n) = \alpha^{(n+1)\mu} \int_1^\infty (z-1)^{\mu-1} (1 - \alpha + \alpha z)^{X_n} \exp(-\Psi_\alpha(1 - \alpha + \alpha z))dz.$$

The change of variable $v = 1 - \alpha + \alpha z$ implies $E(W_{n+1}^{(\mu)} \mid \mathcal{F}_n) = W_n^{(\mu)}$.

ii) Again by Lemma 2.2, $E(|Z_n|) < \infty$. Using the same arguments as in the proof of i) above, we have for $n \in \mathbb{N}$

$$\begin{aligned} &E(Z_{n+1} \mid \mathcal{F}_n) \\ &\quad = \int_1^\infty (z-1)^{-1}\{(1-\alpha+\alpha z)^{X_n} - (1-\alpha+\alpha z)^{X_0}\}e^{-\Psi_\alpha(1-\alpha+\alpha z)}dz \\ &\quad\quad - \int_1^\infty (z-1)^{-1}\{h(z) - h(1-\alpha+\alpha z)\}dz + (n+1)\log\alpha, \end{aligned}$$

where $h(z) = z^{X_0}\exp(-\Psi_\alpha(z))$. By Frullani's integral (see Novikov (1990)) applied to the function $h(z+1)$, we have

$$\int_1^\infty (z-1)^{-1}\{h(z) - h(1-\alpha+\alpha z)\}dz = h(1)\log\alpha = \log\alpha.$$

The change of variable $v = 1-\alpha+\alpha z$ implies then $E(Z_{n+1} \mid \mathcal{F}_n) = Z_n$. □

3. First entry time

Let $(X_n,\, n \in \mathbb{N})$ be a $\mathbb{Z}_+$-valued $AR(1)$ process and let $\tau(A)$ be the first entry time of X_n over level A, for some $A \in \mathbb{N}$, as defined in (1.3). It is clear that $\tau(A)$ is a stopping time for $(\mathcal{F}_n,\, n \in \mathbb{N})$.

We need to recall a version of the optional sampling theorem which can be found in any graduate text on probability theory.

LEMMA 3.1. *Suppose that $(Y_n,\, n \in \mathbb{N})$ is a martingale and T is a stopping time. If $P(T < \infty) = 1$ and $E(\sup_n |Y_{T\wedge n}|) < \infty$, then $E(Y_T) = E(Y_0)$.*

The following proposition obtains an explicit formula for the expected value of the first entry time. This constitutes the main result of the paper.

PROPOSITION 3.2. *Let $(X_n,\, n \in \mathbb{N})$ be a $\mathbb{Z}_+$-valued $AR(1)$ process such that $X_0 = x_0$, $x_0 \geq 1$. Assume that for $A \in \mathbb{N}$, $A > x_0$,*

$$\int_2^\infty (z-1)^{-1} z^A \exp(-\Psi_\alpha(z))dz < \infty. \tag{3.1}$$

Then $E(\tau(A)) < \infty$ and

$$E(\tau(A)) = (-\log\alpha)^{-1} E(H_\alpha(X_{\tau(A)})). \tag{3.2}$$

PROOF. The proof is a modified version of the proof of Theorem 1 in Novikov (1990). First, note that (3.1) implies $I(0) < \infty$ and hence, by Proposition 2.3, $(Z_n,\, n \in \mathbb{N})$ of (2.9) is a martingale. Essentially, we need to show that Lemma 3.1 applies to $(Z_n,\, n \in \mathbb{N})$ and $\tau(A)$. More specifically, we will establish that $E(\tau(A)) < \infty$ and that $E(\sup_n |H_\alpha(X_{\tau(A)\wedge n})|) < \infty$. These conditions imply those of Lemma 3.1 and, since $Z_0 = 0$, the conclusion will follow straightforwardly. The optional stopping theorem for bounded stopping times implies that for each $n \in \mathbb{N}$,

$$E(\tau(A) \wedge n)(-\log \alpha) = E(H_\alpha(X_{\tau(A)\wedge n})). \tag{3.3}$$

The key step involves deriving from (3.3) the claim that the sequence $(E(\tau(A) \wedge n), n \in \mathbb{N})$ is bounded. This is done by combining a somewhat intricate truncation argument on $\epsilon_{\tau(A)}$ with an application of Wald's equation for iid r.v.'s. In order to insure integrability of the various estimates under assumption (3.1), a suitable function of z is chosen as the truncating bound. Let $C > 0$ and let $g(z)$ be a function defined on $[1, \infty)$ and such that

$$\lim_{z \to 1^+} z^{g(z)} = \lim_{z \to \infty} z^{g(z)} = 1 \tag{3.4}$$

(for e.g., one can choose $g(z) = \frac{1}{\log z} \frac{z-1}{(z-1)^2+1}$ and $g(1) = 1$). Partitioning the domain of integration Ω on the right-hand side of (3.3) into $[\tau(A) > n]$, $[\tau(A) \leq n, \epsilon_{\tau(A)} \geq Cg(z)]$ and $[\tau(A) \leq n, \epsilon_{\tau(A)} < Cg(z)]$, and using the inequalities $X_n < A$ on $[\tau(A) > n]$, and

$$\alpha \circ X_{\tau(A)-1} = \sum_{i=1}^{X_{k-1}} Y_i^{(k-1)} \leq \sum_{i=1}^{A} Y_i^{(k-1)} \leq A \tag{3.5}$$

on $[\tau(A) = k]$, $k \leq n$, we have

$$\begin{aligned} &E(\tau(A) \wedge n) \left(\log \frac{1}{\alpha}\right) \\ &\leq H_\alpha(A) + E\left(\sum_{k=1}^{\tau(A)\wedge n} L_k(C)\right) \\ &+ \int_1^\infty (z-1)^{-1}(z^{A+Cg(z)} - z^{x_0}) \exp(-\Psi_\alpha(z))dz, \end{aligned} \tag{3.6}$$

where

$$L_k(C) = \int_1^\infty (z-1)^{-1}(z^{\sum_{i=1}^{A} Y_i^{(k-1)} + \epsilon_k} - z^{x_0})^+ e^{-\Psi_\alpha(z)} I[\epsilon_k \geq Cg(z)]dz.$$

The first term on the right-hand side of (3.6) is finite by (3.1). The last term is also finite by (3.1) and (3.4). Both terms are independent of n. Since $(Y_i^{(k-1)}, i = 1, 2, \ldots, A; \epsilon_k)$ are iid random vectors in $\mathbb{R}^{A+1}$, the sequence $(L_k(C), k \geq 1)$ is also iid. By independence, (2.4), and the change of variable $v = 1 - \alpha + \alpha z$, we have

$$E(L_1(C)) \leq B_1 + \int_2^\infty (z-1)^{-1}(z^A + z^{x_0})e^{-\Psi_\alpha(z)}dz < \infty,$$

for some constant $B_1 \geq 0$. It follows that $\lim_{C\to\infty} E(L_1(C)) = 0$. Moreover, an application of Wald's identity for iid r.v.'s yields

$$E\left(\sum_{k=1}^{\tau(A)\wedge n} L_k(C)\right) = E(\tau(A)\wedge n)E(L_1(C)).$$

Therefore, for $0 < \eta < 1$ and C large enough,

$$E(\tau(A)\wedge n)(-\log\alpha) \leq B + \eta E(\tau(A)\wedge n)(-\log\alpha),$$

for some constant B. The claim is thus proved and, by letting $n \to \infty$, we have $E(\tau(A)) < \infty$. We conclude the proof of the proposition by showing that the sequence $(H_\alpha(X_{\tau(A)\wedge n}),\ n \in \mathbb{N})$ has integrable upper and lower bounds. By repeating part of the argument above, with $\tau(A)$ in place of $\tau(A)\wedge n$ and $C = 0$, it can be deduced that

$$E(H_\alpha(X_{\tau(A)})) \leq E\left(\sum_{k=1}^{\tau(A)} L_k(0)\right) = E(\tau(A))E(L_1(0)) < \infty. \tag{3.7}$$

By combining (3.7) with the inequality

$$H_\alpha(X_{\tau(A)\wedge n}) \leq H_\alpha(X_{\tau(A)}) + \int_1^\infty (z-1)^{-1}(z^A - z^{x_0})e^{-\Psi_\alpha(z)}dz$$

we obtain that $H_\alpha(X_{\tau(A)\wedge n})$ has an integrable upper bound. Next, it can be easily shown that

$$\begin{aligned} -H_\alpha(X_{\tau(A)\wedge n}) \leq & \int_2^\infty (z-1)^{-1} z^{x_0} e^{-\Psi_\alpha(z)}dz \\ & + \sup_{1\leq z\leq 2} (z^{x_0}e^{-\Psi_\alpha(z)})|X_{\tau(A)\wedge n} - x_0|. \end{aligned} \tag{3.8}$$

Furthermore, applying (1.2) iteratively leads to

$$|X_{\tau(A)\wedge n} - x_0| \leq 2x_0 + \sum_{i=1}^{\tau(A)} \epsilon_i. \tag{3.9}$$

By Wald's identity, the right-hand side of (3.9) is integrable. It follows then by (3.1), (3.8) and (3.9) that $H_\alpha(X_{\tau(A)\wedge n})$ has an integrable lower bound. □

Proposition 3.2 readily provides a lower bound for $E(\tau(A))$.

COROLLARY 3.3. *Under the assumptions of Proposition* 3.2, *we have*

$$E(\tau(A)) \geq (-\log\alpha)^{-1} H_\alpha(A). \tag{3.10}$$

PROOF. Straightforward from the inequality $X_{\tau(A)} \geq A$. □

Remarks. 1) It can be easily shown by using (2.5)–(2.7) that assumption (3.1) in Proposition 3.2 holds for the binomial distribution, the hypergeometric distribution, and the discrete stable (and in particular the Poisson) distribution.

2) It does not appear that Proposition 3.2 can be derived in a simpler fashion. One would have hoped that by first establishing that $E(\sup_n |H_\alpha(X_{\tau(A)\wedge n})|) < \infty$, then $E(\tau(A)) < \infty$ would have followed from (3.3). The difficulty mainly resides in the fact that the truncation argument used in the proof does not carry over to $|H_\alpha(X_{\tau(A)\wedge n})|$.

4. The $\mathbb{Z}_+$-valued $AR(1)$ process with a thinned Poisson innovation

In this section, we study the special $\mathbb{Z}_+$-valued $AR(1)$ process described by the equation

$$X_{n+1} = \alpha \circ X_n + (1-\alpha) \circ \epsilon_{n+1}, \qquad X_0 = x_0, \ n \geq 0, \tag{4.1}$$

where $0 < \alpha < 1$ and $x_0 \geq 1$. In this case X_{n+1} results from a convex combination (with respect to the operator $\circ$) of X_n and ϵ_{n+1}. More precisely, the innovation sequence $(\epsilon_n,\ n \geq 1)$ of the $AR(1)$ process undergoes thinning at each $n \geq 1$ prior to being superimposed to the thinned X_n. We assume that the iid Bernoulli variables needed in the definition of $(1-\alpha)\circ\epsilon_n$ are independent of the $Y_i^{(k)}$'s of Section 2. The filtration $(\mathcal{F}_n,\ n \geq 0)$ is suitably redefined as to include these new Bernoulli variables. We further assume that ϵ_n has a Poisson distribution with parameter $\lambda > 0$. This implies in particular that for each $n \geq 1$, $(1-\alpha)\circ\epsilon_n$ is itself Poisson-distributed with parameter $(1-\alpha)\lambda$.

The following result gives an approximating formula for the expected first entry time of $(X_n,\ n \geq 0)$ for α near 1.

PROPOSITION 4.1. *Let $A \in \mathbb{N}$, $A > x_0$, and let $\tau(A)$ be as in (1.3). Then $E(\tau(A)) < \infty$, and*

$$\lim_{\alpha \to 1} (-\log\alpha) E(\tau(A)) = \sum_{j=0}^{A-1} \sum_{i=(j-x_0)^+}^{A-x_0-1} \frac{(x_0+i)!}{(x_0+i-j)!} \lambda^{-(j+1)}. \tag{4.2}$$

PROOF. Since $\Psi_\alpha(z) = \lambda(z-1)$, condition (3.1) holds, and hence by Proposition 3.2, $E(\tau(A)) < \infty$. Simple calculations lead to

$$H_\alpha(A) = \sum_{j=0}^{A-1} \sum_{i=(j-x_0)^+}^{A-x_0-1} \frac{(x_0+i)!}{(x_0+i-j)!} \lambda^{-(j+1)}. \tag{4.3}$$

Further, by Corollary 3.3, we have

$$(-\log\alpha) E(\tau(A)) \geq H_\alpha(A). \tag{4.4}$$

Let $N(\alpha) = [\![\delta(1-\alpha)^{-1/2}]\!]$ where $[\![a]\!]$ is the largest integer less than or equal to a and $0 < \delta < \lambda$. Partitioning Ω into $[\epsilon_{\tau(A)} \le N(\alpha)]$ and $[\epsilon_{\tau(A)} > N(\alpha)]$ in (3.2) leads to

$$(4.5)\qquad \left(\log\frac{1}{\alpha}\right)E(\tau(A)) \le \int_1^\infty (z-1)^{-1}(z^A\{\alpha+(1-\alpha)z\}^{N(\alpha)} - z^{x_0})e^{-\lambda(z-1)}dz + E(I[\epsilon_{\tau(A)} > N(\alpha)]H_\alpha(A+(1-\alpha)\circ\epsilon_{\tau(A)})).$$

Since $(\alpha+(1-\alpha)z)^{N(\alpha)} \le e^{\delta(z-1)}$, the first term on the right-hand side of (4.5) converges to $H_\alpha(A)$ as $\alpha \to 1$ and by Wald's identity the second term does not exceed

$$(4.6)\qquad E(\tau(A))E(I[\epsilon_1 > N(\alpha)]H_\alpha(A+(1-\alpha)\circ\epsilon_1)).$$

Now,

$$(4.7)\qquad E(I[\epsilon_1 > N(\alpha)]H_\alpha(A+(1-\alpha)\circ\epsilon_1)) = E\left(I[\epsilon_1 > N(\alpha)] \cdot \int_1^\infty (z-1)^{-1}(z^A\{\alpha+(1-\alpha)z\}^{\epsilon_1} - z^{x_0})e^{-\lambda(z-1)}dz\right),$$

from which it can be easily deduced that

$$(4.8)\qquad \lim_{\alpha\to 1}\left(\log\frac{1}{\alpha}\right)^{-1} E(I[\epsilon_1 > N(\alpha)]H_\alpha(A+(1-\alpha)\circ\epsilon_1)) = 0.$$

Equation (4.2) follows then from (4.3)–(4.8). □

REFERENCES

Aly, E. E. and Bouzar, N. (1994). Explicit stationary distributions for some Galton-Watson processes with immigration, *Comm. Statist. Stochastic Models*, **10**, 499–517.

Harris, T. E. (1963). *The Theory of Branching Processes*, Springer, Berlin.

McKenzie, E. (1986). Autoregressive-moving average processes with negative binomial and geometric marginal distributions, *Advances in Applied Probability*, **18**, 679–705.

McKenzie, E. (1988). Some ARMA models for dependent sequences of Poisson counts, *Advances in Applied Probability*, **20**, 822–835.

Novikov, A. A. (1990). On the first passage time of an autoregressive process over a level and an application to a disorder problem, *Theory Probab. Appl.*, **35**, 269–279.

Steutel, F. W. and van Harn, K. (1979). Discrete analogues of self-decomposability and stability, *Ann. Probab.*, **7**, 893–899.