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ON MOMENTS OF BIVARIATE ORDER STATISTICS

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Abstract. In the present paper, we give the exact explicit expression for the product moments (of any order) of bivariate order statistics (o.s.) from any arbitrary continuous bivariate distribution function (d.f.). Furthermore, for any arbitrary bivariate uniform d.f., universal distribution-free bounds for the differences of any two different product moments (of order $(1,1)$ or $(-1,1)$) are given.

Key words and phrases: Bivariate order statistics, moments of order statistics, dependence function.

1. Introduction

Let $\bar{X}_j = (X_{1j}, X_{2j})$, $j \geq 1$ denote i.i.d. random vectors, with absolutely continuous d.f. $F = F(x_1, x_2) = F(\bar{x})$ which has marginal d.f.'s $F_t(x_t)$, $t = 1, 2$. Write $X_{t,1:n} \leq X_{t,2:n} \leq \cdots \leq X_{t,n:n}$, $t = 1, 2$, for the order marginals of the first n pair $(X_{1,j}, X_{2,j})$. Let $\bar{X} = (X_1, X_2)$ denote a generic $\bar{X}_j = (X_{1j}, X_{2j})$, and define $\mu_{i,j:n}^{(m_i,m_j)} = E(X_{1,i:n}^{m_i} X_{2,j:n}^{m_j})$, where m_i and m_j are any real numbers. Furthermore, let $\nu_{i,j:n}^{(m_i,m_j)}$ be the (m_i, m_j)-th moment of the bivariate o.s. $(U_{1,i:n}, U_{2,j:n})$ from any arbitrary uniform d.f. $D = D(u_1, u_2) = D(\bar{u})$ (i.e., D has uniform $(0,1)$ marginal d.f.'s). For convenience, we shall abbreviate $\mu_{i,j:n}^{(1,1)}(\nu_{i,j:n}^{(1,1)})$ by $\mu_{i,j:n}(\nu_{i,j:n})$ and the components of the numerical vectors $\bar{x} = (x_1, x_2)$, are signified by a subscript. Throughout this paper the densities of the random variables (r.v.'s) $\bar{X}$ and X_t, $t = 1, 2$, are defined, respectively, as $f = f(\bar{x}) = \frac{\partial^2 F}{\partial x_1 \partial x_2}$ and $f_t = f_t(x_t) = \frac{\partial F_t}{\partial x_t}$, $t = 1, 2$. Furthermore, let $F^{1,\cdot} = \frac{\partial F}{\partial x_1}$ and $F^{\cdot,1} = \frac{\partial F}{\partial x_2}$ (i.e., $F^{1,1} = f$). Finally, the abbreviations $\min(a, b) = a \wedge b$ and $\max(a, b) = a \vee b$ will be adopted.

2. Moments of bivariate o.s.

The following theorem gives a basic formula, which is needed in the next section and enable us to derive the single and the product moments (of any order) of bivariate o.s.

THEOREM 2.1. *Let m_i and m_j be any two real values and let $1 \leq i,\ j \leq n$; $n \geq 1$. Then, for any continuous bivariate d.f. $F(\bar{x})$, we have*

$$(2.1)\quad \mu_{i,j:n}^{(m_i,m_j)} = \sum_{0\vee(i+j-n-1)}^{(i-1)\wedge(j-1)} C_1 \iint_0^1 (F_1^{-1}(u_1))^{m_i}(F_2^{-1}(u_2))^{m_j} (u_1-D)^{i-1-r}D^r (u_2-D)^{j-1-r}(1-u_1-u_2+D)^{n-i-j+1+r}D^{1,1}du_1du_2 + \sum_{\varphi=\varphi_*}^{\varphi^*}\sum_{\theta=\theta_*}^{\theta^*}\sum_{r=r_*}^{r^*} C_2 \iint_0^1 (F_1^{-1}(u_1))^{m_i}(F_2^{-1}(u_2))^{m_j} (u_1-D)^{i-1-\theta-r}D^r(u_2-D)^{j-1-\varphi-r} (1-u_1-u_2+D)^{n-i-j+\theta+\varphi+r}(D^{1,\cdot})^{\varphi}(D^{\cdot,1})^{\theta} (1-D^{1,\cdot})^{1-\varphi}(1-D^{\cdot,1})^{1-\theta}du_1du_2,$$

where, $\varphi_* = 0\vee(j-n+1)$; $\varphi^* = 1\wedge(j-1)$; $\theta_* = 0\vee(i-n+1)$; $\theta^* = 1\wedge(i-1)$; $r_* = 0\vee(i+j-n-\theta-\varphi)$; $r^* = (i-\theta-1)\wedge(j-\varphi-1)$; $C_1 = \frac{n!}{(i-1-r)!r!(j-1-r)!(n-i-j+1+r)!}$; $C_2 = \frac{n!}{(i-1-\theta-r)!r!(j-1-\varphi-r)!(n-i-j+\theta+\varphi+r)!}$; $F_t^{-1}(u_t) = \sup\{x_t : F_t(x_t) \preceq u_t\}$, $0 < u_t < 1$, *is the inverse of the d.f.* $F_t(x_t)$ *and* $D = D(\bar{u})$ *is the dependence function of* F, *i.e.,* $D = D(\bar{x}) = D(F_1(x_1), F_2(x_2))$ (*see Galambos* (1978/87) *and Leadbeter and Rootzen* (1988)).

Remark 2.1. By using the well known δ-method (see Arnold *et al.* (1992), Chap. 5) we can derive an approximate formula for the product moments for any bivariate d.f. F as

$$\mu_{i,j:n} = F_1^{-1}(\ell_i)F_2^{-1}(\ell_j)\left[1 + \frac{i+j-2ij}{2(n+1)(n+2)} + \frac{1}{2}\nu_{i,j:n}\right] + o(n^{-2}),$$

where $\ell_t = \frac{t}{n+1}$, $t = i, j$ and $\nu_{i,j:n}$ is determined from (2.1) by putting $F_t^{-1}(u_t) = u_t$, $t = 1, 2$, $m_i = m_j = 1$ and $D = F(F_1^{-1}, F_2^{-1})$.

PROOF OF THEOREM 2.1. The key ingredient of the proof is the observation that the event $E = \{x_1 < X_{1,i:n} \leq x_1 + \delta x_1; x_2 < X_{2,j:n} \leq x_2 + \delta x_2\}$ may be realized as follows: r; φ_1; s_1; θ_1; w; θ_2; s_2; φ_2 and t observations must fall respectively in the regions $I_1 = (-\infty, x_1] \cap (-\infty, x_2]$; $I_2 = (x_1, x_1+\delta x_1] \cap (-\infty, x_2]$; $I_3 = (x_1+\delta x_1, \infty) \cap (-\infty, x_2]$; $I_4 = (-\infty, x_1] \cap (x_2, x_2+\delta x_2]$; $I_5 = (x_1, x_1+\delta x_1] \cap (x_2, x_2+\delta x_2]$; $I_6 = (x_1+\delta x_1, \infty) \cap (x_2, x_2+\delta x_2]$; $I_7 = (-\infty, x_1] \cap (x_2+\delta x_2, \infty)$; $I_8 = (x_1, x_1+\delta x_1] \cap (x_2+\delta x_2, \infty)$ and $I_9 = (x_1+\delta x_1, \infty) \cap (x_2+\delta x_2, \infty)$. Therefore, the joint density function $f_{i,j:n}(\bar{x})$ of $(X_{1,i:n}, X_{2,j:n})$ is the limit of $\frac{P(E)}{\delta x_1 \delta x_2}$, as $\delta x_1, \delta x_2 \to 0$, where $P(E)$ can be derived by noting that $\theta_1+\theta_2+w = \varphi_1+\varphi_2+w = 1$; $r+\theta_1+s_2 = i-1$; $r+\varphi_1+s_1 = j-1$; $r+\theta_1+s_2+\varphi_1+w+\theta_2+s_1+\varphi_2+t = n$; $r, \theta_1, s_2, \varphi_1, w, \theta_2, s_1, \varphi_2, t \geq 0$; $P(I_1) = F$; $P(I_2) \simeq F^{1,\cdot}\delta x_1$; $P(I_3) \simeq (F_2 - F)$; $P(I_4) \simeq F^{\cdot,1}\delta x_2$; $P(I_5) \simeq F^{1,1}\delta x_1\delta x_2 = f\delta x_1\delta x_2$; $P(I_6) \simeq (f_2 - F^{\cdot,1})\delta x_2$; $P(I_7) \simeq (F_1 - F)$; $P(I_8) \simeq (f_1 - F^{1,\cdot})\delta x_1$ and $P(I_9) \simeq (1 - F_1 - F_2 + F)$. Finally, (2.1) follows from the relation $\iint_{-\infty}^{\infty} x_1^{m_i}x_2^{m_j} f_{i,j:n}(\bar{x})dx_1dx_2$ by taking the transformation $u_t = F_t(x_t)$, $t = 1, 2$.

3. Distribution-free bounds

THEOREM 3.1. *For any bivariate uniform d.f.* $D = D(\bar{u})$ *and all* $2 \leq i, j \leq n-1$ *we have*

$$\frac{-i}{(n+1)(n+2)} \leq \nu_{i,j:n} - \nu_{i,j-1:n} \leq \frac{3i+1}{(n+1)(n+2)}$$

and

$$\frac{-j}{(n+1)(n+2)} \leq \nu_{i,j:n} - \nu_{i-1,j:n} \leq \frac{3j+1}{(n+1)(n+2)}.$$

Moreover, for all $n = 1, 2, \ldots,$ *we have*

$$1 - \frac{3}{(n+1)} \leq \nu_{n,n:n} \leq 1 - \frac{1}{(n+1)(n+2)};$$

$$\nu_{1,1:n} \leq \frac{n+3}{(n+1)(n+2)}; \quad \text{and} \quad \frac{n-1}{(n+1)(n+2)} \leq \nu_{1,n:n}, \nu_{n,1:n} \leq \frac{1}{(n+1)}.$$

Remark 3.1. For any bivariate uniform d.f. D, in view of Theorem 3.1, it is easy to show that $U_{1,i:n}$ and $U_{2,j:n}$, for fixed $i, j = 1, n$ are asymptotically uncorrelated.

The following corollary is an easy consequence of the mentioned theorem.

COROLLARY 3.1. *For* $2 \leq i, j \leq n-1$, *we have*

$$\nu_{i,1:n} - \frac{(j-1)i}{(n+1)(n+2)} \leq \nu_{i,j:n} \leq \nu_{i,1:n} + \frac{(j-1)(3i+1)}{(n+1)(n+2)}.$$

COROLLARY 3.2. *For all* $2 \leq i, j \leq n-1$, *we have*

$$\frac{-3j+1}{j(j-1)} \leq \nu^{(1,-1)}_{i-1,j:n} - \nu^{(1,-1)}_{i,j:n} \leq \frac{1}{(j-1)};$$

$$\frac{-3i+1}{i(i-1)} \leq \nu^{(-1,1)}_{i,j-1:n} - \nu^{(-1,1)}_{i,j:n} \leq \frac{1}{(i-1)}$$

and

$$1 - \frac{2}{n} \leq \nu^{(1,-1)}_{n,n:n}, \nu^{(-1,1)}_{n,n:n} \leq 1 + \frac{2n-1}{n(n-1)}.$$

COROLLARY 3.3. *For any bivariate d.f.* $F(\bar{x})$ *the rank concomitant o.s.* $\rho_{i,[j]:n}$ *of* $X_{2,[j]:n} = \sum_{i=1}^{n} X_{2j} I_{[rank(x_{1i})=j]}$, *where* $I_{[A]}$ *denotes the indicator function of the set* A, *satisfies the relation* $\rho_{i,[j]:n} \geq \frac{n-i-j+2}{n+1}\rho_{i,[j]:n+1}$, $\forall 1 \leq i, j \leq n-1$. *The general theory of concomitant o.s. is discussed by many authors (see e.g., Galambos* (1978/87), *David* (1981) *and Bhattacharya* (1984)).

For proving Theorem 3.1, we first present two lemmas.

LEMMA 3.1. *For $2 \leq i, j \leq n-1$ and for any two real values m_i, m_j we have*

$$\frac{2n-i-j+2}{n+1}\nu_{i,j:n+1}^{(m_i,m_j)} + \frac{i}{n+1}\nu_{i+1,j:n+1}^{(m_i,m_j)} + \frac{j}{n+1}\nu_{i,j+1:n+1}^{(m_i,m_j)}$$
$$\geq 2\nu_{i,j:n}^{(m_i,m_j)} + (\nu_{i,j-1:n}^{(m_i,m_j+1)} + \nu_{i-1,j:n}^{(m_i+1,m_j)} - \nu_{i,j:n}^{(m_i+1,m_j)} - \nu_{i,j:n}^{(m_i,m_j+1)})$$
$$\geq \frac{2n-i-j}{n+1}\nu_{i,j:n+1}^{(m_i,m_j)} + \frac{i-1}{n+1}\nu_{i+1,j:n+1}^{(m_i,m_j)} + \frac{j-1}{n+1}\nu_{i,j+1:n+1}^{(m_i,m_j)}.$$

PROOF. For all $2 \leq i, j \leq n-1$, we have, in (2.1), $\varphi_* = \theta_* = 0$ and $\varphi^* = \theta^* = 1$. Let $B_{i,j:n}^{(m_i,m_j)}$, $C_{i,j:n}^{(m_i,m_j)}$, $D_{i,j:n}^{(m_i,m_j)}$ and $E_{i,j:n}^{(m_i,m_j)}$ denote the second term on the right hand side of (2.1) (the three summations over φ, θ and r) when $\varphi = \theta = 1$; $\varphi = \theta = 0$; $\varphi = 0$, $\theta = 1$ and $\varphi = 1$, $\theta = 0$ respectively. Therefore, (2.1) may be written as

$$\nu_{i,j:n}^{(m_i,m_j)} = A_{i,j:n}^{(m_i,m_j)} + (B_{i,j:n}^{(m_i,m_j)} + C_{i,j:n}^{(m_i,m_j)} + D_{i,j:n}^{(m_i,m_j)} + E_{i,j:n}^{(m_i,m_j)}), \tag{3.1}$$

where $A_{i,j:n}^{(m_i,m_j)}$ denotes the first term on the right hand side of (2.1), i.e.,

$$A_{i,j:n}^{(m_i,m_j)} = \sum_{r=0\vee(i+j-n-1)}^{(i-1)\wedge(j-1)} C_1 \iint_0^1 u_1^{m_i} u_2^{m_j} (u_1 - D)^{i-1-r} D^r (u_2 - D)^{j-1-r} (1 - u_1 - u_2 + D)^{n-i-j+1+r} D^{1,1} du_1 du_2. \tag{3.2}$$

Now, upon using the identity $1 = (u_1 - D) + (u_2 - D) + (1 - u_1 - u_2 + D) + D$ in the integrand of (3.2), and simplifying the resulting expression, we get

$$A_{i,j:n}^{(m_i,m_j)} = \left(\frac{i}{n+1}A_{i+1,j:n+1}^{(m_i,m_j)} + \frac{j}{n+1}A_{i,j+1:n+1}^{(m_i,m_j)} + \frac{n-i-j+2}{n+1}A_{i,j:n+1}^{(m_i,m_j)}\right) + (A_{i,j:n}^{\star(m_i,m_j)} + A_{i-1,j-1:n}^{\star(m_i,m_j)} - A_{i-1,j:n}^{\star(m_i,m_j)} - A_{i,j-1:n}^{\star(m_i,m_j)}), \tag{3.3}$$

where

$$A_{i,j:n}^{\star(m_i,m_j)} = \sum_{r=0\vee(i+j-n-1)}^{(i-1)\wedge(j-1)} C_1 \iint_0^1 u_1^{m_i} u_2^{m_j} (u_1 - D)^{i-1-r} D^{r+1} (u_2 - D)^{j-1-r} (1 - u_1 - u_2 + D)^{n-i-j+1+r} D^{1,1} du_1 du_2. \tag{3.4}$$

On the other hand, by writing the term D^{r+1} in the integrand of (3.4) as $(1 - u_1 - u_2 + D)D^r - (1 - u_1 - u_2)D^r$; $u_1 D^r - (u_1 - D)D^r$ and $u_2 D^r - (u_2 - D)D^r$, we get, respectively

$$A^{\star(m_i,m_j)}_{i-1,j:n} = \left(\frac{n-i-j+2}{n+1} A^{(m_i,m_j)}_{i,j:n+1} + A^{\star(m_i,m_j)}_{i-1,j-1:n}\right) - (A^{(m_i,m_j)}_{i,j:n} - A^{(m_i+1,m_j)}_{i,j:n} - A^{(m_i,m_j+1)}_{i,j:n}); \tag{3.5}$$

$$A^{\star(m_i,m_j)}_{i-1,j:n} = A^{(m_i+1,m_j)}_{i-1,j:n} - \left(\frac{i-1}{n+1} A^{(m_i,m_j)}_{i,j:n+1} - A^{\star(m_i,m_j)}_{i-1,j-1:n}\right) \tag{3.6}$$

and

$$A^{\star(m_i,m_j)}_{i,j-1:n} = A^{(m_i,m_j+1)}_{i,j-1:n} - \left(\frac{j-1}{n+1} A^{(m_i,m_j)}_{i,j:n+1} - A^{\star(m_i,m_j)}_{i-1,j-1:n}\right). \tag{3.7}$$

Combining (3.3) with (3.5), (3.6) and (3.7) we deduce that

$$2A^{(m_i,m_j)}_{i,j:n} = \left(\frac{2n-i-j+2}{n+1} A^{(m_i,m_j)}_{i,j:n+1} + \frac{i}{n+1} A^{(m_i,m_j)}_{i+1,j:n+1} + \frac{j}{n+1} A^{(m_i,m_j)}_{i,j+1:n+1}\right) + (A^{(m_i+1,m_j)}_{i,j:n} + A^{(m_i,m_j+1)}_{i,j:n} - A^{(m_i+1,m_j)}_{i-1,j:n} - A^{(m_i,m_j+1)}_{i,j-1:n}).$$

Similarly, we can easily get analogous relations for $B^{(m_i,m_j)}_{i,j:n}, \ldots, E^{(m_i,m_j)}_{i,j:n}$. Namely,

$$2B^{(m_i,m_j)}_{i,j:n} = \left(\frac{2n-i-j+2}{n+1} B^{(m_i,m_j)}_{i,j:n+1} + \frac{i-1}{n+1} B^{(m_i,m_j)}_{i+1,j:n+1} + \frac{j-1}{n+1} B^{(m_i,m_j)}_{i,j+1:n+1}\right) + (B^{(m_i+1,m_j)}_{i,j:n} + B^{(m_i,m_j+1)}_{i,j:n} - B^{(m_i+1,m_j)}_{i-1,j:n} - B^{(m_i,m_j+1)}_{i,j-1:n});$$

$$2C^{(m_i,m_j)}_{i,j:n} = \left(\frac{2n-i-j}{n+1} C^{(m_i,m_j)}_{i,j:n+1} + \frac{i}{n+1} C^{(m_i,m_j)}_{i+1,j:n+1} + \frac{j}{n+1} C^{(m_i,m_j)}_{i,j+1:n}\right) + (C^{(m_i+1,m_j)}_{i,j:n} + C^{(m_i,m_j+1)}_{i,j:n} - C^{(m_i+1,m_j)}_{i-1,j:n} - C^{(m_i,m_j+1)}_{i,j-1:n});$$

$$2D^{(m_i,m_j)}_{i,j:n} = \left(\frac{2n-i-j+1}{n+1} D^{(m_i,m_j)}_{i,j:n+1} + \frac{i-1}{n+1} D^{(m_i,m_j)}_{i+1,j:n+1} + \frac{j}{n+1} D^{(m_i,m_j)}_{i,j+1:n+1}\right) + (D^{(m_i+1,m_j)}_{i,j:n} + D^{(m_i,m_j+1)}_{i,j:n} - D^{(m_i+1,m_j)}_{i-1,j:n} - D^{(m_i,m_j+1)}_{i,j-1:n})$$

and

$$2E^{(m_i,m_j)}_{i,j:n} = \left(\frac{2n-i-j+1}{n+1} E^{(m_i,m_j)}_{i,j:n+1} + \frac{i}{n+1} E^{(m_i,m_j)}_{i+1,j:n+1} + \frac{j-1}{n+1} E^{(m_i,m_j)}_{i,j+1:n+1}\right) + (E^{(m_i+1,m_j)}_{i,j:n} + E^{(m_i,m_j+1)}_{i,j:n} - E^{(m_i+1,m_j)}_{i-1,j:n} - E^{(m_i,m_j+1)}_{i,j-1:n}).$$

By combining the above derived five relations (for $A^{(m_i,m_j)}_{i,j:n}, \ldots, E^{(m_i,m_j)}_{i,j:n}$) with (3.1) we can easily obtain the following two relations

$$2\nu^{(m_i,m_j)}_{i,j:n} = \left(\frac{2n-i-j}{n+1} \nu^{(m_i,m_j)}_{i,j:n+1} + \frac{i-1}{n+1} \nu^{(m_i,m_j)}_{i+1,j:n+1} + \frac{j-1}{n+1} \nu^{(m_i,m_j)}_{i,j+1:n+1}\right) + (\nu^{(m_i+1,m_j)}_{i,j:n} + \nu^{(m_i,m_j+1)}_{i,j:n} - \nu^{(m_i+1,m_j)}_{i-1,j:n} - \nu^{(m_i,m_j+1)}_{i,j-1:n})$$

$$+\frac{1}{n+1}(2A_{i,j:n+1}^{(m_i,m_j)}+2B_{i,j:n+1}^{(m_i,m_j)}+D_{i,j:n+1}^{(m_i,m_j)}+E_{i,j:n+1}^{(m_i,m_j)}$$
$$+A_{i+1,j:n+1}^{(m_i,m_j)}+C_{i+1,j:n+1}^{(m_i,m_j)}+E_{i+1,j:n+1}^{(m_i,m_j)}$$
$$+A_{i,j+1:n+1}^{(m_i,m_j)}+C_{i,j+1:n+1}^{(m_i,m_j)}+D_{i,j+1:n+1}^{(m_i,m_j)})$$

and

$$2\nu_{i,j:n}^{(m_i,m_j)}=\left(\frac{2n-i-j+2}{n+1}\nu_{i,j:n+1}^{(m_i,m_j)}+\frac{i}{n+1}\nu_{i+1,j:n+1}^{(m_i,m_j)}+\frac{j}{n+1}\nu_{i,j+1:n+1}^{(m_i,m_j)}\right)$$
$$+(\nu_{i,j:n}^{(m_i+1,m_j)}+\nu_{i,j:n}^{(m_i,m_j+1)}-\nu_{i-1,j:n}^{(m_i+1,m_j)}-\nu_{i,j-1:n}^{(m_i,m_j+1)})$$
$$-\frac{1}{n+1}(D_{i,j:n+1}^{(m_i,m_j)}+E_{i,j:n+1}^{(m_i,m_j)}+2C_{i,j:n+1}^{(m_i,m_j)}$$
$$+B_{i+1,j:n+1}^{(m_i,m_j)}+D_{i+1,j:n+1}^{(m_i,m_j)}+B_{i,j+1:n+1}^{(m_i,m_j)}+E_{i,j+1:n+1}^{(m_i,m_j)}),$$

which yield immediately the desired result after noting that $0\le A_{i,j:n}^{(m_i,m_j)},\ldots,E_{i,j:n}^{(m_i,m_j)}\le 1$. The lemma is proved.

LEMMA 3.2. *For any two real values m_i and m_j, we have*

$$\frac{1}{n+1}\left(\nu_{n,n+1:n+1}^{(m_i,m_j)}+\nu_{n+1,n:n+1}^{(m_i,m_j)}+2n\nu_{n+1,n+1:n+1}^{(m_i,m_j)}\right)\ge\nu_{n,n:n}^{(m_i+1,m_j)}+\nu_{n,n:n}^{(m_i,m_j+1)}$$
$$\ge\frac{2(n-1)}{n+1}\nu_{n+1,n+1:n+1}^{(m_i,m_j)};$$
$$2\nu_{1,1:n}^{(m_i,m_j)}-\frac{n-1}{n+1}\nu_{1,1:n+1}^{(m_i,m_j)}\ge\nu_{1,1:n}^{(m_i+1,m_j)}+\nu_{1,1:n}^{(m_i,m_j+1)}$$
$$\ge\nu_{1,1:n}^{(m_i,m_j)}-\frac{n}{n+1}\nu_{1,1:n+1}^{(m_i,m_j)};$$
$$\frac{n}{n+1}\nu_{1,n+1:n+1}^{(m_i,m_j)}+\frac{1}{n+1}\nu_{2,n+1:n+1}^{(m_i,m_j)}\ge\nu_{1,n:n}^{(m_i,m_j+1)}\ge\frac{n-1}{n+1}\nu_{1,n+1:n+1}^{(m_i,m_j)}$$

and

$$\frac{n}{n+1}\nu_{n+1,1:n+1}^{(m_i,m_j)}+\frac{1}{n+1}\nu_{n+1,2:n+1}^{(m_i,m_j)}\ge\nu_{n,1:n}^{(m_i+1,m_j)}\ge\frac{n-1}{n+1}\nu_{n+1,1:n+1}^{(m_i,m_j)}.$$

PROOF. Let us prove the first relation of the lemma. Upon using the identity $u_1+u_2=(u_1-D)+(u_2-D)+2D$, in (3.2) (with $i=j=n$) we can easily, after simple calculations, derive the identity

$$A_{n,n:n}^{(m_i+1,m_j)}+A_{n,n:n}^{(m_i,m_j+1)}=\frac{1}{n+1}(A_{n,n+1:n+1}^{(m_i,m_j)}+A_{n+1,n:n+1}^{(m_i,m_j)})+\frac{2n}{n+1}A_{n+1,n+1:n+1}^{(m_i,m_j)}.$$

Similarly, we can obtain analogous identity for $B_{n,n:n}$. Namely,

$$B_{n,n:n}^{(m_i+1,m_j)}+B_{n,n:n}^{(m_i,m_j+1)}=\frac{1}{n+1}(B_{n,n+1:n+1}^{(m_i,m_j)}+B_{n+1,n:n+1}^{(m_i,m_j)})+\frac{2n-2}{n+1}B_{n+1,n+1:n+1}^{(m_i,m_j)}.$$

Combining the last two relations with the facts that (in view of (2.1)) $\nu_{n,n:n}^{(m_i,m_j)}=A_{n,n:n}^{(m_i,m_j)}+B_{n,n:n}^{(m_i,m_j)}$; $\nu_{n,n+1:n+1}^{(m_i,m_j)}=A_{n,n+1:n+1}^{(m_i,m_j)}+B_{n,n+1:n+1}^{(m_i,m_j)}+E_{n,n+1:n+1}^{(m_i,m_j)}$ and

$\nu_{n+1,n:n+1}^{(m_i,m_j)} = A_{n+1,n:n+1}^{(m_i,m_j)} + B_{n+1,n:n+1}^{(m_i,m_j)} + D_{n+1,n:n+1}^{(m_i,m_j)}$, we can easily, after routine calculations, get the two relations

$$\begin{aligned}\nu_{n,n:n}^{(m_i+1,m_j)} &+ \nu_{n,n:n}^{(m_i,m_j+1)} \\ &= \frac{1}{n+1}(\nu_{n,n+1:n+1}^{(m_i,m_j)} + \nu_{n+1,n:n+1}^{(m_i,m_j)} + 2n\nu_{n+1,n+1:n+1}^{(m_i,m_j)}) \\ &- \frac{1}{n+1}(2B_{n+1,n+1:n+1}^{(m_i,m_j)} + D_{n+1,n:n+1}^{(m_i,m_j)} + E_{n,n+1:n+1}^{(m_i,m_j)});\end{aligned}$$

and

$$\begin{aligned}\nu_{n,n:n}^{(m_i+1,m_j)} &+ \nu_{n,n:n}^{(m_i,m_j+1)} \\ &= \frac{2(n-1)}{n+1}\nu_{n+1,n+1:n+1}^{(m_i,m_j)} + \frac{1}{n+1}(A_{n,n+1:n+1}^{(m_i,m_j)} + A_{n+1,n:n+1}^{(m_i,m_j)} \\ &+ 2A_{n+1,n+1:n+1}^{(m_i,m_j)} + B_{n,n+1:n+1}^{(m_i,m_j)} + B_{n+1,n:n+1}^{(m_i,m_j)}),\end{aligned}$$

which immediately yield the desired result. For proving the three remaining relations, we first notice that $\nu_{1,1:n}^{(m_i,m_j)} = A_{1,1:n}^{(m_i,m_j)} + C_{1,1:n}^{(m_i,m_j)}$; $\nu_{1,n:n}^{(m_i,m_j)} = A_{1,n:n}^{(m_i,m_j)} + E_{1,n:n}^{(m_i,m_j)}$ and $\nu_{n,1:n}^{(m_i,m_j)} = A_{n,1:n}^{(m_i,m_j)} + D_{n,1:n}^{(m_i,m_j)}$. Therefore, upon using, respectively, the identities $u_1 + u_2 = 1 - (1 - u_1 - u_2 + D) + D$; $u_2 = (u_2 - D) + D$ and $u_1 = (u_1 - D) + D$ and by proceeding on similar lines as we did in the proving of the first relation (with only the obvious changes) we can easily get, respectively, the desired relations. The lemma is proved.

PROOF OF THEOREM 3.1. An easy application of Lemma 3.1 with $m_i = 1$, $m_j = 0$ and $m_i = 0$, $m_j = 1$ leads, respectively to the first two relations in the theorem (note that $\nu_{i,j:n}^{(0,1)} = \frac{j}{n+1}$, $\nu_{i,j:n}^{(1,0)} = \frac{i}{n+1}$, $\nu_{i,j:n}^{(0,2)} = \frac{j(j+1)}{(n+1)(n+2)}$ and $\nu_{i,j:n}^{(2,0)} = \frac{i(i+1)}{(n+1)(n+2)}$, see Gibbons (1970)). The remaining relations of the theorem follow, respectively, from the relations of Lemma 3.2, by putting $m_i = 1$, $m_j = 0$ in the first three relations of the lemma and $m_i = 0$, $m_j = 1$ in the last relation. The theorem is established.

PROOF OF COROLLARY 3.2. We again appeal to Lemma 3.1 with $m_i = 0$, $m_j = -1$ and $m_i = -1$, $m_j = 0$ to get, respectively the first two relations of the corollary (note that $\nu_{i,j:n}^{(0,-1)} = \frac{n}{j-1}$ and $\nu_{i,j:n}^{(-1,0)} = \frac{n}{i-1}$, $\forall i,j > 1$). The last two relations of the corollary (for $\nu_{n,n:n}^{(1,-1)}$ and $\nu_{n,n:n}^{(-1,1)}$) follow from the third and the fourth relations of Lemma 3.2 by putting, respectively $m_i = 0$, $m_j = -1$ and $m_i = -1$, $m_j = 0$. The proof is completed.

PROOF OF COROLLARY 3.3. The proof follows by noting that $\rho_{i,[j]:n} = A_{i,j:n}^{(0,0)}$ and

$$\rho_{i,[j]:n} = A_{i,j:n}^{(0,0)} \geq \sum_{r=0\vee(i+j-n-1)}^{(i-1)\wedge(j-1)} C_1 \iint_0^1 (u_1 - D)^{i-1-r} D^r (u_2 - D)^{j-1-r}$$

$$(1-u_1-u_2+D)^{n-i-j+2+r}D^{1,1}du_1du_2 = \frac{n-i-j+2}{n+1}A^{(0,0)}_{i,j:n+1} + A^{\star(0,0)}_{i-1,j-1:n}$$
$$\geq \frac{n-i-j+2}{n+1}A^{(0,0)}_{i,j:n+1}.$$

The proof is completed.

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