Ann. Inst. Statist. Math.
Vol. 51, No. 2, 345–349 (1999)

EQUALITY IN DISTRIBUTION IN A CONVEX ORDERING FAMILY

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(Received March 7, 1996; revised October 23, 1997)

Abstract. Let F and G be the respective distributions of nonnegative random variables X and Y satisfying the convex ordering. We investigate the class of functions h for which the equality $E[h(X)] = E[h(Y)]$ guarantees $F = G$. It leads to extensions of some existing results and at the same time offers a somewhat simpler proof.

Key words and phrases: Characterization of distribution, convex ordering, concave ordering, moment, Laplace-Stieltjes transform, moment generating function, probability generating function, order statistics.

1. Introduction

A variability ordering $<^p$ for life distributions, introduced by Bhattacharjee and Sethuraman (1990), was later generalized by Bhattacharjee (1991) to $<^{(p)}$. If X and Y are nonnegative random variables with distributions F and G (and survival functions $\bar{F}$ and $\bar{G}$), respectively, we say that for $p > 0$

$$X <^{(p)} Y \text{ (or } F <^{(p)} G)$$
$$\text{if and only if } \int_x^\infty t^{p-1}\bar{F}(t)dt \le \int_x^\infty t^{p-1}\bar{G}(t)dt \quad \text{for all} \quad x \ge 0,$$
$$X <_{(p)} Y \text{ (or } F <_{(p)} G)$$
$$\text{if and only if } \int_0^x t^{p-1}\bar{F}(t)dt \ge \int_0^x t^{p-1}\bar{G}(t)dt \quad \text{for all} \quad x \ge 0,$$
$$X <^p Y \text{ (or } F <^p G)$$
$$\text{if and only if } X <^{(p)} Y \quad \text{and} \quad E(X^p) = E(Y^p) < \infty.$$

The ordering $<^{(p)}$ is meant as a generalization of $<^{(1)}$, which was called convex ordering by Stoyan ((1983), p. 8), who used the notation $\le_c$. Before Stoyan, the latter was called *stochastically smaller in mean* by Bessler and Veinott (1966). In reliability theory it is known that a life distribution F is harmonic new better than used in expectation (HNBUE) if and only if $F <^1$-precedes an exponential distribution (Klefsjö (1982)). Various properties of the generalized orderings were studied by Bhattacharjee (1991), Cai (1994) and Li and Zhu (1994). It was shown for example that if $X <^{(p)} Y$ and $EX^r = EY^r < \infty$ for some $r > p$, then $F = G$. In other words, within the $<^{(p)}$-ordered family, one moment, *of order higher than* p, is sufficient to identify the distribution.

In this paper we extend some of their results, and at the same time offer a simpler proof. We begin with a simple observation: if $X <^{(p)} Y$ for some $p > 0$, then $X^r <^{(p/r)} Y^r$ for each $r > 0$. Hence, for $p > 0$, $X <^{(p)} Y$ if and only if $X^p <^{(1)} Y^p$ (see also Bhattacharjee (1991), p. 377). Because the distribution of X^p uniquely determines that of X and vice versa, there is therefore no gain in generality in studying the orderings $<^{(p)}$ (and similarly, $<_{(p)}$) for any p other than $p = 1$. In the next section we shall forsake $<^{(p)}$ and consider only the random variables X and Y such that $X <^{(1)} Y$ (or $X <_{(1)} Y$). We investigate the class of functions h for which the equality $E[h(X)] = E[h(Y)]$ guarantees $X =^d Y$ (namely X and Y have the same distribution).

2. The main results

To prove the main results (Theorems 2.1, 2.2 and 2.3) below, we need some representations of Lebegue-Stieltjes integral $E[h(X)]$, which are fundamental and require no proof.

LEMMA 2.1. *Let X be a nonnegative random variable with distribution F.*

(i) *If h is a nonnegative, nondecreasing and continuous function defined on $[0,\infty)$ such that $E[h(X)] < \infty$, then $\bar{F}(x)h(x) \to 0$ as $x \to \infty$, and*

$$E[h(X)] = \int_{(0,\infty)} \bar{F}(x)dh(x) + h(0).$$

(ii) *If in addition, $EX < \infty$ and h is twice differentiable, with $h'' \ge 0$, on $(0,\infty)$, then $h'(x)\int_x^\infty \bar{F}(t)dt \to 0$ as $x \to \infty$, and*

$$E[h(X)] = \int_0^\infty h''(x)\left[\int_x^\infty \bar{F}(t)dt\right]dx + h(0) + h'(0^+)EX,$$

where $h'(0^+) \equiv \lim_{x\to 0} h'(x)$ is finite.

(iii) *If instead, $h'' \le 0$ on $(0,\infty)$, then $h'(x)\int_0^x \bar{F}(t)dt \to 0$ as $x \to 0$. Moreover, if $EX < \infty$, then*

$$E[h(X)] = -\int_0^\infty h''(x)\left[\int_0^x \bar{F}(t)dt\right]dx + h(0) + h'(\infty)EX,$$

where $h'(\infty) \equiv \lim_{x\to\infty} h'(x)$ is finite.

Our first result deals with the convex ordering $<^{(1)}$ and nondecreasing convex h.

THEOREM 2.1. *Let X and Y be nonnegative random variables such that $X <^{(1)} Y$ and $EY < \infty$ (hence $EX < \infty$). Suppose that there is a continuous function h defined on $[0, \infty)$ such that* (i) $h \geq 0$, $h' \geq 0$, $h'' > 0$ *on* $(0, \infty)$, (ii) $E[h(X)] = E[h(Y)] < \infty$, *and* (iii) *either* $h'(0^+) = 0$ *or* $EX = EY$, *then* $X =^d Y$.

PROOF. Let F and G be the respective distributions of X and Y. By Lemma 2.1(ii),

$$E[h(X)] - E[h(Y)] = \int_0^\infty h''(x) \left\{ \int_x^\infty [\bar{F}(t) - \bar{G}(t)] dt \right\} dx + h'(0^+)[EX - EY],$$

and therefore,

$$\int_0^\infty h''(x) \left\{ \int_x^\infty [\bar{F}(t) - \bar{G}(t)] dt \right\} dx = 0. \tag{2.1}$$

Recall that $X <^{(1)} Y$ implies $\int_x^\infty [\bar{F}(t) - \bar{G}(t)] dt \leq 0$ for all $x > 0$. This, together with (2.1) and the assumption that $h''(x) > 0$ on $(0, \infty)$, implies $\int_x^\infty [\bar{F}(t) - \bar{G}(t)] dt = 0$ a.e. on $(0, \infty)$, namely, $F = G$. The proof is complete.

COROLLARY 2.1. *If* (a) $X <^{(1)} Y$ *and* $E(X^r) = E(Y^r) < \infty$ *for some* $r > 1$, (b) $X <^1 Y$ *and* $E(e^{sX}) = E(e^{sY}) < \infty$ *for some* $s > 0$, *or* (c) $X <^1 Y$ *and* $E(a^X) = E(a^Y) < \infty$ *for some* $a > 1$, *then* $X =^d Y$.

PROOF. In Theorem 2.1, take $h(x) = x^r$ for (a), and $h(x) = e^{sx}$ for (b). (c) follows from (b) by noting that $a^x = e^{sx}$ with $s = \ln a > 0$.

Bhattacharjee (1991) showed that $<^{(p)}$ ($<_{(p)}$, resp.)-ordering is preserved under formation of parallel (series, resp.) systems. Cai ((1994), Theorem 3.3(II)) used it to characterize the parent distributions in such systems. Our next corollary offers a trade-off. When the X_i's (and the Y_i's) are *identically* distributed, the population distribution is uniquely determined by that of a single order statistic, say, the maximum order statistic $X_{n,n}$ (see Galambos (1975), p. 79), which enables us to drop Cai's requirement of $F(x) \in (0, 1)$, for $x > 0$. The proof is straightforward and omitted.

COROLLARY 2.2. *Let $X_1, X_2, \ldots, X_n$ ($n \geq 2$) be i.i.d. $\sim F$, and $Y_1, Y_2, \ldots, Y_n$ be i.i.d. $\sim G$. If* (i) $F <^{(1)} G$ *and* (ii) $EX_{n,n}^r = EY_{n,n}^r < \infty$ *for some* $r > 1$, *then* $F = G$.

Our next theorem and its corollaries are a variant of the above for the *concave ordering* $<_{(1)}$ (see Stoyan (1983), pp. 11–12), and require no new proof.

THEOREM 2.2. *Let X and Y be nonnegative random variables such that $X <_{(1)} Y$ and $EX < \infty$ (hence $EY < \infty$). Suppose that there is a continuous function h defined on $[0, \infty)$ such that* (i) $h \geq 0$, $h' \geq 0$, $h'' < 0$ *on* $(0, \infty)$, (ii) $E[h(X)] = E[h(Y)] < \infty$, *and* (iii) *either* $h'(\infty) = 0$ *or* $EX = EY$, *then* $X =^d Y$.

COROLLARY 2.3. *Let $X <_{(1)} Y$ and $EX < \infty$. If* (a) $E(X^r) = E(Y^r)$ *for some* $r \in (0,1)$, (b) $E(e^{-sX}) = E(e^{-sY})$ *for some* $s > 0$, *or* (c) $E(a^X) = E(a^Y)$ *for some* $a \in (0,1)$, *then* $X =^d Y$.

COROLLARY 2.4. *Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be i.i.d. $\sim F$, and $Y_1, Y_2, \ldots, Y_n$ be i.i.d. $\sim G$. If* (i) $F <_{(1)} G$ *and* (ii) *the minimum order statistics satisfy $EX_{1,n} < \infty$ and $EX_{1,n}^r = EY_{1,n}^r$ for some $r \in (0,1)$, then $F = G$.*

Remark. Corollary 2.1(a) was obtained separately by Stoyan ((1983), p. 10), Cai ((1994), Theorem 2.2(II)) and Li and Zhu ((1994), Theorem 2.3). Also, Corollary 2.3(a) was obtained separately by Cai ((1994), Theorem 2.2(I)) and Li and Zhu ((1994), Theorem 2.4). To see Corollary 2.3(b), take $h(x) = 1 - e^{-sx}$ in Theorem 2.2. Actually, Corollary 2.3(b) extends a result of Lin ((1994), Theorem 4.1).

Theorem 2.2 fails to cover the full generality of Cai ((1994), Theorem 2.2(I)) who also considered negative r values. Our next result, which deals with the concave ordering $<_{(1)}$ and nonincreasing convex h, fills the gap.

THEOREM 2.3. *Let X and Y be positive random variables such that $X <_{(1)} Y$ and $EX < \infty$. Suppose that there is a twice differentiable function h defined on $(0, \infty)$ such that* (i) $h \geq 0$, $h' \leq 0$, $h'' > 0$ *on* $(0, \infty)$, (ii) $E[h(X)] = E[h(Y)] < \infty$, *and* (iii) *either $h'(0^+)$ is finite, or $h'(0^+) = -\infty$ with $(h'(x))^2/h''(x) = O(h(x))$ as $x \to 0$, then $X =^d Y$.*

PROOF. Let F and G be the distributions of X and Y, respectively. First note that $h(x)F(x) \to 0$ as $x \to 0$, by the conditions (i) and (ii). Thus

$$E[h(X)] = h(\infty) - \int_0^\infty F(x)h'(x)dx,$$

where $h(\infty) \equiv \lim_{x\to\infty} h(x)$ is finite due to the condition (i). Let $W(x) \equiv \int_0^x [\bar{F}(t) - \bar{G}(t)]dt \geq 0$ for $x \geq 0$, then

$$0 = E[h(Y)] - E[h(X)] = \int_0^\infty h'(x)[F(x) - G(x)]dx \tag{2.2}$$

$$= \int_0^\infty [-h'(x)][\bar{F}(x) - \bar{G}(x)]dx = \int_0^\infty [-h'(x)]dW(x) \tag{2.3}$$

$$= -h'(x)W(x)\,|_0^\infty + \int_0^\infty h''(x)W(x)dx. \tag{2.4}$$

Note that $h'(x)W(x) \to 0$ as $x \to \infty$, because $h'(\infty) = 0$ by the condition (i). Also, $h'(x)W(x) \to 0$ as $x \to 0$, by the condition (iii) and the L'Hospital's rule. It follows that the integral in (2.4) vanishes. The integrand, which is nonnegative because of $X <_{(1)} Y$ and $h'' > 0$ on $(0, \infty)$, then forces $W(x)$ to vanish a.e. Thus $F = G$.

COROLLARY 2.5. *Let X and Y be positive random variables such that $X <_{(1)} Y$. If* (a) *$EX < \infty$ and $E(X^r) = E(Y^r) < \infty$ for some $r < 0$,* (b) *$E(\coth X) = E(\coth Y) < \infty$, or* (c) *$EX < \infty$ and $E(\operatorname{csch} X) = E(\operatorname{csch} Y) < \infty$, then $X =^d Y$.*

Finally, we give another form of Theorem 2.3 for *nonnegative* random variables, which also extends Corollaries 2.3(b) and 2.3(c).

THEOREM 2.3$'$. *Let X and Y be nonnegative random variables such that $X <_{(1)} Y$ and $EX < \infty$. Suppose that there is a continuous function h defined on $[0, \infty)$ such that* (i) *$h \geq 0$, $h' \leq 0$, $h'' > 0$ on $(0, \infty)$,* (ii) *$E[h(X)] = E[h(Y)] < \infty$, and* (iii) *either $h'(0^+)$ is finite, or $h'(0^+) = -\infty$ with $(h'(x))^2/h''(x) \to 0$ as $x \to 0$, then $X =^d Y$.*

PROOF. Let F be the distribution of X. Then for nonnegative X,

$$\begin{aligned} E[h(X)] &= h(0)F(0) + \int_{(0,\infty)} h(x)dF(x) \\ &= h(0)F(0) + h(x)F(x) \mid_0^\infty - \int_0^\infty F(x)h'(x)dx \\ &= h(\infty) - \int_0^\infty F(x)h'(x)dx. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.3 and is omitted.

COROLLARY 2.6. *Let $X <_{(1)} Y$ and $EX < \infty$. If* (a) *$E[(1 + X^\alpha)^{-1}] = E[(1 + Y^\alpha)^{-1}]$ for some $\alpha \in (0, 1]$,* (b) *$E[\exp(-X^\alpha)] = E[\exp(-Y^\alpha)]$ for some $\alpha \in (0, 1]$, or* (c) *$E(\cot^{-1} X) = E(\cot^{-1} Y) < \infty$, then $X =^d Y$.*

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