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ON WAITING TIME PROBLEMS ASSOCIATED WITH RUNS IN MARKOV DEPENDENT TRIALS

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Abstract. A general technique is developed to study the waiting time distribution for the r-th occurrence of a success run of length k in a sequence of Markov dependent trials. Sooner and later waiting time problems are also discussed.

Key words and phrases: Probability mass function, probability generating function, waiting time, run, sooner and later problems, distributions of order k.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of Markov dependent trials, each trial being a success (1) or a failure (0), with transition probabilities defined by

$$(1.1) \qquad p_{ij} = P(X_{n+1} = j \mid X_n = i), \qquad n \geq 1, \ \ 0 \leq i, j \leq 1$$

and initial probabilities $P(X_1 = j) = p_j$, $j = 0, 1$. Presently, we use this framework for the study of run-related waiting time problems. There are various schemes for counting success runs of length k. Four of the most frequently used schemes are: (I) Non-overlapping scheme: Recounting starts immediately after a success run of length exactly k occurs (Feller (1968)); (II) At least k scheme: A success run of length at least k is counted only once (Goldstein (1990)); (III) Overlapping scheme: A success run of length $m (\geq k)$ accounts for $m - k + 1$ runs (Ling (1989)); (IV) Exactly k scheme: We count only success runs of length exactly k followed by a failure (Mood (1940)).

Let $W_{r,k}^{(a)}$ ($a =$ I, II, III, IV) be a random variable (r.v.) denoting the number of Markov dependent trials until the occurrence of the r-th success run of length k (r, k positive integers) where the superscript denotes the counting scheme

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employed. It is clear that for $r = 1$ the aforementioned r.v.'s are identically distributed and in this case the r.v. will be denoted by W_k. The notation $W_k(p_0, p_1)$ will be occasionally used when the initial probabilities of the Markov dependent trials are taken into consideration. Also, let W_S (W_L) be a r.v. denoting the waiting time for a run of k successes or (and) a run of s failures whichever comes sooner (later).

The above waiting time problems for $a =$ I, III, either for i.i.d. or Markov dependent trials, have been studied by Philippou *et al.* (1983), Philippou (1984), Ling (1989), Ebneshahrashoob and Sobel (1990), Aki (1992), Aki and Hirano (1993), Balasubramanian *et al.* (1993), Mohanty (1994), Aki *et al.* (1996), etc. Recently, Uchida and Aki (1995) derived the probability generating function (p.g.f.) of $W_{r,k}^{(a)}$ ($a =$ I, II and III) using the method of generalized probability generating functions (g.p.g.f.). They also studied generalized sooner and later waiting time problems. Koutras (1997) studied $W_{r,k}^{(a)}$ ($a =$ I, II and III) by means of the method of finite Markov chain imbedding developed by Fu and Koutras (1994) and subsequently refined by Koutras and Alexandrou (1995). Fu (1996) extended this method for multistate trials and obtained the waiting time distribution of a general compound pattern.

This article develops a general technique for the study of $W_{r,k}^{(a)}$, W_S and W_L. Methodologically our technique is equivalent with the method of g.p.g.f. In Section 2, by employing simple concepts of Markov chain theory some general results are established which provide tools for the study of waiting time distributions. In Section 3, by exploiting the Markov chain imbedding technique we derive the probability mass function (p.m.f.) and the p.g.f. of $W_{r,k}^{(a)}$, W_S and W_L. A disadvantage of Koutras' (1997) approach is that it does not seem to be suitable for the study of $W_{r,k}^{(\mathrm{IV})}$. A disadvantage of Fu's (1996) approach is the use of matrices whose dimension tends to infinity as the value of r increases. These disadvantages are surmounted with the present technique.

2. General results

Let $\{Y_n, n \geq 1\}$ be a time homogeneous Markov chain defined on a finite state space $\Omega = \{0, 1, \ldots, s\}$ with transition probability matrix $\boldsymbol{P} = [\boldsymbol{C}_0, \boldsymbol{C}_1, \ldots, \boldsymbol{C}_s]$, where $\boldsymbol{C}_j$ denotes the $(j+1)$-th column vector of $\boldsymbol{P}$ $(0 \leq j \leq s)$, and initial probability distribution $\boldsymbol{a}_1 = [P(Y_1 = 0), P(Y_1 = 1), \ldots, P(Y_1 = s)]$. For each $i, j \in \Omega$, let

$$F_{ij}(n) = \Pr\{Y_{n+1} = j, Y_\nu \neq j, \nu = 2, 3, \ldots, n \mid Y_1 = i\}, \qquad n \geq 1.$$

Clearly, $F_{ij}(1) = p_{ij}$, and we define $F_{ij}(0) = 0$. Consider the following probability (column) vectors

$$\boldsymbol{F}_j(n) = [F_{0j}(n), F_{1j}(n), \ldots, F_{sj}(n)]', \qquad n \geq 0, \ j \in \Omega.$$

Upon using the relation

$$F_{ij}(n) = \sum_{0 \leq m \neq j \leq s} p_{im} F_{mj}(n-1), \qquad n \geq 2,$$

we get

$$F_j(n) = M_j F_j(n-1), \qquad n \geq 2, \tag{2.1}$$

where $M_j = [C_0, \ldots, C_{j-1}, \mathbf{0}, C_{j+1}, \ldots, C_s]$, and $\mathbf{0}$ denotes the column vector of R^{s+1} with all its entries being 0. Making use of (2.1), we conclude to

$$F_j(n) = \begin{cases} \mathbf{0}, & n = 0, \\ C_j, & n = 1, \\ M_j^{n-1} C_j, & n \geq 2. \end{cases} \tag{2.2}$$

Next, consider the (column) vector generating functions

$$G_j(t) = [G_{0j}(t), G_{1j}(t), \ldots, G_{sj}(t)]' = \sum_{n=0}^{\infty} F_j(n) t^n, \qquad j \in \Omega.$$

Denoting by $\mathbf{I}$ the $(s+1) \times (s+1)$ identity matrix, and making use of (2.2), we get

$$G_j(t) = [\mathbf{I} + M_j t + M_j^2 t^2 + \cdots] C_j t,$$

and under proper conditions for the series to converge (e.g. restricting t in a proper neighborhood of zero) we are immediately led to

$$G_j(t) = [\mathbf{I} - M_j t]^{-1} C_j t. \tag{2.3}$$

For each $i, j \in \Omega$, let $F_{ij}^{(r)}(n)$, $n \geq 0$, be a sequence defined by

$$\{F_{ij}^{(r)}(n)\} = \begin{cases} \{F_{ij}(n)\}, & \text{if } r = 1, \\ \{F_{ij}(n)\} * \{F_{jj}^{[r-1]}(n)\}, & \text{if } r \geq 2, \end{cases} \tag{2.4}$$

where $\{a_n\} * \{b_n\}$ denotes the convolution of the sequences $\{a_n\}$ and $\{b_n\}$, and $\{b_n^{[r]}\}$ denotes the r-th convolution of $\{b_n\}$. It follows that $F_{ij}^{(r)}(n)$ is the probability that, starting from state i the r-th entry to state j occurs at the n-th transition. Let

$$F_j^{(r)}(n) = [F_{0j}^{(r)}(n), F_{1j}^{(r)}(n), \ldots, F_{sj}^{(r)}(n)]', \qquad n \geq 0, \ j \in \Omega,$$

and

$$G_j^{(r)}(t) = [G_{0j}^{(r)}(t), G_{1j}^{(r)}(t), \ldots, G_{sj}^{(r)}(t)]' = \sum_{n=0}^{\infty} F_j^{(r)}(n) t^n, \qquad j \in \Omega.$$

It can be easily checked that

$$G_j^{(r)}(t) = G_j(t) [G_{jj}(t)]^{r-1}. \tag{2.5}$$

Next, for each $\omega, i, j \in \Omega$, we introduce

$$F_{ij;\omega}(n) = \Pr\{Y_{n+1} = j, Y_\nu \neq \omega, Y_\nu \neq j, \nu = 2, 3, \ldots, n \mid Y_1 = i\}, \qquad n \geq 1.$$

Clearly, $F_{ij;\omega}(1) = p_{ij}$, and we define $F_{ij;\omega}(0) = 0$. Let

$$\boldsymbol{F}_{j;\omega}(n) = [F_{0j;\omega}(n), F_{1j;\omega}(n), \ldots, F_{sj;\omega}(n)]', \qquad n \geq 0, \ \omega, j \in \Omega.$$

Then

$$\boldsymbol{F}_{j;\omega}(n) = \boldsymbol{M}_{j,\omega}\boldsymbol{F}_{j;\omega}(n-1), \qquad n \geq 2,$$

where $\boldsymbol{M}_{j,\omega}$ is the matrix obtained from $\boldsymbol{P}$ by replacing the columns $\boldsymbol{C}_j$ and $\boldsymbol{C}_\omega$ by $\boldsymbol{0}$. It may be easily checked that

$$\boldsymbol{F}_{j;\omega}(n) = \begin{cases} \boldsymbol{0}, & n = 0, \\ \boldsymbol{C}_j, & n = 1, \\ \boldsymbol{M}_{j,\omega}^{n-1}\boldsymbol{C}_j, & n \geq 2, \end{cases} \tag{2.6}$$

and

$$\begin{aligned} \boldsymbol{G}_{j;\omega}(t) &= [G_{0j;\omega}(t), G_{1j;\omega}(t), \ldots, G_{sj;\omega}(t)]' \\ &= \sum_{n=0}^{\infty} \boldsymbol{F}_{j;\omega}(n)t^n \\ &= [\mathbf{I} - \boldsymbol{M}_{j,\omega}t]^{-1}\boldsymbol{C}_j t. \end{aligned} \tag{2.7}$$

3. Finite Markov chain imbedding and run waiting times

Let $\{X_n, n \geq 1\}$ be the sequence of Markov dependent trials defined by (1.1). In this section we employ the results of Section 2 in order to study $W_{r,k}^{(\mathrm{I})}$, $W_{r,k}^{(\mathrm{IV})}$, W_S and W_L ($W_{r,k}^{(\mathrm{II})}$ and $W_{r,k}^{(\mathrm{III})}$ may be studied analogously). This is accomplished by imbedding the r.v. into a proper Markov chain $\{Y_n, n \geq 1\}$ defined on a finite state space Ω.

3.1 *Non-overlapping scheme*

Let $k \geq 2$ and let $\{Y_n, n \geq 1\}$ be a Markov chain with state space $\Omega = \{0, 1, \ldots, k\}$ defined as follows: If $X_n = 0$ we shall say that the process is in state 0 at time n, i.e. $Y_n = 0$, and if at the n-th trial the number of last consecutive successes counting backwards is m, then $Y_n = k$ if $m = sk$ ($s \geq 1$), and $Y_n = j$ if $m = sk + j$ ($s \geq 0$, $1 \leq j \leq k-1$). Clearly, the resulting Markov chain is homogeneous, $\boldsymbol{a}_1 = [p_0, p_1, 0, \ldots, 0]_{1\times(k+1)}$ and the non vanishing entries of the transition probability matrix $\boldsymbol{P} = [a_{ij}]_{(k+1)\times(k+1)}$, $0 \leq i, j \leq k$, are $a_{00} = p_{00}$, $a_{01} = p_{01}$, $\alpha_{k0} = p_{10}$, $a_{k1} = p_{11}$,

$$a_{ij} = \begin{cases} p_{10} & \text{if } \ j = 0 \\ p_{11} & \text{if } \ j = i + 1 \end{cases} \qquad 1 \leq i \leq k-1. \tag{3.1}$$

It follows that every time a "1"-run of length k occurs in the sequence $\{X_n, n \geq 1\}$ the process enters in state k. Therefore,

$$P(W_{r,k}^{(\mathrm{I})} = n+1) = p_0 F_{0k}^{(r)}(n) + p_1 F_{1k}^{(r)}(n), \qquad n \geq 0. \tag{3.2}$$

Relations (2.2) and (2.4) provide a workable procedure for the evaluation of the p.m.f. of $W_{r,k}^{(\mathrm{I})}$. Denote by $H_{r,k}^{(\mathrm{I})}(t)$ the p.g.f. of $W_{r,k}^{(\mathrm{I})}$. From (2.5) and (3.2) it follows that

$$H_{r,k}^{(\mathrm{I})}(t) = [(p_0t)G_{0k}(t) + (p_1t)G_{1k}(t)][G_{kk}(t)]^{r-1}.$$

The evaluation of $H_{r,k}^{(\mathrm{I})}(t)$ may be easily performed through (2.3) after some routine calculations. On introducing the notation

$$\begin{aligned} A_0(t) &= (p_{01}t)(p_{11}t)^{k-1} \\ A_1(t) &= (1-p_{00}t)(p_{11}t)^{k-1} \\ B(t) &= 1 - p_{00}t - (p_{01}t)(p_{10}t)\sum_{i=2}^{k}(p_{11}t)^{i-2} \end{aligned} \tag{3.3}$$

we get

$$H_{r,k}^{(\mathrm{I})}(t) = \left[p_0t\frac{A_0(t)}{B(t)} + p_1t\frac{A_1(t)}{B(t)}\right]\left[p_{10}t\frac{A_0(t)}{B(t)} + p_{11}t\frac{A_1(t)}{B(t)}\right]^{r-1}. \tag{3.4}$$

We note that the p.g.f. of $W_k(p_0, p_1)$ is

$$H_k(t) = p_0t\frac{A_0(t)}{B(t)} + p_1t\frac{A_1(t)}{B(t)}. \tag{3.5}$$

The above p.g.f.'s are consistent with analogous results obtained by Aki and Hirano (1993), Mohanty (1994) and Uchida and Aki (1995). Some apparent differences are due to the different set-up used there. It is mentioned that Koutras (1997) showed that $W_{r,k}^{(\mathrm{I})} = \sum_{i=1}^{r} T_i$, where T_1 is a r.v. distributed as $W_k(p_0, p_1)$, T_i $(2 \le i \le r)$ is a r.v. distributed as $W_k(1, 0)$, and T_i $(1 \le i \le r)$ are independent r.v.'s. The correct is that T_i $(2 \le i \le r)$ is distibuted as $W_k(p_{10}, p_{11})$ as relations (3.4) and (3.5) imply.

Finally, we mention that the special case $k = 1$ may be studied by means of

$$P(W_{r,1}^{(\mathrm{I})} = n+1) = p_0F_{01}^{(r)}(n) + p_1F_{11}^{(r-1)}(n), \qquad n \ge 0,$$

where $F_{11}^{(0)}(n) = \delta_{0,n}$, $n \ge 0$ (δ_{ij} is the Kronecker's delta function).

3.2 *Exactly k scheme*

Let $r \ge 2$ and let $\{Y_n, n \ge 1\}$ be a Markov chain with state space $\Omega = \{0, 1, \dots, k+2\}$ defined as follows: If at the n-th trial the number of last consecutive successes counting backwards is m, then $Y_n = m$ if $1 \le m \le k$, and $Y_n = k+1$ if $m > k$. If the outcome of the n-th trial is a failure then $Y_n = 0$, unless the failure is preceded by exactly k successes and in this case we define $Y_n = k+2$. Clearly, the resulting Markov chain is homogeneous, $\boldsymbol{a}_1 = [p_0, p_1, 0, \dots, 0]_{1\times(k+3)}$ and the non vanishing entries of the transition probability matrix $\boldsymbol{P} = [a_{ij}]_{(k+3)\times(k+3)}$, $0 \le i, j \le k+2$, are given by (3.1) and $a_{00} = a_{k+2,0} = p_{00}$, $a_{01} = a_{k+2,1} = p_{01}$, $a_{k,k+1} = a_{k+1,k+1} = p_{11}$, $a_{k,k+2} = a_{k+1,0} = p_{10}$.

The above definition of the Markov chain implies that in order to register the r-th success run of length exactly k we have to wait for $(r-1)$ visits to state $(k+2)$ and then for the first visit to state k. Set

$$\{V_i^{(r)}(n)\} = \{F_{i,k+2}^{(r-1)}(n)\} * \{F_{k+2,k}(n)\}, \qquad i = 0,1, \ n \geq 0.$$

Denote by $H_{r,k}^{(\mathrm{IV})}(t)$ the p.g.f. of $W_{r,k}^{(\mathrm{IV})}$. Then, it may be seen that

$$P(W_{r,k}^{(\mathrm{IV})} = n+1) = p_0 V_0^{(r)}(n) + p_1 V_1^{(r)}(n), \qquad n \geq 0,$$

and

$$H_{r,k}^{(\mathrm{IV})}(t) = [(p_0 t) G_{0,k+2}(t) + (p_1 t) G_{1,k+2}(t)][G_{k+2,k+2}(t)]^{r-2} G_{k+2,k}(t).$$

On introducing the notation

$$C_0(t) = (1-p_{11}t)(p_{01}t)(p_{11}t)^{k-1}(p_{10}t)$$
$$C_1(t) = (1-p_{11}t)(1-p_{00}t)(p_{11}t)^{k-1}(p_{10}t)$$
$$D(t) = (1-p_{00}t)(1-p_{11}t) - (p_{01}t)(p_{10}t)[1-(p_{11}t)^{k-1} + (p_{11}t)^k]$$

and making use of (2.3) and (3.3), we get

$$H_{r,k}^{(\mathrm{IV})}(t) = \left[p_0 t \frac{C_0(t)}{D(t)} + p_1 t \frac{C_1(t)}{D(t)}\right] \left[p_{00} t \frac{C_0(t)}{D(t)} + p_{01} t \frac{C_1(t)}{D(t)}\right]^{r-2} \cdot \left[p_{00} t \frac{A_0(t)}{B(t)} + p_{01} t \frac{A_1(t)}{B(t)}\right].$$

The above p.g.f. does not seem to have been noticed before.

3.3 *Sooner and later waiting time problems*

Let $r, k \geq 2$ and let $\{Y_n, n \geq 1\}$ be a Markov chain with state space $\Omega = \{1, 2, \ldots, s+k\}$ defined as follows: If at the n-th trial the number of last consecutive failures (successes) counting backwards is m, then $Y_n = s$ ($Y_n = s+k$) if $m = ls$ ($m = lk$), $l \geq 1$, and $Y_n = j_0$ ($Y_n = s + j_1$) if $m = ls + j_0$ ($m = lk + j_1$) for $1 \leq j_0 \leq s-1$ ($1 \leq j_1 \leq k-1$), $l \geq 0$. Clearly, the resulting Markov chain is homogeneous, $\boldsymbol{a}_1 = [p_0, 0, \ldots, 0, p_1, 0, \ldots, 0]_{1\times(s+k)}$ where p_1 is in the $(s+1)$-th coordinate of $\boldsymbol{a}_1$, and the non vanishing entries of the transition probability matrix $\boldsymbol{P} = [a_{ij}]_{(s+k)\times(s+k)}$, $1 \leq i,j \leq s+k$, are $a_{s1} = a_{i,i+1} = p_{00}$ and $a_{s,s+1} = a_{i,s+1} = p_{01}$ for $1 \leq i \leq s-1$, and $a_{s+k,1} = a_{s+j,1} = p_{10}$ and $a_{s+k,s+1} = a_{s+j,s+j+1} = p_{11}$ for $1 \leq j \leq k-1$.

It follows that every time a failure (success) run of length s (k) occurs in the sequence $\{X_n, n \geq 1\}$ the process enters in state s ($s+k$). Let E_0 (E_1) be the event that a failure (success) run of length s (k) occurs. For $m = 0, 1$, let $P_S^{(m)}(n)$ ($P_L^{(m)}(n)$) be the probability that at the n-th trial the sooner (later) event between E_0 and E_1 occurs and the sooner (later) event is E_m. Then

$$P(W_x = n) = P_x^{(0)}(n) + P_x^{(1)}(n), \qquad n \geq 1, \ x = S, L.$$

Now, consider the vectors $\boldsymbol{F}_{j;\omega}(n)$ of Section 2 defined on $\Omega = \{1, 2, \dots, s+k\}$. For $n \geq 0$, we observe that

$$P_S^{(0)}(n+1) = p_0 F_{1,s;s+k}(n) + p_1 F_{s+1,s;s+k}(n),$$
$$P_S^{(1)}(n+1) = p_0 F_{1,s+k;s}(n) + p_1 F_{s+1,s+k;s}(n).$$

The evaluation of the p.m.f. of W_S may be performed through (2.6), while its p.g.f. may be derived by (2.7) and coincides with the respective one obtained by Balasubramanian *et al.* (1993) (see also Aki and Hirano (1993) and Uchida and Aki (1995)).

We proceed now the later waiting time problem. Denote by $\bar{W}_s$ the waiting time for the occurrence of a failure run of length s in $\{X_n, n \geq 1\}$. The p.m.f. and the p.g.f. of $\bar{W}_s$ can be obtained by using the results of Subsection 3.1. For $n \geq 1$, the p.m.f. and p.g.f. of W_L may be obtained by

$$\{P_L^{(0)}(n)\} = \{P_S^{(1)}(n)\} * \{P(\bar{W}_s(p_{10}, p_{11}) = n)\},$$
$$\{P_L^{(1)}(n)\} = \{P_S^{(0)}(n)\} * \{P(W_k(p_{00}, p_{01}) = n)\}.$$

We note in ending that the memory space requirements for the numerical evaluation of the p.m.f.'s presented in this paper depends mainly on the dimension of the vectors $\boldsymbol{F}_j(\cdot)$ and $\boldsymbol{F}_{j;\omega}(\cdot)$. Therefore, roughly speaking, our memory availability should be enough in order to register these entries. Finally we mention that our technique could be routinely extended in order to accommodate a higher order Markov model (cf. Aki *et al.* (1996)), sequences involving non identical trials (cf. Fu and Kutras (1994)), the waiting time distribution of a general pattern (cf. Fu (1996)), etc. Related work will be reported soon.

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