Ann. Inst. Statist. Math.
Vol. 51, No. 2, 301–322 (1999)

ON THE DISTRIBUTION OF THE EXTREMES OF UNEQUALLY CORRELATED NORMAL VARIABLES WITH APPLICATIONS TO ANTEDEPENDENT CLUSTER DATA*

KRISHNA K. SAHA AND B. C. SUTRADHAR

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada A1C 5S7

(Received December 16, 1996; revised August 21, 1997)

Abstract. Based on a small correlations approach, this paper deals with the probability distributions of the order statistics under a general Gaussian model with unequal (positive or negative) correlations. Illustrations are given for the cluster data those follow suitable antedependence models.

Key words and phrases: Unequal but small correlations, distribution of order statistics, tail probability.

1. Introduction

Let $y_i = (y_{i1}, \ldots, y_{ij}, \ldots, y_{in})^T$ be a vector of n observations for the i-th cluster ($i = 1, 2, \ldots, q$) on a response variable Y. Also, let $Y_{ij} = \mu_i + v_i + u_{ij}$, $i = 1, \ldots, q$, $j = 1, \ldots, n$, where μ_i is the fixed effect due to the i-th cluster, $v_i \overset{\text{iid}}{\sim} N(0, \sigma_v^2)$, $u_{ij} \overset{\text{iid}}{\sim} N(0, \sigma_u^2)$, and v_i's are independent of u_{ij}'s. It then follows that

$$Y_i \sim N(\mu_i 1_n, \sigma^2 V),$$

where $\sigma^2 = \sigma_v^2 + \sigma_u^2$ and $V = (1 - \rho)I_n + \rho U_n$ with $\rho = \sigma_v^2/\sigma^2$ as the common intercluster correlation, 1_n is the $n \times 1$ unit vector, I_n is the $n \times n$ identity matrix and U_n is the $n \times n$ unit matrix. In such cluster models, one usually makes inferences about μ_i which requires the knowledge of the intercluster correlation ρ. In practice, this correlation parameter takes small values, see for example, Cochran ((1977), pp. 233–248), Scott and Holt (1982), Wu *et al.* (1988), and Rao *et al.* (1993).

There are other practical situations, where the observations in a cluster may not have the same correlation, and where one may be interested in inferences about the distribution of the extremes (maxima or minima). For example, in an

* The research was partially supported by a grant from the Natural Science and Engineering Research Council of Canada.

agricultural study, a large number of agricultural zones in West Africa may be considered to determine the effects of the extreme temperatures exceeding a prefixed maximum of weekly temperatures on the production of potatoes. Here, for a given cluster or zone i, one would be interested to study the pattern of $Y_{i(n)} = \max(Y_{ij}, j = 1, \ldots, n)$, where Y_{ij} is the maximum temperature in the i-th ($i = 1, \ldots, q$) zone during the j-th ($j = 1, \ldots, n$) week. Note that as the observations $Y_{i1}, \ldots, Y_{ij}, \ldots, Y_{in}$ of the i-th zone are collected successively over weeks, it is reasonable to assume that these n observations are correlated and correlations at different lags will be generally unequal. Consequently, this problem reduces to the order statistics problem where observations will be unequally correlated.

Another interesting situation where extreme value analysis for unequally correlated data is useful, is the repeated cluster data analysis where observations in a cluster follow an antedependence (non-stationary) model. For example, consider the calf data analyzed by Kenward (1987) (see also Gabriel (1961, 1962), Albert (1992)). In this problem, the main objective is to compare two or more methods for controlling the intestinal parasites in cattle. During the grazing season, from spring to autumn, cattle can ingest roundworm larvae, which have developed from eggs previously deposited on the pasture in the faces of infected cattle. Once infected an animal is deprived of nutrients and its resistence to other disease is lowered, which in turn can greatly affect its growth. In order to control the disease, an infected calf is usually assigned to a particular treatment. For monitoring the effects of a treatment for the disease, the response of interest, weight, is recorded for an infected calf at n time points and it is examined whether the maximum of these weights ($y_{(n)}$) are less than a standard weight (h, say) of an uninfected calf of the same age (at the initial level of the experiments). That is, we require to compute the probability $\Pr(Y_{(n)} \leq h)$, for known h, which indicates the failure of the treatment. Alternatively, one may find the probability $\Pr(Y_{(1)} \geq h)$ to see whether the treatment is working effectively. Here the observations $y_1, \ldots, y_n$ will most likely be a realization of the sample $Y_1, \ldots, Y_n$ that follow the antedependence (nonstationary) model (cf. Section 6), as weights are likely to vary with repeated time, equally or unequally spaced.

Note that in both of the above examples, the correlations among observations in a cluster are generally unequal. In these or other similar situations, one may often wish to make certain inferences by exploiting the extremes, such as maxima or minima. But, there does not appear any such inferential study based on extremes of unequally correlated data. We further note that as the clusters in the above two examples are considered to be independent, the extension of the inferences based on a single cluster to the case of multi-clusters is very straightforward. Consequently, in this paper we deal with inferences for the extremes of a single cluster. Because of this, in what follows, without any loss of generality, we drop the subscript i and rewrite $y_i = (y_{i1}, \ldots, y_{ij}, \ldots, y_{in})^T$ as $y = (y_1, \ldots, y_j, \ldots, y_n)^T$. For the special case where $Y = (Y_1, \ldots, Y_j, \ldots, Y_n)^T \sim N(0, R)$ with $R = \{\rho_{jk}\}$ as the variance-covariance matrix of Y with $\rho_{jk} = 1$, for $j = k$ and $\rho_{jk} = \rho$, for $j \neq k$, the distribution function of the r-th order statistic has been studied by many authors over the last four decades. For example, we refer to Gupta *et al.* (1964), Gupta *et al.* (1973), Owen and Steck (1962), Hoffman and Saw (1975),

and Rawlings (1976), for such inferences. But, the order statistics inferences for the antedependent types of data, where correlations can be positive or negative but unequal, have not been adequately discussed in the literature.

The main objective of the present paper is to derive the distributions of the extremes under a general Gaussian model with unequal correlations. This is achieved by using a small correlations asymptotic approach. Applications of the distributional results are discussed in the context of antedependence model. The performance of the proposed approach is examined, first for the equi-correlation case, by comparing the tail probabilities of the extremes with those of Gupta *et al.* (1973). Next, for the unequally correlated case, we conduct a limited simulation study to verify the accuracy of the tail probabilities obtained by the present approach. The proposed approach is also compared with the existing Bonferroni bounds approximation used by Ellenberg (1973, 1976), among others, for the computations of the tail probabilities of the extremes.

2. An approximate joint probability model for order statistics: a small correlations approach

Suppose that $Y = (Y_1, \ldots, Y_j, \ldots, Y_n)^T \sim N(0, \Sigma)$, with $\Sigma = D^{1/2} R D^{1/2}$, where $D = \text{diag}(\sigma_1^2, \ldots, \sigma_j^2, \ldots, \sigma_n^2)$ and R is the correlation matrix of Y written as $R = (\rho_{jk})$ with $\rho_{jj} = 1$ and $\rho_{jk} \neq 0$ for $j \neq k$. Also suppose that ρ_{jk}'s are small in magnitude. This assumption about the small correlations is reasonable for many practical situations, for example, in cluster sampling problems as mentioned in the previous section. Then by expanding the joint probability density function (p.d.f.) of Y, namely, $f(y_1, \ldots, y_j, \ldots, y_n; \Sigma) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\{-(1/2) y^T \Sigma^{-1} y\}$ about $\Sigma = D$, one easily obtains an approximate joint p.d.f. given by

$$
\begin{aligned}
(2.1) \quad & f_a(y_1, \ldots, y_j, \ldots, y_n; \Sigma) \\
&= \Bigg[k_0 - (1/2) \sum_{j=1}^{n} a_{jj} y_j^2 + \sum_{j<k}^{n} b_{jk} y_j y_k + (1/2) \sum_{j<k}^{n} c_{jjkk} y_j^2 y_k^2 \\
&\qquad + \sum_{j \neq k \neq l}^{n} c_{jjkl} y_j^2 y_k y_l + \sum_{j \neq k \neq l \neq m}^{n} d_{jklm} y_j y_k y_l y_m \Bigg] \\
&\quad \times f(y_1, \ldots, y_j, \ldots, y_n; D)
\end{aligned}
$$

where $k_0 = 1 + (1/2) \sum_{j<k}^{n} \rho_{jk}^2$, $a_{jk} = \frac{1}{\sigma_j \sigma_k} \sum_{l \neq j, l \neq k}^{n} \rho_{jl} \rho_{kl}$, $b_{jk} = \frac{\rho_{jk}}{\sigma_j \sigma_k} - a_{jk}$, $c_{jjkl} = \frac{\rho_{jk} \rho_{jl}}{\sigma_j^2 \sigma_k \sigma_l}$, $d_{jklm} = \frac{\rho_{jk}\rho_{lm} + \rho_{jl}\rho_{km} + \rho_{jm}\rho_{kl}}{\sigma_j \sigma_k \sigma_l \sigma_m}$, and $f(y_1, \ldots, y_j, \ldots, y_n; D) = (2\pi)^{-n/2} |D|^{-1/2} \exp\{-(1/2) y^T D^{-1} y\} = \prod_{j=1}^{n} f(y_j; \sigma_j^2)$ with $f(y_j; \sigma_j^2) = (2\pi\sigma_j^2)^{-1/2} \exp\{-y_j^2 / 2\sigma_j^2\}$.

Let $Y_{(1)} \leq \cdots \leq Y_{(j)} \leq \cdots \leq Y_{(n)}$ be the correlated order statistics of the original variables $Y_1, \ldots, Y_j, \ldots, Y_n$. Then, given the realizations of the order statistics to be $y_{(1)} \leq \cdots \leq y_{(j)} \leq \cdots \leq y_{(n)}$, the original variables Y_j $(j = 1, \ldots, n)$ are constrained to take on the values $y_{(j_k)}$ which yields the same expression for the similar terms in equation (2.1) for all $n!$ permutations $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$. This

ordering mechanism, consequently, yields the joint probability density function $g^*(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; \Sigma)$ of the n order statistics $Y_{(1)}, \ldots, Y_{(j)}, \ldots, Y_{(n)}$ given by

$$
\begin{aligned}
(2.2) \qquad & g^*(y_{(1)}, y_{(2)}, \ldots, y_{(n)}; \Sigma) \\
& = \Bigg[k_0^* - a_g^* \sum_{j=1}^{n} y_{(j)}^2 + b_g^* \sum_{j<k}^{n} y_{(j)} y_{(k)} + c_g^* \sum_{j<k}^{n} y_{(j)}^2 y_{(k)}^2 \\
& \qquad + c_g^{**} \sum_{j \neq k \neq l}^{n} y_{(j)}^2 y_{(k)} y_{(l)} + d_g^* \sum_{j \neq k \neq l \neq m}^{n} y_{(j)} y_{(k)} y_{(l)} y_{(m)} \Bigg] \\
& \qquad \times \prod_{r=1}^{n} f(y_{(r)}; \sigma_{[r]}^2)
\end{aligned}
$$

with $k_0^* = n! k_0$, and for general covariance structure, the coefficients are given by $a_g^* = (1/2)(n-1)! \sum_{j=1}^{n} a_{jj}$, $b_g^* = 2(n-2)! \sum_{j<k}^{n} b_{jk}$, $c_g^* = (n-2)! \sum_{j<k}^{n} c_{jjkk}$, $c_g^{**} = 2(n-3)! \sum_{j \neq k \neq l}^{n} c_{jjkl}$, $d_g^* = 4!(n-4)! \sum_{j \neq k \neq l \neq m} d_{jklm}$, where k_0, a_{jj}, b_{jk}, c_{jjkk}, c_{jjkl}, and d_{jklm} are defined in (2.1) for appropriate j, k, l, and m. In (2.2), $f(y_{(r)}; \sigma_{[r]}^2)$ is the p.d.f. of a normal variable with mean zero and variance $\sigma_{[r]}^2$ where $\sigma_{[r]}^2 = \sigma_l^2$ for $y_{(r)} = y_l$ $(l = 1, \ldots, n)$.

The above joint p.d.f. (2.2) of the n order statistics is exploited in Section 3 to obtain the marginal distribution of a single order statistic. In Section 4, we provide the marginal distributions of the extremes. Applications of the distributional results of the extremes are shown in Sections 5 and 6, respectively, for the equally (positive or negative) and unequally (positive or negative) correlated normal variables.

3. Distribution of a single order statistic

We now turn to the distribution of a single order statistic, say $Y_{(r)}$ $(1 \leq r \leq n)$. To derive this distribution, one requires to integrate out $Y_{(1)}, \ldots, Y_{(r-1)}$, $Y_{(r+1)}, \ldots, Y_{(n)}$ from the approximate joint p.d.f. of all n order statistics given in (2.2). Note, however, that this integration is straightforward but lengthy. For simplicity, the following U and L functions will be used to define four sets of λ functions, which will be exploited later on to write the p.d.f. of $Y_{(r)}$.

For real t_j, t_k, t_l, and t_m, we define necessary U and L functions as follows:

$$
(3.1) \qquad \begin{cases} \lambda_{jU}^{(t_j)}(y_{(r)}) = \displaystyle\int_{-\infty}^{y_{(r)}} \cdots \int_{-\infty}^{y_{(2)}} y_{(j)}^{t_j} f(y_{[1,r-1]}) dy_{[1,r-1]}, \\ \qquad\qquad \text{for} \quad j < r \\ \lambda_{jklmU}^{(t_j t_k t_l t_m)}(y_{(r)}) = \displaystyle\int_{-\infty}^{y_{(r)}} \cdots \int_{-\infty}^{y_{(2)}} y_{(j)}^{t_j} y_{(k)}^{t_k} y_{(l)}^{t_l} y_{(m)}^{t_m} f(y_{[1,r-1]}) dy_{[1,r-1]}, \\ \qquad\qquad \text{for} \quad j,k,l,m < r \end{cases}
$$

and

$$(3.2)\quad \begin{cases} \lambda_{jL}^{(t_j)}(y_{(r)}) = \displaystyle\int_{y_{(r)}}^{\infty}\cdots\int_{y_{(n-1)}}^{\infty} y_{(j)}^{t_j} f(y_{[r+1,n]})dy_{[r+1,n]}, \\ \qquad\qquad \text{for} \quad j > r \\ \lambda_{jklmL}^{(t_j t_k t_l t_m)}(y_{(r)}) = \displaystyle\int_{y_{(r)}}^{\infty}\cdots\int_{y_{(n-1)}}^{\infty} y_{(j)}^{t_j} y_{(k)}^{t_k} y_{(l)}^{t_l} y_{(m)}^{t_m} f(y_{[r+1,n]})dy_{[r+1,n]}, \\ \qquad\qquad \text{for} \quad j,k,l,m > r \end{cases}$$

where $f(y_{[1,r-1]}) \equiv \{f(y_{(1)};\sigma^2_{[1]})\cdots f(y_{(r-1)};\sigma^2_{[r-1]})\}$, $f(y_{[r+1,n]}) \equiv \{f(y_{(n)};\sigma^2_{[n]})\cdots f(y_{(r+1)};\sigma^2_{[r+1]})\}$, $dy_{[1,r-1]} \equiv dy_{(1)}\cdots dy_{(r-1)}$, and $dy_{[r+1,n]} \equiv dy_{(n)}\cdots dy_{(r+1)}$.

Although the above U and L functions are defined for all real t_j, t_k, t_l, and t_m, in what follows, each of them will take values 0, 1 or 2, only.

Function 1. $\lambda_j^{(2)}(y_{(r)})$; $j,r = 1,2,\ldots,n$.

In terms of the U and L functions $\lambda_{jU}^{(t_j)}(y_{(r)})$ and $\lambda_{jL}^{(t_j)}(y_{(r)})$ defined in (3.1)–(3.2), this Function 1 is defined as

$$(3.3)\quad \lambda_j^{(2)}(y_{(r)}) = \begin{cases} f(y_{(r)};\sigma^2_{[r]})\lambda_{jU}^{(2)}(y_{(r)})\lambda_{jL}^{(0)}(y_{(r)}), & j < r \\ y_{(r)}^2 f(y_{(r)};\sigma^2_{[r]})\lambda_{jU}^{(0)}(y_{(r)})\lambda_{jL}^{(0)}(y_{(r)}), & j = r \\ f(y_{(r)};\sigma^2_{[r]})\lambda_{jU}^{(0)}(y_{(r)})\lambda_{jL}^{(2)}(y_{(r)}), & j > r. \end{cases}$$

Note that the superscript $t_j = 2$ in the left hand side of (3.3) is reflected, respectively, in the U and L functions in the right hand side of (3.3) based on $j < r$ and $j > r$. If $t_j = 2$ appears in the U function, t_j takes zero value in the L function and vice versa. For the case when $j = r$, the superscript $t_j = 2$ becomes the power of $y_{(j)} = y_{(r)}$. In this $j = r$ case, the superscript t_j of the U and L functions takes zero value.

Function 2. $\lambda_{jk}^{(11)}(y_{(r)})$; $j,k,r = 1,2,\ldots,n$.

Similar to Function 1, for $j,k < r$; $j < r, k = r$; $j < r, k > r$, we now define Function 2 in terms of $\lambda_{jkU}^{(t_j t_k)}(y_{(r)})$ and $\lambda_{jkL}^{(t_j t_k)}(y_{(r)})$ as

$$(3.4)\quad \lambda_{jk}^{(11)}(y_{(r)}) = \begin{cases} f(y_{(r)};\sigma^2_{[r]})\lambda_{jkU}^{(11)}(y_{(r)})\lambda_{jkL}^{(00)}(y_{(r)}), & j < k < r \\ y_{(r)}^1 f(y_{(r)};\sigma^2_{[r]})\lambda_{jkU}^{(10)}(y_{(r)})\lambda_{jkL}^{(00)}(y_{(r)}), & j < r,\ k = r \\ f(y_{(r)};\sigma^2_{[r]})\lambda_{jkU}^{(10)}(y_{(r)})\lambda_{jkL}^{(01)}(y_{(r)}), & j < r,\ k > r. \end{cases}$$

Here, the superscripts $t_j = 1$ and $t_k = 1$ in the left hand side of (3.4) is reflected in the right hand side functions depending on the position of j and k with respect to r. More specifically, for the case when $j,k < r$, the superscripts $t_j = 1$ and $t_k = 1$ in the left hand side of (3.4) are reflected in the U function in the right

hand side. Both of the superscripts of the L function in this case will take zero values. For $j < r$, $k = r$, the superscript $t_j = 1$ is reflected in the U function and $t_k = 1$ becomes the power of $y_{(k)} = y_{(r)}$. In this case, the superscripts t_k in the U function will take zero value but both superscripts in the L function will take zero values. For the third case when $j < r$, $k > r$, $t_j = 1$ is reflected as the superscript of the U function and $t_k = 1$ is reflected as the superscript of the L function. The remaining superscripts in the U and L functions will take zero values.

We further note that for other possible positions of j and k with respect to r, the pattern of assigning the superscripts t_j and t_k in the U and L functions required to define $\lambda_{jk}^{(11)}(y_{(r)})$ will remain the same. The all possible positions for j and k with respect to r excluding the positions shown in (3.4) are $[j = r, k < r; j = r, k > r]$, $[j > r, k < r; j > r, k = r; j > r, k > r]$.

A set of functions, namely $\lambda_{jk}^{(22)}(y_{(r)})$, similar to Function 2, will also be necessary to define the p.d.f. of $Y_{(r)}$. These functions are defined by replacing the superscripts $t_j = 1$ and $t_k = 1$ in the expressions for $\lambda_{jk}^{(11)}(y_{(r)})$ with $t_j = 2$ and $t_k = 2$.

Function 3. $\lambda_{jkl}^{(211)}(y_{(r)})$; $j, k, l, r = 1, 2, \ldots, n$.

Similar to the definitions of Functions 1 and 2, Function 3 is defined by using the U and L functions $\lambda_{jklU}^{(t_j t_k t_l)}(y_{(r)})$ and $\lambda_{jklL}^{(t_j t_k t_l)}(y_{(r)})$ from (3.1)–(3.2) as follows:

$$\lambda_{jkl}^{(211)}(y_{(r)}) = \begin{cases} f(y_{(r)}; \sigma_{[r]}^2)\lambda_{jklU}^{(211)}(y_{(r)})\lambda_{jklL}^{(000)}(y_{(r)}), \\ \qquad j, k, l < r \\ y_{(r)}^1 f(y_{(r)}; \sigma_{[r]}^2)\lambda_{jklU}^{(210)}(y_{(r)})\lambda_{jklL}^{(000)}(y_{(r)}), \\ \qquad j, k < r,\ l = r \\ f(y_{(r)}; \sigma_{[r]}^2)\lambda_{jklU}^{(200)}(y_{(r)})\lambda_{jklL}^{(011)}(y_{(r)}), \\ \qquad j < r, k \text{ and } l > r. \end{cases} \tag{3.5}$$

The technique of assigning the values of t_j, t_k, and t_l from the left hand side of (3.5) to the expressions in the right hand side, is quite similar to the techniques used for Function 1 and Function 2. In this case, all possible positions for j, k, and l with respect to r excluding the positions shown in (3.5) are $[j, k > r, l = r; j, k > r, l < r; j, k, l > r]$, $[j < r, k = r, l > r; j, l < r, k = r]$, $[j > r, k = r, l < r; j, l > r, k = r]$, $[j = r, k < r, l > r; j = r, k, l < r]$, $[j = r, k > r, l < r; j = r, k, l > r]$, $[j < r, k > r, l = r; j, l < r, k > r; j < r, k, l > r]$, and $[j > r, k < r, l = r; j > r, k, l < r; j, l > r, k < r]$. Note that for any of these positions, one can easily obtain the appropriate expressions for Function 3 in the similar way as we have obtained the expressions for this function for the particular position of j, k, and l with respect to r shown in (3.5).

Function 4. $\lambda_{jklm}^{(1111)}(y_{(r)})$; $j, k, l, m, r = 1, 2, \ldots, n$.

In terms of U and L functions $\lambda_{jklmU}^{(t_j t_k t_l t_m)}(y_{(r)})$ and $\lambda_{jklmL}^{(t_j t_k t_l t_m)}(y_{(r)})$ from (3.1)–(3.2), we can define Function 4, following the definition of Functions 1, 2 and 3,

as follows:

$$\lambda_{jklm}^{(1111)}(y_{(r)}) = \begin{cases} f(y_{(r)};\sigma_{[r]}^2)\lambda_{jklmU}^{(1111)}(y_{(r)})\lambda_{jklmL}^{(0000)}(y_{(r)}), \\ \qquad j,k,l,m < r \\ y_{(r)}^1 f(y_{(r)};\sigma_{[r]}^2)\lambda_{jklmU}^{(1110)}(y_{(r)})\lambda_{jklmL}^{(0000)}(y_{(r)}), \\ \qquad j,k,l < r,\ m = r \\ f(y_{(r)};\sigma_{[r]}^2)\lambda_{jklmU}^{(1100)}(y_{(r)})\lambda_{jklmL}^{(0011)}(y_{(r)}), \\ \qquad j,k < r, l \text{ and } m > r. \end{cases} \tag{3.6}$$

Here again, the technique of assigning the values of t_j, t_k, t_l, and t_m from the left side of (3.6) to the expressions in the right side, is quite similar to the previous techniqes used to define Functions 1, 2 and 3. One may also write very easily all possible positions for j, k, l, and m with respect to r. Note that in this case, there are 56 possible positions.

Turning back to the distribution of the single order statistic, $Y_{(r)}$ $(1 \le r \le n)$, a lengthy integration, as mentioned earlier, yields the marginal density function $g_r^*(y_{(r)})$ of the statistic $Y_{(r)}$. More specifically, by integrating the joint p.d.f. (2.2) over the range of $y_{(1)},\ldots,y_{(r-1)}$, $y_{(r+1)},\ldots,y_{(n)}$, one obtains

$$\begin{aligned} g_r^*(y_{(r)}) \simeq k_0^*\phi(y_{(r)}) - a_g^* \sum_{j=1}^{n} \lambda_j^{(2)}(y_{(r)}) \\ + b_g^* \sum_{j<k}^{n} \lambda_{jk}^{(11)}(y_{(r)}) + c_g^* \sum_{j<k}^{n} \lambda_{jk}^{(22)}(y_{(r)}) \\ + c_g^{**} \sum_{j\neq k\neq l}^{n} \lambda_{jkl}^{(211)}(y_{(r)}) + d_g^* \sum_{j\neq k\neq l\neq m}^{n} \lambda_{jklm}^{(1111)}(y_{(r)}), \\ -\infty \le y_{(r)} \le \infty \end{aligned} \tag{3.7}$$

where a_g^*, b_g^*, c_g^*, c_g^{**}, and d_g^* are defined as in equation (2.2). Further in (3.7), $\phi(y_{(r)}) = [(r-1)!(n-r)!]^{-1}[F(y_{(r)})]^{r-1}[1-F(y_{(r)})]^{n-r} f(y_{(r)};\sigma_{[r]}^2)$ with $F(y_{(r)}) = \int_{-\infty}^{y_{(r)}} f(y_j;\sigma_j^2)dy_j$, $f(y_j;\sigma_j^2)$ being the p.d.f. of normal Y_j with mean zero and variance σ_j^2, for all $j = 1,\ldots,n$.

4. Distribution of the maxima: a special case

The value of the largest order statistic may be of special interest in numerous practical situations. The p.d.f. of this order statistic directly follows from the p.d.f. of the r-th order statistic given in (3.7), by putting $r = n$. For this special case, the expressions for Functions 1 through 4 used in (3.7) may be simplified to a great extent. Since the computation of the p.d.f. of $Y_{(n)}$ requires the integrations over the remaining $n-1$ ordered variables, the L functions do not appear at all in any expression of those four functions given in (3.3) to (3.6). Consequently, Function 1 defined in (3.3) takes the form

$$\lambda_j^{(2)}(y_{(n)}) = \begin{cases} f(y_{(n)};\sigma_{[n]}^2)\lambda_{jU}^{(2)}(y_{(n)}), & j < n \\ y_{(n)}^2 f(y_{(n)};\sigma_{[n]}^2)\lambda_{jU}^{(0)}(y_{(n)}), & j = n. \end{cases} \tag{4.1}$$

Note that the subscript j now can take positions with respect to n in the form $j < n$ and $j = n$ only as shown in (4.1).

Similarly, only for three types of positions of j and k with respect to n, Function 2 in (3.4) takes the form

$$\lambda_{jk}^{(11)}(y_{(n)}) = \begin{cases} f(y_{(n)};\sigma_{[n]}^2)\lambda_{jkU}^{(11)}(y_{(n)}), & j,k < n \\ y_{(n)}^1 f(y_{(n)};\sigma_{[n]}^2)\lambda_{jkU}^{(10)}(y_{(n)}), & j < n,\ k = n \\ y_{(n)}^1 f(y_{(n)};\sigma_{[n]}^2)\lambda_{jkU}^{(10)}(y_{(n)}), & k < n,\ j = n. \end{cases} \tag{4.2}$$

In the manner similar to that of (4.1) and (4.2), one may very easily define Functions 3 and 4 following (3.5) and (3.6) respectively. Note that now in Function 3, j, k, and l with respect to n have the positions $[j,k,l < n;\ j,k < n,\ l = n;\ j,l < n,\ k = n;\ k,l < n,\ j = n]$, and j, k, l, and m with respect to n have the positions $[j,k,l,m < n;\ j,k,l < n,\ m = n;\ j,k,m < n,\ l = n;\ j,l,m < n,\ k = n;\ k,l,m < n,\ j = n]$, in Function 4.

4.1 *Computation of the integral for U functions*

To represent all possible U functions defined in (3.1), we write a general function for the case $r = n \geq 2$ as

$$I(y_{(n)}, n-1; \sigma_{[1]}^2, \ldots, \sigma_{[n-1]}^2) = \int_{-\infty}^{y_{(n)}} \cdots \int_{-\infty}^{y_{(2)}} \left\{ \prod_{a=1(1)\overline{n-1}} y_{(a)}^{t_a} \right\} f(y_{[1,n-1]}) dy_{[1,n-1]}, \tag{4.3}$$

where t_a for all $a = 1,\ldots,n-1$ can take values 0, 1 or 2, $f(y_{[1,n-1]}) \equiv \{f(y_{(1)};\sigma_{[1]}^2)\cdots f(y_{(n-1)};\sigma_{[n-1]}^2)\}$, and $dy_{[1,n-1]} \equiv dy_{(1)}\cdots dy_{(n-1)}$ as in (3.1). In the general function $I(y_{(n)}, n-1; \sigma_{[1]}^2, \ldots, \sigma_{[n-1]}^2)$ in (4.3), the index $n-1$ represents the number of integrations necessary to obtain this function. For example, $I(y_{(n)}, n-1; \sigma_{[1]}^2, \ldots, \sigma_{[n-1]}^2)$ in (4.3) is identical to $\lambda_{jU}^{(2)}(y_{(n)})$, a U function used to define Function 1 in Section 3, when $t_a = t_j = 2$ for $a = j$ and $t_a = 0$ for $a \neq j$. Similarly $I(y_{(n)}, n-1; \sigma_{[1]}^2, \ldots, \sigma_{[n-1]}^2)$ in (4.3) is identical to $\lambda_{jkU}^{(11)}(y_{(n)})$, a U function used to define Function 2 in Section 3, when $t_a = t_j = 1$ for $a = j$, $t_a = t_k = 1$ for $a = k$, and $t_a = 0$ for $a \neq j,k$.

Now turning back to the integration (4.3), it is easy to see that

$$I(y_{(m)}, m-1; \sigma_{[1]}^2, \ldots, \sigma_{[m-1]}^2) = \int_{-\infty}^{y_{(m)}} I(y_{(m-1)}, m-2; \sigma_{[1]}^2, \ldots, \sigma_{[m-2]}^2) \times f(y_{(m-1)};\sigma_{[m-1]}^2) dy_{(m-1)}, \tag{4.4}$$

for $m = 2,\ldots,n$. As the integration in (4.3) has to be done successively, the application of the relation (4.4) requires the solution of a basic integral of the form

$$\eta(x, t_j, \sigma_{[j]}^2) = \int_{-\infty}^{x} u^{t_j} e^{-u^2/2\sigma_{[j]}^2} du, \qquad x \in \Re^1, \tag{4.5}$$

at different stages. More specifically, this integration for all $j = 1, \ldots, n-1$ will be necessary to solve the integral $I(y_{(n)}, n-1; \sigma^2_{[1]}, \ldots, \sigma^2_{[n-1]})$. Note that we have written the integral form (4.5) in terms of t_j and $\sigma^2_{[j]}$ defined in (4.3) for the convenience of obtaining the final result for the integral in (4.3). In reality, in (4.5), t_j can take any real value and $\sigma^2_{[j]}$ needs to be a positive quantity. Now the solution of $\eta(x, t_j, \sigma^2_{[j]})$ may be achieved by using the following partial sum form of the incomplete gamma function, that is,

$$I^*(\mu, \alpha) = \int_0^\mu s^{\alpha-1} e^{-\alpha} ds = \Gamma(\alpha) e^{-\mu} \sum_{r=\alpha}^{\infty} \frac{\mu^r}{r!} = \Gamma(\alpha) e^{-\mu} \sum_{r=0}^{\infty} \frac{\mu^{r+\alpha}}{(r+\alpha)!}$$

(cf. Gupta (1960, 1962) and Prescott (1974)). For $x > 0$, some straightforward computations yield

$$\eta(x, t_j, \sigma^2_{[j]}) = Q_j(t_j, \sigma^2_{[j]}) + \sum_{r_k=0}^{\infty} G_{jr_k}(t_j, \sigma^2_{[j]}; r_k) x^{2r_k + t_j + 1} e^{-x^2/2\sigma^2_{[j]}} \tag{4.6}$$

where

$$Q_j(t_j, \sigma^2_{[j]}) = (-1)^{t_j} 2^{(t_j-1)/2} \sigma^{t_j+1}_{[j]} \Gamma\left(\frac{t_j+1}{2}\right)$$

$$G_{jr_k}(t_j, \sigma^2_{[j]}; r_k) = \Gamma\left(\frac{t_j+1}{2}\right) \Big/ \left\{ 2^{r_k+1} \sigma^{2r_k}_{[j]} \left(r_k + \frac{t_j+1}{2}\right)! \right\}.$$

These $Q_j(\cdot)$ and $G_{jr_k}(\cdot)$ functions, for convenience, will be denoted by Q_j and G_{jr_k} functions respectively. When $x < 0$, the expression (4.6) still remains valid except that the G_{jr_k} function is now multiplied by $(-1)^{t_j+1}$.

Next, for example, consider a product containing m G functions defined as

$$G_{1r_1} G_{2d_1 r_2} \cdots G_{md_{m-1} r_m} Q_{(m+1)d_m} \tag{4.7}$$

where the subscript d_j $(j = 1, \ldots, m)$ either for G or Q function denotes the sum of the r's from it's (G or Q) preceding j consecutive G functions. In the above product function (4.7), the first G function is the same as G_{1r_1} function defined in (4.6), and other $m-1$ 'G' functions are similar to but different than the G functions defined in (4.6). Further, in (4.7), $d_1 = r_1, d_2 = r_1 + r_2, \ldots, d_m = \sum_{j=1}^{m} r_j$. If a similar product of G and Q functions such as $G_{1r_1} Q_{2d_1} G_{3r_2} Q_{4d_1}$ is considered, then d_1 in Q_{2d_1} is simply r_1 and d_1 in Q_{4d_1} is r_2. The Q and G functions in (4.7) with d_j as one of the subscripts, unlike in (4.6), are defined as follows

$$Q_{(m+1)d_m}(d_m, t_{(m+1)m}, \sigma^2_{(m+1)m}) \tag{4.8}$$

$$= \frac{(-1)^{t_{(m+1)m}} 2^{d_m + (t_{(m+1)m}-1)/2} \Gamma\left(d_m + \frac{t_{(m+1)m}+1}{2}\right)}{\sigma^{d_m + (t_{(m+1)m}+1)/2}_{(m+1)m}}$$

$$G_{kd_j r_h}(d_j, t_{kj}, \sigma^2_{kj}; r_h) \tag{4.9}$$

$$= \Gamma\left(d_j + \frac{t_{kj}+1}{2}\right) \Big/ \left\{ 2^{r_h+1} \left(d_j + \frac{t_{kj}+1}{2} + r_h\right)! \right\}$$

with $t_{kj} = \sum_{l=k-j}^{k-1}(t_l + 1) + t_k$, and $\sigma_{kj}^2 = \sum_{l=k-j}^{k-1}(1/\sigma_{[l]}^2) + (1/\sigma_{[j]}^2)$, for $k = 2, \ldots, m$, $j = 1, \ldots, m-1$, and $h = 2, \ldots, m$.

Note that varieties of product functions such as (4.7) will appear in all different terms in the solution of the main integral $I(y_{(n)}, n-1; \sigma_{[1]}^2, \ldots, \sigma_{[n-1]}^2)$. In some of the terms of the integration result for $I(y_{(n)}, n-1; \sigma_{[1]}^2, \ldots, \sigma_{[n-1]}^2)$, these product functions will represent constants and in some other terms, they will be the coefficients of integrals of the form

$$\xi(z, d_j, t_{kj}, \sigma_{kj}^2) = \int_{-\infty}^{z} w^{2d_j + t_{kj}} e^{-(u^2/2)\sigma_{kj}^2} dw \tag{4.10}$$

for $j = 1, \ldots, n-2$, and $k = 2, \ldots, n-1$, which, as a multiple of suitable coefficients containing the product of appropriate number of G or Q functions, is yielded by the multiplication of integral results such as $\eta(x, t_j, \sigma_{[j]}^2)$ defined in (4.6), by appropriate functions of order statistics defined under various integrations in (4.3). Now, for $z > 0$, in the manner similar to that of (4.6), the integral in (4.10) reduces to

$$\xi(z, d_j, t_{kj}, \sigma_{kj}^2) = Q_{kd_j}(d_j, t_{kj}, \sigma_{kj}^2) + \sum_{r_h=0}^{\infty} G_{kd_j r_h}(d_j, t_{kj}, \sigma_{kj}^2; r_h) \times z^{2(d_j + r_h) + t_{kj} + 1} \exp\left\{-\frac{z^2}{2}\sigma_{kj}^2\right\}, \tag{4.11}$$

where Q_{kd_j}'s and $G_{kd_j r_h}$'s are defined similar to the Q and G functions in (4.8) and (4.9) respectively. For $z < 0$, the expression (4.11) still remains valid except that the $G_{kd_j r_h}$ function is now multiplied by $(-1)^{t_{kj}+1}$.

Further, for positive Y_j $(j = 1, \ldots, n)$ (implying $x > 0$ in (4.6), and $z > 0$ in (4.11)) and for $n_2 > n_1$, let ${}^{n_1}H^{**(n_2)}(Q, G)$ denote a single combination of the product of n_1 'G' functions and $n_2 - n_1$ 'Q' functions. As G and Q functions can be arranged in ${}^{n_1}C_{n_2} = q_{n_1, n_2}$ (say) possible ways to make such a product, in order to obtain the summation of all these product combinations, we, for convenience, label them as ${}^{n_1}H_1^{*(n_2)}(Q, G), \ldots, {}^{n_1}H_l^{*(n_2)}(Q, G), \ldots, {}^{n_1}H_{q_{n_1,n_2}}^{*(n_2)}(Q, G)$. For example, for $n_1 = 3$ and $n_2 = 2$, all possible combinations are ${}^3H_1^{*(2)}(Q, G) = Q_1 G_{2r_1} G_{3d_1 r_2}$, ${}^3H_2^{*(2)}(Q, G) = G_{1r_1} Q_{2d_1} G_{3r_2}$, and ${}^3H_3^{*(2)}(Q, G) = G_{1r_1} G_{2d_1 r_2} Q_{3d_2}$. Note that without any loss of generality, one may label the second product $G_{10r_1} Q_{2d_1} G_{3r_2}$ by ${}^3H_1^{*(2)}(Q, G)$ or ${}^3H_3^{*(2)}(Q, G)$. In this case, $d_1 = r_1$, $d_2 = r_1 + r_2$ and in ${}^3H_1^{*(2)}(Q, G)$, $G_{3d_1 r_2} = G(d_1, t_{31}, \sigma_{31}^2; r_2)$, and for $n_1 = 4$ and $n_2 = 2$, all possible combinations are ${}^4H_1^{*(2)}(Q, G) = Q_1 Q_2 G_{3r_1} G_{4d_1 r_2}$, ${}^4H_2^{*(2)}(Q, G) = Q_1 G_{2r_1} Q_{3d_1} G_{4r_2}$, ${}^4H_3^{*(2)}(Q, G) = G_{1r_1} Q_{2d_1} G_{3r_2} Q_{4d_1}$, ${}^4H_4^{*(2)}(Q, G) = Q_1 G_{2r_1} G_{3d_1 r_2} Q_{4d_2}$, ${}^4H_5^{*(2)}(Q, G) = G_{1r_1} G_{2d_1 r_2} Q_{3d_2} Q_4$, and ${}^4H_6^{*(2)}(Q, G) = G_{1r_1} Q_{2d_1} Q_3 G_{4r_2}$. Here d_1 may be r_1 or r_2, $d_2 = r_1 + r_2$ and in ${}^4H_1^{*(2)}(Q, G)$, $G_{4d_1 r_2} = G(d_1, t_{41}, \sigma_{41}^2; r_2)$. Note that any $G_{kd_j r_h}$ or Q_{kd_j} function will appear in the product combination only if it is preceded by a G function. More specifically, in any product combination, $G_{kd_j r_h}$ function will be preceded by $(k-1)$ 'Q' or 'G'

functions and j G functions. Similarly, in any product combination, Q_{kd_j} function will also be preceded by $(k-1)$ 'Q' or 'G' functions and j G functions. Further note that, for the case when smaller order statistics take negative values, that is, $Y_j < 0$ (implying $x < 0$ in (4.6), and $z < 0$ in (4.11)), the G functions in each term will be multiplied by $(-1)^{t_j+1}$ or $(-1)^{t_{kj}+1}$, where t_j or t_{kj} are the corresponding values in G functions.

Now, by using the above notations, after a lengthy calculation, the integral (4.3) reduces to

$$
\begin{aligned}
(4.12)\quad & I(y_{(n)}, n-1; \sigma^2_{[1]}, \ldots, \sigma^2_{[n-1]}) \\
&= C_{n-1}\Delta_1^{(n-1)} + \sum_{r_1=0}^{\infty}\left\{\Theta_1^{*(n-1)}(r_1) y_{(n)}^{2r_1+t_{n-1}} \exp\left\{-\frac{y^2_{(n)}}{2\sigma^2_{[n-1]}}\right\} + \sum_{l=2}^{n-1}\Theta_l^{*(n-1)}(r_1)\right\} \\
&\quad + \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} {}^{n-1}B^{*(2)}(r_1, r_2) + \cdots \\
&\quad + \sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty} {}^{n-1}B^{*(m)}(r_1, \ldots, r_m) + \cdots \\
&\quad + \sum_{r_1=0}^{\infty}\cdots\sum_{r_{n-2}=0}^{\infty} {}^{n-1}B^{*(n-2)}(r_1, \ldots, r_{n-2}) \\
&\quad + \sum_{r_1=0}^{\infty}\cdots\sum_{r_{n-1}=0}^{\infty} \Psi_1^{(n-1)}(r_1, \ldots, r_{n-1}) \\
&\quad \times y_{(n)}^{2(d_{n-2}+r_{n-1})+t_{(n-1)(n-2)}+1} \exp\left\{-\frac{y^2_{(n)}}{2}\sigma^2_{(n-1)(n-2)}\right\}
\end{aligned}
$$

where, in general,

$$
C_u = \frac{1}{(2\pi)^{u/2}|D|^{1/2}}
$$

$$
\Delta_1^{(u)} = C_u \prod_{j=1}^{u} Q_j
$$

and

$$
\Psi_1^{(u)}(r_1, \ldots, r_u) = C_u \prod_{j=1}^{u} G_{jd_{j-1}r_j} \quad \text{with} \quad d_0 = 0
$$

$$
\Theta_l^{*(n-1)}(r_1) = C_{n-1}\Delta_1^{((n-1)-l)} G_{[(n-1)-l+1]r_1}[Q_{[(n-1)-l+2]d_1}]^{I_{s_1}}[\Delta^{(n-1)}_{(n-1)-l+3}]^{I_{s_2}}
$$

with $I_{s_1} = 0$ if $(n-1)-l+2 > n-1$ and $I_{s_1} = 1$, otherwise. Similarly $I_{s_2} = 0$ if $(n-1)-l+3 > n-1$ and $I_{s_2} = 1$, otherwise. Furthermore, in (4.12), for

$m = 2, 3, \ldots, n-2$, we have

$$ {}^{n-1}B^{*(m)}(r_1, \ldots, r_m) = C_{n-1} \sum_{j=1}^{q_{n-1}} q_{n-1} {}^{n-1}H_j^{**(m)}(Q, G) $$

where

$$ q_{n-1,m} = {}^{n-1}C_m $$

and

$$ {}^{n-1}H_j^{**(m)}(Q, G) = {}^{n-1}H_j^{*(m)}(Q, G)[y_{(n)}^{2(d^*+r^*)+t^*+1} e^{-(y_{(n)}^2/2)\sigma^{2^*}}]^{I_G} $$

with $I_G = 1$ if the product function ${}^{n-1}H_j^{**(m)}$ is ended by a G function and $I_G = 0$, otherwise. For the case when $I_G = 1$ and the product function is ended by a G function of the form $G_{kd_jr_h}$, d^*, r^*, t^* and σ^{2^*} in the square bracket [ ] will take the values of the d_j, r_h, t_{kj} and σ_{kj}^2, respectively, those are used in the last G function involved in the product function ${}^{n-1}H_j^{*^m}(Q, G)$. However, if $I_G = 1$ but the product function is ended by a G function of the form G_{jr_h}, then $d^* = 0$, $r^* = r_h$, $t^* = t_j$ and $\sigma^{2^*} = \sigma_{[j]}^2$, where r_h, t_j and $\sigma_{[j]}^2$ are used in the last G_{jr_h} function in the product function ${}^{n-1}H_j^{*^m}(Q, G)$. Note that one may exploit the integration result in (4.12) to obtain any U function necessary in (3.7) for maxima by putting appropriate values of t_a. For example, for $\lambda_{jkU}^{(11)}(y_{(n)})$ in (4.2), we require to put $t_a = 1$ for $a = j, k$ and $t_a = 0$ for all $a \neq j, k$ in equation (4.12).

5. Percentile points of the maxima for homoscedastic equi-correlated (positive or negative) normal variables

For the equi-correlated normal variables $Y_1, Y_2, \ldots, Y_n$ with $E(Y_j) = 0$, $E(Y_j^2) = \sigma^2$, for all $j = 1, 2, \ldots, n$ and $E(Y_jY_k) = \rho$, for all $j \neq k$, the probability density function of the maxima directly follows from (3.7) by using $r = n$ and adjusting the constant coefficients in (3.7) for $\rho_{jk} = \rho$ for all $j \neq k$ and $\sigma_j^2 = \sigma^2$ for $j = 1, 2, \ldots, n$. This density function for this special case will be denoted by $g_n^*(y_{(n)}, \rho, \sigma^2)$ with adjusted coefficients for equal correlations as a_s^*, b_s^*, c_s^*, c_s^{**}, and d_s^*. The interpretation of the λ functions in the density function of $y_{(n)}$ are quite similar to those λ functions (Functions 1 to 4) given in Section 4.

Now, let h_α be the $(1-\alpha)$ percentile points for the maxima $Y_{(n)}$. Then the distribution function of the maxima, that is, $W_n^*(h_\alpha; \rho, \ldots, \rho, \ldots, \rho; \sigma^2, \ldots, \sigma^2) = \Pr(Y_{(n)} \leq h_\alpha)$ may be easily computed by performing the integral $\int_{-\infty}^{h_\alpha} g_n^*(y_{(n)}, \rho, \sigma^2) dy_{(n)}$. In carrying out this integral, for notational convenience, we now turn back to the U functions given in (3.1) and define necessary Λ functions as follows:

$$ (5.1) \qquad \begin{cases} \Lambda_j^{(t_j)}(v, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2) = \displaystyle\int_{-\infty}^{v} \lambda_{jU}^{(t_j)}(y_{(n)}) dy_{(n)} \\ \Lambda_{jklm}^{(t_jt_kt_lt_m)}(v, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2) = \displaystyle\int_{-\infty}^{v} \lambda_{jklmU}^{(t_jt_kt_lt_m)}(y_{(n)}) dy_{(n)}. \end{cases} $$

Table 1. The probabilities for the maxima of positive equi-correlated normal variables based on CTM and SCA for selected α, ρ and n with $\sigma^2 = 1$, corresponding to the nominal $100(1-\alpha)\%$ probabilities with $\alpha = 0.01$, 0.025 and 0.05.

n	ρ	$\alpha =$	0.010	0.025	0.050	n	ρ	$\alpha =$	0.010	0.025	0.050
2	0.100	h_α	2.5739	2.2368	1.9508	3	0.200	h_α	2.7078	2.3829	2.1080
		CTM	0.989799	0.974801	0.949806			CTM	0.989801	0.974807	0.949813
		SCA	0.989832	0.974685	0.950282			SCA	0.991875	0.975631	0.952218
	0.125	h_α	2.5736	2.2361	1.9497		0.250	h_α	2.7058	2.3795	2.1029
		CTM	0.989801	0.974801	0.949812			CTM	0.989802	0.974807	0.949811
		SCA	0.989842	0.974728	0.949266			SCA	0.994132	0.976198	0.954731
	0.200	h_α	2.5722	2.2336	1.9456	4	0.100	h_α	2.8041	2.4907	2.2276
		CTM	0.989802	0.974807	0.949809			CTM	0.989801	0.974804	0.949806
		SCA	0.989877	0.974877	0.949708			SCA	0.990062	0.975166	0.950091
	0.250	h_α	2.5709	2.2314	1.9423		0.125	h_α	2.8034	2.4894	2.2255
		CTM	0.989802	0.974807	0.949809			CTM	0.989799	0.974801	0.949804
		SCA	0.989906	0.974993	0.950045			SCA	0.990824	0.975913	0.950789
3	0.100	h_α	2.7105	2.3878	2.1158		0.200	h_α	2.7078	2.3829	2.1080
		CTM	0.989801	0.974802	0.949803			CTM	0.989802	0.974804	0.949814
		SCA	0.988399	0.974687	0.948522			SCA	0.994899	0.975946	0.954433
	0.125	h_α	2.7099	2.3829	2.1141		0.250	h_α	2.7083	2.4804	2.2116
		CTM	0.989799	0.974805	0.949804			CTM	0.989802	0.974807	0.949804
		SCA	0.988554	0.974806	0.949234			SCA	0.994904	0.975926	0.957025

Note that the integrals in the right hand side of (5.1) are similar to the integral $\int_{-\infty}^{v} I(y_{(n)}, n-1; \sigma^2_{[1]}, \ldots, \sigma^2_{[n-1]}) dy_{(n)}$, where $I(\cdot)$ is also a function of the auxiliary parameters $t_j, t_k, t_l, \ldots$, etc., although these parameters were not shown for simplicity in the integral (4.3).

By using above notations, it is easily seen that the distribution function $W_n^*(h_\alpha; \rho, \ldots, \rho, \ldots, \rho; \sigma^2, \ldots, \sigma^2)$ has the expression given by

$$
\begin{aligned}
(5.2)\quad & W_n^*(h_\alpha; \rho, \ldots, \rho, \ldots, \rho; \sigma^2, \ldots, \sigma^2) \\
& \simeq k_0^* \Phi(h_\alpha, \sigma^2) - a_s^* \sum_{j=1}^{n} \Lambda_j^{(2)}(h_\alpha, n; \sigma^2, \ldots, \sigma^2) \\
& \quad + b_s^* \sum_{j<k}^{n} \Lambda_{jk}^{(11)}(h_\alpha, n; \sigma^2, \ldots, \sigma^2) + c_s^* \sum_{j<k}^{n} \Lambda_{jk}^{(22)}(h_\alpha, n; \sigma^2, \ldots, \sigma^2) \\
& \quad + c_s^{**} \sum_{j \neq k \neq l}^{n} \Lambda_{jkl}^{(211)}(h_\alpha, n; \sigma^2, \ldots, \sigma^2) \\
& \quad + d_s^* \sum_{j \neq k \neq l \neq m}^{n} \Lambda_{jklm}^{(1111)}(h_\alpha, n; \sigma^2, \ldots, \sigma^2),
\end{aligned}
$$

where $\Phi(h_\alpha, \sigma^2) = \Phi(h_\alpha)\,|_{\sigma_1^2 = \cdots = \sigma_n^2 = \sigma^2}$ with $\Phi(h_\alpha) = \frac{1}{n!}[F(h_\alpha)]^n$, $F(h_\alpha)$ be-

ing the distribution function of the normal variable. For example, $\Lambda_{jk}^{(11)}(h_\alpha, n; \sigma^2, \ldots, \sigma^2) = I(h_\alpha, n; \sigma^2, \ldots, \sigma^2)$ when $t_j = 1$, $t_k = 1$, $t_a = 0$ for $a \neq j, k$.

In order to examine the performance of the proposed small correlations approach, we now compute the probability $W_n^*(h_\alpha; \rho, \ldots, \rho, \ldots, \rho; 1, \ldots, 1)$ in (5.2) for selected values of $\alpha = 0.010$, 0.025 and 0.050, and small values of correlation, namely, $\rho = 0.100$, 0.125, 0.200 and 0.250 and compare the results with those given in Gupta *et al.* (1973). More specifically, we use the h_α values from Gupta *et al.* (1973) for $n = 2$, 3, and 4 and compute $W_n^*(h_\alpha; \rho, \ldots, \rho, \ldots, \rho; 1, \ldots, 1)$ by using the formula given in (5.2). These probabilities, computed based on small correlations approach (SCA), along with the corresponding probabilities obtained by correlation transformation method (CTM) of Gupta *et al.* (1973), are shown in Table 1.

It is clear from Table 1 that the SCA based probabilities are close to the CTM based probabilities given in Gupta *et al.* (1973), for all $\rho \leq 0.250$. Note, however, that for $\rho > 0.250$, our SCA based results do not agree with the CTM based results, as our approach is developed based on small values of correlations.

Remark that for negative equi-correlations case, the CTM of Gupta *et al.* (1973) is not suitable to compute the percentile points of the maxima. But, as our SCA is very general, one may apply this approach to compute the h_α values for small negative ρ as well. In Table 2, we show the h_α values for $n = 2$, 3, 4, $\alpha = 0.01$, 0.025, 0.05, and the selected values of $\rho = -0.100$, -0.125, -0.200 and -0.250. Note that the h_α values for these negative equi-correlations $\rho = -0.100$, -0.125, -0.200 and -0.250 are generally different than the h_α values for $\rho = 0.100$, 0.125, 0.200 and 0.250 respectively.

Table 2. The SCA based $100(1 - \alpha)\%$ percentile points h_α of the maxima for negative equi-correlations and selected α and n with $\sigma^2 = 1$.

n	ρ	$\alpha =$	0.010	0.025	0.050
2	-0.100		2.5123	2.2519	1.9878
	-0.125		2.5067	2.2408	1.9707
	-0.200		2.4956	2.2397	1.9667
	-0.250		2.4841	2.2251	1.9513
3	-0.100		2.5405	2.3065	2.0907
	-0.125		2.4799	2.2799	2.0509
	-0.200		2.3778	2.1829	2.0218
	-0.250		2.2858	2.1295	1.9837
4	-0.100		2.7279	2.4567	2.2156
	-0.125		2.6801	2.3912	2.1967
	-0.200		2.5478	2.3369	2.1613
	-0.250		2.4765	2.2801	2.1409

6. Application to antedependence models

The s-th order antedependence model of size n has the structure

$$
\begin{aligned}
Y_1 &= \delta_1 \eta_1 \\
Y_j &= \sum_{j'=1}^{s_j} \gamma_{j,j-j'} y_{j-j'} + \delta_j \eta_j, \qquad j = 2, \ldots, n
\end{aligned} \tag{6.1}
$$

where $s_j = \min(s, j-1)$, $\gamma_{j,j-j'}$, $j' = 1, 2, \ldots, s_j$, $j > j'$, are antedependent parameters, δ_j, $j = 1, 2, \ldots, n$ are n-scale parameters, and the errors η_j are independent and normally distributed with zero mean and unit variance. For $s = 1$, the covariances among the n observations are given by

$$\sigma_{jj}(1) = \gamma^2_{j,j-1}\sigma_{(j-1)(j-1)}(1) + \delta_j^2, \quad j = 1, 2, \ldots, n \quad \text{with} \quad \gamma_{1,0} = 0$$

and

$$\sigma_{jk}(1) = \gamma_{k,k-j}\sigma_{j(k-j)}(1), \qquad j < k,$$

where $\sigma_{jj}(1)$ is the variance of Y_j, and $\sigma_{jk}(1)$ is the covariance of Y_j and Y_k under the first order antedependent model. Observe that here all the variances and covariances are functions of different scale and antedependent parameters, those may vary due to repetition. Thus, the correlations among observations in a cluster under such an antedependence model are generally unequal.

For a higher order antedependence model, the structure of the covariances for n observations in general is complicated, but the covariances can be computed directly by using the interdependent relationship of the observations. For example, when $s = 2$ and $n = 3$, it directly follows from the model that

$$
\begin{aligned}
\mathrm{Var}(Y_1) &= \sigma_{11(2)} = \delta_1^2, \\
E(Y_1 Y_2) &= \sigma_{12(2)} = \gamma_{2,1}\delta_1^2, \\
E(Y_1 Y_3) &= \sigma_{13(2)} = \gamma_{3,2}\gamma_{2,1}\delta_1^2 + \gamma_{3,1}\delta_1^2, \\
E(Y_2^2) &= \sigma_{22(2)} = \gamma_{2,1}^2\delta_1^2 + \delta_2^2,
\end{aligned}
$$

and

$$E(Y_2 Y_3) = \sigma_{23(2)} = \gamma_{3,2}(\gamma_{2,1}^2\delta_1^2 + \delta_2^2) + \gamma_{3,1}\gamma_{2,1}\delta_1^2.$$

It is clear that the variances and covariances for the second ($s = 2$) order antedependence model also depend on different scale and antedependent parameters, showing the nonstationarity among the components of Y. In the similar fashion, one may show the nonstationarity under the antedependence model for any suitable s and n.

Since the p.d.f. of $Y_{(n)}$ in (3.7) is given for general correlation structures, by performing the integration in the similar manner to that of (5.2), we obtain the $W_n^*(h_\alpha; \rho_{12}, \ldots, \rho_{kj}, \ldots, \rho_{n-1,n}; \sigma_1^2, \ldots, \sigma_n^2) = \Pr(Y_{(n)} \le h)$ from the equation (3.7) as

$$W_n^*(h_\alpha; \rho_{12}, \ldots, \rho_{kj}, \ldots, \rho_{n-1,n}; \sigma_1^2, \ldots, \sigma_n^2) \tag{6.2}$$

$$\simeq k_0^*\Phi(h_\alpha) - a_{gan}^* \sum_{j=1}^{n} \Lambda_j^{(2)}(h_\alpha, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2)$$
$$+ b_{gan}^* \sum_{j<k}^{n} \Lambda_{jk}^{(11)}(h_\alpha, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2)$$
$$+ c_{gan}^* \sum_{j<k}^{n} \Lambda_{jk}^{(22)}(h_\alpha, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2)$$
$$+ c_{gan}^{**} \sum_{j\neq k\neq l}^{n} \Lambda_{jkl}^{(211)}(h_\alpha, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2)$$
$$+ d_{gan}^* \sum_{j\neq k\neq l\neq m}^{n} \Lambda_{jklm}^{(1111)}(h_\alpha, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2),$$

where the coefficients a_{gan}^*, b_{gan}^*, c_{gan}^*, c_{gan}^{**} and d_{gan}^* are obtained from a_g^*, b_g^*, c_g^*, c_g^{**} and d_g^* in (2.2) by considering σ_j^2 as $\sigma_{jj(s)}$ and calculating ρ_{jk} by using $\rho_{jk} = \sigma_{jk(s)}/\{\sigma_{jj(s)}\sigma_{kk(s)}\}^{1/2}$, $j, k = 1, 2, \ldots, n$ and $j \neq k$. In (6.2), $\Phi(h_\alpha)$ is defined as in equation (5.2), and all Λ functions in (6.2) are obtained from equations (5.1). Here again, these Λ functions are comparable with $I(\cdot)$ functions. For example, $\Lambda_{jk}^{(11)}(h_\alpha, n, \sigma_{[j]}^2, \ldots, \sigma_{[n]}^2) = I(h_\alpha, n; \sigma_{[1]}^2, \ldots, \sigma_{[n]}^2)$ when $t_j = 1$, $t_k = 1$, $t_a = 0$ for $a \neq j, k$, irrespective of $j \leq k$ or $j > k$.

6.1 *Illustration of numerical computations of percentile points*

Before computing the percentile points of the maxima, we first compute the correlation coefficients among the repeated observations those are generated following an antedependence model of order $s = 1$ and 2 having the variance-covariance matrix structure discussed in the previous section. In this numerical computation, we consider $n = 3$ and 4, and scale parameters of the antedependence model $\delta_j = 1$ for all j. We also consider the values of the antedependent parameters $\gamma_{j,j-j'}$ as $\gamma_{j,j-j'} \leq 0.250$ for $j = 2, \ldots, n$, $j' = 1, \ldots, s_j$, $j > j'$ yielding all possible ρ_{jk} as $\rho_{jk} \leq 0.250$ for $j \neq k$, which are small in magnitude. Now the computation of the 95% percentile points of $Y_{(n)}$ for the selected n, ρ_{jk}, and σ_j^2 is done by using the distribution function $W_n^*(h_\alpha; \rho_{12}, \ldots, \rho_{kj}, \ldots, \rho_{n-1,n}; \sigma_1^2, \ldots, \sigma_n^2)$ given in equation (6.2). The percentile points are shown in Table 3.

Note that the h_α values computed for $\rho_{jk} \leq \rho_0$ are generally different from h_α values obtained in Gupta *et al.* (1973) for $\rho = \rho_0$. More specifically, our calculations show that for $n = 3$, the h_α's for the cases with unequal correlations are greater than the h_α's for the cases with equal correlations. On the other hand, for $n = 4$, some of the h_α values for the cases with unequal correlations are greater, and some of them are less than the h_α values for the cases with equal correlations. In the next subsection, we examine the performance of the proposed small correlations based approximation by conducting a limited simulation study.

Table 3. The SCA based 95% percentile points of the maxima for antedependence data with order $s = 1, 2$ and selected ρ_{jk} $(j, k = 1, \ldots, n)$, σ_j^2 $(j = 1, \ldots, n)$ and n.

n	s	σ_1^2	σ_2^2	σ_3^2	σ_4^2	ρ_{12}	ρ_{13}	ρ_{23}	h_α
						ρ_{14}	ρ_{24}	ρ_{34}	
3	1	1	1.0046	1.0100		0.0677	0.0068	0.0997	2.1598
		1	1.0225	1.0164		0.1483	0.0189	0.1271	2.1423
		1	1.0312	1.0412		0.1738	0.0346	0.1990	2.1357
		1	1.0625	1.0562		0.2425	0.0559	0.2308	2.1283
3	2	1	1.0100	1.0123		0.0995	0.0752	0.0878	2.1512
		1	1.0163	1.0303		0.1268	0.1129	0.1425	2.1398
		1	1.0400	1.0469		0.1961	0.1403	0.1931	2.1257
		1	1.0455	1.0789		0.2087	0.1543	0.2495	2.1167
4	1	1	1.0077	1.0044	1.0104	0.0872	0.0057	0.0658	2.2409
						0.0006	0.0066	0.0997	
		1	1.0138	1.0184	1.0229	0.1167	0.0157	0.1343	2.2345
						0.0023	0.0201	0.1497	
		1	1.0400	1.0320	1.0260	0.1961	0.0347	0.1771	2.2256
						0.0056	0.0284	0.1601	
		1	1.0465	1.0660	1.0409	0.2108	0.0525	0.2489	2.2043
						0.0104	0.0494	0.1984	
4	2	1	1.0100	1.0118	1.0184	0.0995	0.0745	0.0854	2.2335
						0.0156	0.0915	0.1059	
		1	1.0129	1.0290	1.0306	0.1126	0.0997	0.1455	2.2226
						0.0238	0.1157	0.1432	
		1	1.0224	1.0325	1.0529	0.1481	0.1153	0.1505	2.2177
						0.0371	0.1395	0.1944	
		1	1.0246	1.0520	1.0794	0.1549	0.1326	0.1969	2.2112
						0.0473	0.1561	0.2482	

6.1.1 *Verification of percentile points: a simulation study*

To examine the accuracy of the percentile values shown in Table 3, we have conducted a small simulation study. In the simulation study, we generated 5000 n-dimentional ($n = 3, 4$) observations from the normal distribution with zero mean and the variance-covariance matrix corresponding to the antedependent correlations given in Table 3. We now refer to this zero mean case as the common mean (CM) case, and compute the proportion of simulation runs with maxima of the run greater than or equal to h_α. This proportion is denoted by $p_0 = p(y_{(n)} \geq h_\alpha \mid \text{CM})$, and is shown in Table 4. Since h_α was chosen from Table 3 such that $\Pr\{Y_{(n)} \leq h_\alpha\} = 0.95$, these proportion p_0 should be compared with $\alpha = 0.5$. It is clear from Table 4 that p_0's are very close to 0.05, showing that the SCA based computations for h_α works quite well. We also verify the performance of our small correlations approach to compute the proportion of simulation runs, with maximum of the runs greater than or equal to h_α. For the purpose, we first generate 5000 n-dimentional ($n = 3, 4$) observations from the normal dis-

Table 4. The proportion p_0 under the case CM and the proportions p_{11} and p_{12} under the two different cases EM_1 and EM_2, respectively, for selected h_α and n, based on 5000 simulations.

n	s	h_α	p_0	p_{11}	p_{12}
3	1	2.1598	0.0466	0.9966	0.9664
		2.1423	0.0488	0.9970	0.9670
		2.1357	0.0504	0.9963	0.9668
		2.1283	0.0530	0.9970	0.9672
3	2	2.1512	0.0482	0.9980	0.9690
		2.1398	0.0500	0.9980	0.9698
		2.1257	0.0506	0.9982	0.9706
		2.1167	0.0540	0.9982	0.9712
4	1	2.2409	0.0536	0.9956	0.9668
		2.2345	0.0546	0.9954	0.9664
		2.2246	0.0564	0.9956	0.9668
		2.2043	0.0608	0.9958	0.9680
4	2	2.2335	0.0512	0.9954	0.9662
		2.2226	0.0532	0.9954	0.9666
		2.2177	0.0559	0.9954	0.9666
		2.2112	0.0580	0.9954	0.9672

tribution when one observation in a run is generated with extreme mean equal to 4 (EM_1) or 5 (EM_2), the means of other observations in the run being the same as zero. Now, similar to the CM case, we compute the proportion of simulation runs which satisfies $y_{(n)} \geq h_\alpha$ under EM_1 or EM_2. These proportions under the two different situations EM_1 or EM_2 are denoted by $p_{11} = p(y_{(n)} \geq h_\alpha \mid EM_1)$ or $p_{12} = p(y_{(n)} \geq h_\alpha \mid EM_2)$ respectively, and they are shown in columns five and six of Table 4. Note that the computed proportions for both cases EM_1 or EM_2 appear to be quite high, indicating that the SCA based computations of percentile values is also quite powerful.

6.2 *A comparison with Bonferroni bound approximation*

In this subsection, we compare the performance of Bonferroni bound approximation with our small correlations approach, to compute the percentile points h_α for the maxima. We do this for general unequal correlations cases with both equal and unequal variances. In the latter case, we compute the variance and covariances following certain antedependence models, those are different from the models discussed in Subsection 6.1.1. We first compute the h_α values such that $\Pr\{Y_{(n)} \geq h_\alpha\} = \alpha$ by using our small correlations approach, as in the previous sections. These h_α values are then used to examine the performance of the well known Bonferroni bounds approximation. The upper and lower bounds, as functions of h_α values, are defined as follows:

$$LB(h, \sigma_j^2) = \sum_{j=1}^{n} \Pr(Y_j \geq h_\alpha) - \sum_{j<k}^{n} \Pr(Y_j \geq h_\alpha, Y_k \geq h_\alpha) \tag{6.3}$$

and

$$UB(h_\alpha, \sigma_j^2) = \sum_{j=1}^{n} \Pr(Y_j \geq h_\alpha) \tag{6.4}$$

where $E(Y_j^2) = \sigma_j^2$, and Y_j and Y_k are correlated such that $E(Y_j Y_k) = \rho_{jk}$ with $E(Y_j) = 0$. Now by using Z_j for $\frac{Y_j}{\sigma_j}$, we obtain the lower bound as

$$\begin{aligned} LB(h_\alpha, \sigma_j^2) &= \sum_{j=1}^{n} \Pr\left(Z_j \geq \frac{h_\alpha}{\sigma_j}\right) - \sum_{j<k}^{n} \Pr\left(Z_j \geq \frac{h_\alpha}{\sigma_j}, Z_k \geq \frac{h_\alpha}{\sigma_k}\right) \\ &= \sum_{j=1}^{n} \left[1 - F\left(\frac{h_\alpha}{\sigma_j}\right)\right] - \sum_{j<k}^{n} \Pr\left(Z_j \geq \frac{h_\alpha}{\sigma_j}, Z_k \geq \frac{h_\alpha}{\sigma_k}\right) \end{aligned} \tag{6.5}$$

and the upper bound is given by

$$UB(h_\alpha, \sigma_j^2) = \sum_{j=1}^{n} \left[1 - F\left(\frac{h_\alpha}{\sigma_j}\right)\right] \tag{6.6}$$

with

$$\Pr\left(Z_j \geq \frac{h_\alpha}{\sigma_j}, Z_k \geq \frac{h_\alpha}{\sigma_k}\right) = \int_{h_\alpha/\sigma_k}^{\infty} \int_{h_\alpha/\sigma_j}^{\infty} f(z_j, z_k; \rho_{jk}) dz_j dz_k \tag{6.7}$$

which is cumbersome to compute directly. Note, however, that our numerical computations as discussed below, show that it is enough to compare our SCA based percentile values with Bonferroni upper bound only.

Similar to the case for unequal variances, we also obtain the lower and upper bounds, for the cases with equal variance, that is for $E(Y_j^2) = \sigma^2$ $(j = 1, 2, \ldots, n)$, as

$$\begin{aligned} LB(h, \sigma^2) &= n\left[1 - F\left(\frac{h}{\sigma}\right)\right] - \sum_{j<k}^{n} \Pr\left(Z_j \geq \frac{h}{\sigma}, Z_k \geq \frac{h}{\sigma}\right) \\ &= n\left[1 - F\left(\frac{h}{\sigma}\right)\right] - \sum_{j<k}^{n} \Phi_2\left(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{jk}\right) \end{aligned} \tag{6.8}$$

and

$$UB(h, \sigma^2) = n\left[1 - F\left(\frac{h}{\sigma}\right)\right]. \tag{6.9}$$

For $\Phi_2(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{jk})$, we obtain an expression provided by Greig (1967) as

$$\Phi_2\left(\frac{h}{\sigma}, \frac{h}{\sigma}, \rho_{ij}\right) = [1 - (1 - \rho_{jk})^{1/2}]\Phi_1\left(\frac{h}{\sigma}\right) + (1 - \rho_{jk})^{1/2}\Phi_1^2\left(\frac{h}{\sigma}\right)$$

with

$$\Phi_1\left(\frac{h}{\sigma}\right) = \left[1 - F\left(\frac{h}{\sigma}\right)\right].$$

Table 5. The SCA based percentile values h_α for selected tail probabilities (TP) and the corresponding Bonferroni upper and lower bounds for selected ρ_{jk} $(j,k = 1,\ldots,n)$ and n, for homoscedastic normal variable case.

n	σ^2	ρ_{12}	ρ_{13}	ρ_{23}	h_α	TP	LB(h_α)	UB(h_α)
3	1.10	0.1000	0.0100	0.1000	2.2667	0.050291	0.055770	0.059006
		0.1500	0.0225	0.1500	2.2438	0.049813	0.057373	0.062053
		0.2000	0.0400	0.2000	2.1989	0.049611	0.061697	0.068413
		0.2500	0.0625	0.2500	2.1532	0.048262	0.066191	0.075441
	2.15	0.1000	0.0100	0.1000	4.4297	0.050230	0.055812	0.059051
		0.1500	0.0225	0.1500	4.3918	0.048976	0.056985	0.061624
		0.2000	0.0400	0.2000	4.3115	0.048059	0.060795	0.067389
		0.2500	0.0625	0.2500	4.2101	0.048583	0.066080	0.075312
	3.20	0.1000	0.0100	0.1000	6.6001	0.050666	0.055519	0.058736
		0.1500	0.0225	0.1500	6.5012	0.049813	0.058490	0.063287
		0.2000	0.0400	0.2000	6.3912	0.049611	0.061948	0.068697
		0.2500	0.0625	0.2500	6.2591	0.048906	0.066415	0.075703
	4.25	0.1000	0.0100	0.1000	8.7789	0.050065	0.055112	0.058296
		0.1500	0.0225	0.1500	8.5913	0.050759	0.059898	0.064845
		0.2000	0.0400	0.2000	8.4478	0.050188	0.063327	0.070264
		0.2500	0.0625	0.2500	8.2796	0.050286	0.067605	0.077097

Next, to compute the tail probability of percentile points h_α such that $\Pr\{Y_{(n)} \geq h_\alpha\} = 0.05$ for general unequal correlations with unequal variance cases, we have used the distribution functions $W_n^*(h_\alpha; \rho_{12},\ldots,\rho_{kj},\ldots,\rho_{n-1,n}; \sigma_1^2,\ldots,\sigma_n^2)$ given in equations (6.2). We have also computed similar tail probabilities for the equal variance case by using the distribution function $W_n^*(h_\alpha; \rho_{12},\ldots,\rho_{kj},\ldots,\rho_{n-1,n}; \sigma_1^2,\ldots,\sigma_n^2)$ which is a special case of the distribution function given in (6.2). In this special case, the Λ's used in the expression for $W_n^*(h_\alpha; \rho_{12},\ldots,\rho_{kj},\ldots,\rho_{n-1,n}; \sigma_1^2,\ldots,\sigma_n^2)$ in (6.2) reduces to the Λ's those used previously in (5.2). For the same percentile points h_α, we also compute the upper and lower bounds by using the Bonferroni bounds approximation in equations (6.8)–(6.9) for homoscedastic normal variable case. In the heteroscedastic normal variable cases, the computation of the lower bound is complicated. Consequently, we have used the upper bound given by (6.6) to compare the bound approximation with our SCA based results. It is interesting to note that our numerical computations show that in some situations the upper bounds are seen to be lower than 0.05, indicating that the lower bounds calculations are not necessary in such cases. In other situations, however, the lower bounds would have been much better representative than the upper bounds, but they were not calculated because of the technical difficulty as mentioned above. The results are reported in Tables 5 and 6, respectively, for the cases with equal and unequal variances. Note that in both of the homoscedastic and heteroscedastic normal variable cases, the bounds for different percentile points h_α based on the Bonferroni approach, are seen to deviate

Table 6. The SCA based percentile points h_α for selected tail probabilities (TP) and the corresponding Bonferroni upper bounds and tail probabilities (TP) for selected σ_j^2 $(j = 1, \ldots, n)$, ρ_{jk} $(j, k = 1, \ldots, n)$ and n, for heteroscedastic case.

s	n	σ_1^2	σ_2^2	σ_3^2	ρ_{12}	ρ_{13}	ρ_{23}	h_α	TP	UB(h_α)
	2	0.9801	1.5273		0.0971			2.5978	0.0508	0.0488
		2.5921	4.6520		0.1577			7.9756	0.0509	0.0449
		2.2500	0.5849		0.1959			3.9162	0.0500	0.0412
		2.8561	2.2423		0.2498			5.4434	0.0502	0.0363
1	3	1.5625	2.2697	4.0401	0.0932	0.0093	0.0996	6.3514	0.0499	0.0612
		0.5625	1.0207	3.0952	0.1426	0.0146	0.1027	3.6814	0.0507	0.1183
		3.0625	1.6488	0.5862	0.2288	0.0459	0.2009	5.0589	0.0492	0.0509
		2.2500	1.0835	3.9359	0.2419	0.0152	0.0629	3.5908	0.0492	0.2369
2	3	0.2971	0.8111	2.9509	0.0814	0.0396	0.0609	4.0968	0.0492	0.0833
		1.9074	1.0586	0.6044	0.1478	0.0901	0.1413	3.2108	0.0499	0.0481
		0.9902	1.2506	0.3245	0.0357	0.2083	0.1856	2.8416	0.0493	0.0143
		3.4044	1.3159	0.9151	0.2253	0.2448	0.1473	6.2689	0.0507	0.0334

in time to a great extent from the nominal probability 0.05. Furthermore, it was found that this deviation increases as variances and correlations increase in general. But the corresponding tail probabilities for the same percentile points based on our small correlations approach were found to be very close to the nominal probability 0.05.

Acknowledgements

The authors would like to thank a referee for constructive comments.

References

Albert, J. M. (1992). A corrected likelihood ratio statistic for the multivariate regression model with antedependent errors, *Comm. Statist. Theory Mathods*, **21**(7), 1823–1843.

Cochran, W. G. (1977). *Sampling Techniques*, Wiley, New York.

Ellenberg, J. H. (1973). The joint distribution of the standardized least squares residuals from a general linear regression, *J. Amer. Statist. Assoc.*, **68**, 941–943.

Ellenberg, J. H. (1976). Testing for a single outlier from a general linear regression, *Biometrics*, **32**, 637–645.

Gabriel, K. R. (1961). The model of antedependence for data of biological growth, *Bull. Inst. Internat. Statist.*, 33rd Session, 253–264, Paris.

Gabriel, K. R. (1962). Antedependence analysis of an ordered set of variables, *Ann. Math. Statist.*, **33**, 201–212.

Greig, M. (1967). Extremes in a random assembly, *Biometrika*, **54**, 273–282.

Gupta, S. S. (1960). Order statistics from the gamma distribution, *Technometrics*, **2**, 243–262.

Gupta, S. S. (1962). Gamma distribution, *Contributions to Order Statistics* (eds. A. E. Sarhan and B. G. Greenberg), 431–450, Wiley, New York.

Gupta, S. S., Pillai, K. C. S. and Steck, G. P. (1964). On the distribution of linear functions and ratios of linear functions of ordered correlated normal random variables with emphasis on range, *Biometrika*, **51**, 143–151.

Gupta, S. S., Nagel, N. and Panchapakesan, S. (1973). On the order statistics from equally correlated normal random variables, *Biometrika*, **60**(2), 403–413.

Hoffman, T. R. and Saw, J. G. (1975). Distribution of the largest of a set of equi-correlated normal variables, *Comm. Statist.*, **4**(1), 49–55.

Kenward, M. G. (1987). A method for comparing profiles of repeated measurements, *Appl. Statist.*, **36**(3), 296–308.

Owen, D. B. and Steck, G. P. (1962). Moments of order statistics from the equicorrelated multivariate normal distribution, *Ann. Math. Statist.*, **33**, 1286–1291.

Prescott, P. (1974). Variances and covariances of order statistics from the gamma distribution, *Biometrika*, **61**(3), 607–613.

Rao, J. N. K., Sutradhar, B. C. and Yue, K. (1993). Generalized least squares F test in regression analysis with two-stage cluster samples, *J. Amer. Statist. Assoc.*, **88**, 1388–1391.

Rawlings, J. O. (1976). Order statistics for a special class of unequally correlated multi-normal variates, *Biometrics*, **32**, 875–887.

Scott, A. C. and Holt, D. (1982). The effect of two-stage sampling on ordinary least squares methods, *J. Amer. Statist. Assoc.*, **77**, 848–854.

Wu, C. F. J., Holt, D. and Holmes, D. J. (1988). The effect of two-stage sampling on the F statistics, *J. Amer. Statist. Assoc.*, **83**, 150–159.