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PARAMETRIZATION INVARIANCE WITH RESPECT TO SECOND ORDER ADMISSIBILITY UNDER MEAN SQUARED ERROR

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Abstract. Our main concern is about second order admissibility under mean squared error. A sufficient condition and a necessary condition for a modified maximum likelihood estimator to be second order admissible regardless of parametrization are obtained. In addition, some procedures for characterizing such estimators are provided.

Key words and phrases: Second order admissibility, parametrization invariance, mean squared error.

1. Introduction

Second order asymptotic efficiency of the maximum likelihood estimator (MLE) has been discussed by many authors under several criteria. In particular, the theory of second order optimality on the basis of a mean squared error has been studied by Rao (1963), Ghosh and Subramanyam (1974) and Efron (1975). Pfanzagl and Wefelmeyer (1978) and Ghosh *et al.* (1980) showed the optimality of the MLE $\hat{\theta}_{ML}$ for regular loss functions including quadratic loss under general conditions. Their works can be summarized as follows: any first order efficient estimator is inferior to a certain modified MLE $\hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML})$ with the same asymptotic bias structure under regular situations. This assertion suggests that any estimator in practical use can be chosen from the class of estimators of the form $\hat{\theta}_{ML} + \frac{1}{n}c(\theta_{ML})$, but gives no idea of how to choose the second term $c(\cdot)$.

We are concerned with finding out, if possible, an *optimal* estimator from a class of estimators of the form $\hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML})$. One serious problem is, however, that no optimal estimator exists in the sense that it minimizes the second order term of the mean squared error uniformly in a parameter θ. As one approach to solving this difficulty, Ghosh and Sinha (1981) and Levit (1980, 1985) discussed the optimality of estimators under weaker properties such as second order admissibility or minimaxity.

In this paper, we will consider a desirable modification of the MLE based on the concept of second order admissibility. Ghosh and Sinha (1981) have given a necessary and sufficient condition on $c(\cdot)$ for an estimator $\tilde{\theta}_{ML} = \hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML})$ to be second order admissible (see Lemma 3.1). As a result, an optimal estimator in the sense of second order admissibility is characterized under a specified parametrization θ.

Our main concern is to characterize the modification of the MLE with second order admissibility regardless of parametrization. It should be noted that the choice of parametrization has a great influence on the optimality of the estimators in our situation. It is well known, indeed, that in a statistical decision problem under mean squared error, the second order admissibility is not parametrization-invariant.

Thus, our problem is to obtain a set of bias-adjustment factors c such that the estimator $\tilde{\theta}_{ML}$ is parametrization-invariant with respect to second order admissibility. Here it is worth noting that the arbitrariness of the function $c(\cdot)$ will complicate the problem greatly. So we will restrict the discussion to a subclass $\mathcal{T}$ of estimators with the form given in (3.3). We can realize the justification of such a restriction on the grounds that the estimators in the class $\mathcal{T}$ have an asymptotically similar structure up to $o(n^{-1})$ with respect to mean squared error as the MLE itself or the bias-corrected MLE.

Under the general conditions according to Gusev (1975, 1976), we have derived a sufficient condition for the estimator $\tilde{\theta}_{ML}$ in $\mathcal{T}$ to be parametrization-invariant with respect to second order admissibility (see Theorem 3.4). This result suggests an interesting solution to the second order estimation problem under mean squared error. Namely, an estimator of the form (3.9) is, in our situation, not inferior to any estimator regardless of parametrization. This shows that the estimator is one of the optimal estimators under mean squared error and is of great use. On the other hand, a necessary condition is described in Theorem 3.5.

In Section 4, we formulate a procedure for specifying a set $\mathcal{A}$ of the pairs (α, β) such that the estimator $\tilde{\theta}_{ML}$ in $\mathcal{T}$ is always second order admissible for any choice of parametrization θ. As a result, the only work we should do is to check the necessary and sufficient condition given by Ghosh and Sinha (1981) for the set satisfying the condition in Theorem 3.5. Furthermore, as a special case, we establish a simpler procedure (see Theorem 4.2). As seen in Section 5, the procedure is applicable to many statistical models.

2. Preliminaries

Suppose that a parametric model is indexed by a parameter θ and the parameter space Θ is the whole real line. Let $X_1, X_2, \ldots, X_n$ be a sequence of independently and identically distributed real-valued random variables with density $f(x, \theta)$. Throughout this paper, we assume the usual regularity conditions which guarantee the validity of formal asymptotic expansions of the MLE and the Taylor expansion of the mean squared error of the MLE up to $o(n^{-2})$. In particular, we make use of the regularity conditions of Gusev (1975, 1976).

Next, we shall introduce the following notation. The derivative with respect to θ is denoted by a prime, and the Fisher information by I_θ. Hereafter, we shall suppress the parameter θ in writing when no possibility of confusion exists. We define the symbol μ_{ijk} as

$$\mu_{ijk} = E_\theta \left[\left(\frac{f'}{f}\right)^i \left(\frac{f''}{f}\right)^j \left(\frac{f'''}{f}\right)^k \right].$$

Then, in our notation, we have the following lemma about relationships between μ's (see Lemma 4 in Gusev (1976)).

LEMMA 2.1.
(1) $I' = 2\mu_{110} - \mu_{300}$,
(2) $\mu'_{110} = \mu_{020} + \mu_{101} - \mu_{210}$,
(3) $\mu'_{300} = 3\mu_{210} - 2\mu_{400}$.

Now, recall that the expansion of the mean squared error of the MLE is guaranteed up to $o(n^{-2})$. Then, by expressing the results of Gusev (1976) by means of our notation, it follows that

$$E[\sqrt{n}(\hat{\theta}_{ML} - \theta)] = -\frac{\mu_{110}}{2nI^2} + o(n^{-1}),$$

and

$$E[\sqrt{n}\sqrt{I}(\hat{\theta}_{ML} - \theta)]^2 = 1 + \frac{1}{n}\left\{ \frac{15\mu_{110}^2 - 4\mu_{300}^2}{4I^3} - \frac{\mu_{101} + \mu_{210} - \mu_{400} + I^2}{I^2} \right\} + o\left(\frac{1}{n}\right).$$

Thus, we obtain the following expansion for the modified MLE $\tilde{\theta}_{ML} = \hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML})$;

$$\begin{aligned}
(2.1)\qquad & E\left[\sqrt{n}\sqrt{I}\left(\hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML}) - \theta\right)\right]^2 \\
&= E[\sqrt{n}\sqrt{I}(\hat{\theta}_{ML} - \theta)]^2 + \frac{2}{\sqrt{n}}E[\sqrt{n}I(\hat{\theta}_{ML} - \theta)c(\hat{\theta}_{ML})] \\
&\quad + \frac{I}{n}E[c(\hat{\theta}_{ML})]^2 \\
&= 1 + \frac{1}{n}\left\{ \frac{15\mu_{110}^2 - 4\mu_{300}^2}{4I^3} - \frac{\mu_{101} + \mu_{210} - \mu_{400} + I^2}{I^2} \right\} \\
&\quad + \frac{1}{n}\left\{ -\frac{\mu_{110}}{I}c + 2\frac{d}{d\theta}c + Ic^2 \right\} + o\left(\frac{1}{n}\right).
\end{aligned}$$

For any reparametrization $\eta = g(\theta)$, let us denote the quantity corresponding to μ_{ijk} by ν_{ijk}. Then some ν's are related to the μ's as

$$(2.2)\qquad I_\eta = \frac{I_\theta}{g'^2}, \quad \nu_{110} = \frac{1}{g'^3}\left(\mu_{110} - \frac{g''}{g'}I_\theta\right), \quad \nu_{300} = \frac{\mu_{300}}{g'^3}.$$

Now, we note that, in view of (2.2), the quantity $\mu_{300}/I_\theta\sqrt{I_\theta}$ is invariant under any parametrization.

3. Some conditions for parametrization invariance with respect to second order admissibility

Our purpose is to characterize the modification of the MLE with second order admissibility regardless of parametrization. For the purpose, the result of Ghosh and Sinha (1981) will play an essential role. Their result gives a necessary and sufficient condition for the estimator $\tilde{\theta}_{ML}$ to be second order admissible under a specified parametrization θ.

LEMMA 3.1. *The estimator $\tilde{\theta}_{ML} = \hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML})$ is second order admissible for a parametrization θ iff, for some $-\infty < \theta_0 < \infty$,*

$$\int_{\theta_0}^{\infty} I_\theta \exp\left\{ -\int_{\theta_0}^{\theta} b(u) I_\theta(u) du \right\} d\theta = \infty \tag{3.1}$$

and

$$\int_{-\infty}^{\theta_0} I_\theta \exp\left\{ \int_{\theta}^{\theta_0} b(u) I_\theta(u) du \right\} d\theta = \infty, \tag{3.2}$$

where $b = c - \mu_{110}/2I_\theta^2$.

Our aim is to obtain a set of the functions $c(\cdot)$ satisfying the criteria (3.1) and (3.2) under any parametrization. However, the problem is very complicated and troublesome on account of the arbitrariness of the function $c(\cdot)$. So, we will consider a subclass $\mathcal{C}_\theta$ of the bias-adjustment factors c:

$$\mathcal{C}_\theta = \left\{ \left(\alpha + \frac{5}{2}\right)\frac{\mu_{110}}{I_\theta^2} + (\beta - 1)\frac{\mu_{300}}{I_\theta^2} \;\middle|\; \alpha, \beta \in R \right\}.$$

Now, we define a subclass $\mathcal{T}$ of the modified MLE as

$$\mathcal{T} = \left\{ \tilde{\theta}_{ML} = \hat{\theta}_{ML} + \frac{1}{n}c(\hat{\theta}_{ML}) \;\middle|\; c(\cdot) \in \mathcal{C}_\theta \right\}. \tag{3.3}$$

A justification of such a restriction to the subclass $\mathcal{T}$ will be realized from Lemma 2.1 and the expansion (2.1). In fact, it is easy to see that the estimators in the class $\mathcal{T}$ have an asymptotically similar structure up to $o(n^{-1})$ with respect to mean squared error as do the MLE and the bias-corrected MLE up to $o(n^{-1})$. In addition, it should be noted that the class $\mathcal{T}$ includes the asymptotically expectation-unbiased estimator with $(\alpha, \beta) = (-3/2, 1)$ and the asymptotically median-unbiased estimator with $(\alpha, \beta) = (-5/2, 7/6)$.

Now, our problem is reduced to how to constitute a set $\mathcal{A}$ of pairs (α, β) such that $\tilde{\theta}_{ML}$ is always second order admissible for any choice of parametrization θ. We shall begin with the following lemma:

LEMMA 3.2. *When $\alpha = -1$, second order admissibility or inadmissibility of the modified MLE $\tilde{\theta}_{ML}$ is preserved under any monotone differentiable reparametrization $\eta = g(\theta)$.*

PROOF. By virtue of Lemma 2.1(1), the left-hand side of the criterion (3.1) for the parametrization θ is reduced to

$$\int_{\theta_0}^{\infty} \exp\left\{-\int_{\theta_0}^{\theta}\left(\alpha\frac{\mu_{110}}{I_\theta}+\beta\frac{\mu_{300}}{I_\theta}\right)du\right\}d\theta. \tag{3.4}$$

On the other hand, criterion (3.1) for the parametrization η is

$$\int_{\eta_0}^{\bar{\eta}} \exp\left\{-\int_{\eta_0}^{\eta}\left(\alpha\frac{\nu_{110}}{I_\eta}+\beta\frac{\nu_{300}}{I_\eta}\right)du\right\}d\eta, \tag{3.5}$$

where the parameter space for η is an interval with end points, $\underline{\eta}$ and $\bar{\eta}$, $(-\infty \le \underline{\eta} \le \bar{\eta} \le \infty)$. Thus, making use of (2.2), criterion (3.5) is, by the transformation, equal to

$$g'(\theta_0)\int_{\theta_0}^{\infty} \exp\left\{-\int_{\theta_0}^{\theta}\left(\alpha\frac{\mu_{110}}{I_\theta}+\beta\frac{\mu_{300}}{I_\theta}-(\alpha+1)\frac{g''}{g'}\right)du\right\}d\theta. \tag{3.6}$$

Now, for $\alpha = -1$, (3.6) is equivalent to (3.4) except for a multiplicative term $g'(\theta_0)$. This shows the equivalence of criterion (3.1) under any parametrization as $\alpha = -1$. We also have a similar consequence for criterion (3.2). These lead to the assertion of this lemma. □

We note that this lemma suggests a key to the solution of our problem. In fact, it will be natural to guess from the result of the lemma that the set in C with $\alpha = -1$ will play a central role hereafter. Furthermore, it will be doubtful whether parametrization invariance is met for $\alpha \ne -1$ because of the form of criterion (3.6) in the proof of Lemma 3.2. Indeed, this fact will be shown later.

The following lemma is easily derived from the regularity condition:

$$\inf_{\tau\in R^1}\int |f(x,\tau)-f(x,\tau-1)|dx > 0. \tag{3.7}$$

LEMMA 3.3. *It holds that*

$$\int_{\theta_0}^{\infty}\sqrt{I_\theta}d\theta = \int_{-\infty}^{\theta_0}\sqrt{I_\theta}d\theta = \infty. \tag{3.8}$$

PROOF. By Cauchy-Schwarz inequality, we have

$$\sqrt{I_\theta} \ge \int |f'(x,\theta)|dx.$$

Accordingly, it follows that

$$\begin{aligned}\int_{\theta_0}^{\theta_0+1} \sqrt{I_\theta} d\theta &\geq \int_{\theta_0}^{\theta_0+1} \int |f'(x,\theta)| dx d\theta \\ &\geq \int \left| \int_{\theta_0}^{\theta_0+1} f'(x,\theta) d\theta \right| dx \\ &= \int |f(x,\theta_0+1) - f(x,\theta_0)| dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{\theta_0}^{\infty} \sqrt{I_\theta} d\theta &\geq \int_{\theta_0}^{\theta_0+n} \sqrt{I_\theta} d\theta \\ &\geq \sum_{i=1}^{n} \int |f(x,\theta_0+i) - f(x,\theta_0+i-1)| dx.\end{aligned}$$

As $n \to \infty$, by the regularity condition (3.7), it holds that

$$\int_{\theta_0}^{\infty} \sqrt{I_\theta} d\theta = \infty.$$

A similar argument leads to the other equality. □

In view of Lemmas 3.2 and 3.3, we have a sufficient condition for the estimator $\tilde{\theta}_{ML}$ in $\mathcal{T}$ to be parametrization-invariant with respect to second order admissibility.

THEOREM 3.4. *The modified MLE* $\tilde{\theta}_{ML}$

$$\tilde{\theta}_{ML} = \hat{\theta}_{ML} + \frac{1}{n}\left(\frac{3\mu_{110}}{2I_\theta^2} - \frac{\mu_{300}}{2I_\theta^2}\right)\Bigg|_{\theta=\hat{\theta}_{ML}}, \tag{3.9}$$

that is, the estimator with the pair $(\alpha, \beta) = (-1, 1/2)$, *is second order admissible under mean squared error, independent of any parametrization.*

This theorem gives an interesting and important solution to the second order estimation problem under mean squared error. Namely, an estimator of the form (3.9) is optimal in the sense that the estimator is not inferior to any estimator in our situation.

Next, we will give a necessary condition for the estimator $\tilde{\theta}_{ML}$ in $\mathcal{T}$ to be parametrization-invariant with respect to second order adimissibility.

Here, we shall notice that, without loss of generality, we can take the parametrization η such that $I_\eta = 1$ on the whole real line. Suppose that the regularity conditions of Gusev are satisfied under a parametrization θ. Now, we consider a reparametrization $\eta = g(\theta)$:

$$\eta = \int_{\theta_0}^{\theta} \sqrt{I_\theta(u)} du.$$

Then, the parameter space for the parametrization η is the whole real line on account of Lemma 3.3. In addition, we have $I_\eta = 1$ on $(-\infty, \infty)$ since $I_\eta = I_\theta / g'^2$ and $g' = \sqrt{I_\theta}$.

At first, we will show that the estimator with the pair $(0,0)$ does not preserve the second order admissibility. We take a parametrization θ with $I_\theta = 1$ on $(-\infty, \infty)$. Let a reparametrization be $\eta = e^\theta$. Then, the criterion (3.2) is

$$\int_0^{\eta_0} I_\eta \exp\left\{\int_\eta^{\eta_0}\left(2\frac{\nu_{110}}{I_\eta} - \frac{\nu_{300}}{I_\eta}\right)du\right\}d\eta = \int_0^{\eta_0} I_{\eta_0} d\eta < \infty,$$

since $I'_\eta = 2\nu_{110} - \nu_{300}$ in view of Lemma 2.1 (1). This shows that the pair $(0,0)$ does not lead to second order admissibility under the parametrization η. Therefore, the estimator $\tilde{\theta}_{ML}$ with the pair $(0,0)$ is not parametrization-invariant with respect to second order admissibility.

Next, we will prove that for $\alpha \neq -1$ the second order admissibility is never preserved under any parametrization. We denote by $K(\beta)$ the region of k such that the pair $(-k, k\beta)$ is second order admissible under any parametrization. Then, under a parametrization θ we have for $k \in K(\beta)$:

$$\int_{\theta_0}^{\infty} \exp\left\{-\int_{\theta_0}^{\theta}\left(-\frac{k\mu_{110}}{I_\theta} + \frac{k\beta\mu_{300}}{I_\theta}\right)du\right\}d\theta = \infty \tag{3.10}$$

and

$$\int_{-\infty}^{\theta_0} \exp\left\{\int_{\theta}^{\theta_0}\left(-\frac{k\mu_{110}}{I_\theta} + \frac{k\beta\mu_{300}}{I_\theta}\right)du\right\}d\theta = \infty.$$

Now, we consider a reparametrization η for any constant κ:

$$\eta = \int_{\theta_0}^{\theta} \exp\left\{-\int_{\tau_0}^{\tau}\left(-\frac{\kappa\mu_{110}}{I_\theta} + \frac{\kappa\beta\mu_{300}}{I_\theta}\right)du\right\}d\tau.$$

Then, it is easy to see that the parametrization η is a monotone transformation. For the parameter η, criterion (3.1) is

$$\int_{\eta_0}^{\bar{\eta}} \exp\left\{-\int_{\eta_0}^{\eta}\left(-\frac{k\nu_{110}}{I_\eta} + \frac{k\beta\nu_{300}}{I_\eta}\right)du\right\}d\eta. \tag{3.11}$$

By variable transformation, (3.11) is reduced to

$$\begin{aligned}\int_{\theta_0}^{\infty} g' \exp&\left\{-\int_{\theta_0}^{\theta}\left(\frac{-k\mu_{110}}{I_\theta} + \frac{k\beta\mu_{300}}{I_\theta} + \frac{kg''}{g'}\right)du\right\}d\theta \\ &= \int_{\theta_0}^{\infty} \exp\left\{-(k + \kappa - k\kappa)\int_{\theta_0}^{\theta}\left(\frac{-\mu_{110}}{I_\theta} + \frac{\beta\mu_{300}}{I_\theta}\right)du\right\}d\theta,\end{aligned} \tag{3.12}$$

since

$$g' = \exp\left\{-\int_{\theta_0}^{\theta}\left(-\frac{\kappa\mu_{110}}{I_\theta} + \frac{\kappa\beta\mu_{300}}{I_\theta}\right)\right\}$$

and

$$\frac{g''}{g'} = \frac{\kappa\mu_{110}}{I_\theta} - \frac{\kappa\beta\mu_{300}}{I_\theta}.$$

Therefore, for the preservation of second order admissibility, it follows from (3.10) and (3.12) that if $k \in K(\beta)$, $k + \kappa - k\kappa = -(k-1)(\kappa-1) \in K(\beta)$ for any κ on $(-\infty, \infty)$. Thus, $K(\beta)$ must be $\{\phi\}, \{1\}$ or $\{(-\infty, \infty)\}$. Now, since $0 \notin K(\beta)$ as shown before, $K(\beta) = \{(-\infty, \infty)\}$ is inappropriate. Therefore, we have $K(\beta) = \{\phi\}$ or $\{1\}$.

Next, we consider the region $\tilde{K}(\beta)$ of k such that an estimator with the pair $(0, k\beta)$ is second order admissible under any parametrization. A similar argument leads to $\tilde{K}(\beta) = \{\phi\}$. These facts show that, for $\alpha \neq -1$, the second order admissibility is never preserved under any parametrization. This leads to the following result:

THEOREM 3.5. *A necessary condition for the preservation of the second order adimissibility is* $\alpha = -1$.

4. A characterization of the estimators with parametrization invariance

According to the argument in Section 3, we can formulate a procedure for characterizing a set $\mathcal{A}$ of pairs (α, β) such that the second order admissibility is preserved under any parametrization, as follows.

First, we consider a parametrization θ satisfying the regularity conditions of Gusev. For the class of estimators $\tilde{\theta}_{ML}$ with the pair $(-1, \beta)$, we check criteria (3.1) and (3.2) and constitute the region $S(\beta)$ of β such that the criteria are simultaneously satisfied. Then, the estimators $\tilde{\theta}_{ML}$ with $(-1, \beta)$ for $\beta \in S(\beta)$ are second order admissible, independent of any parametrization. In addition, no parametrization invariant estimator exists in $\mathcal{T}$ except for those.

It is worth noting that, as a special case when $\mu_{300} = 0$, we have only one estimator in $\mathcal{T}$ with parametrization invariance with respect to second order admissibility, in the form

$$\tilde{\theta}_{ML} = \hat{\theta}_{ML} + \frac{1}{n}\frac{3\mu_{110}}{2I_\theta^2}\bigg|_{\theta=\hat{\theta}_{ML}}. \tag{4.1}$$

The procedure above is applicable to many models. However, there are some cases where it is not easy to check criteria (3.1) and (3.2). So, we will introduce an easier method for finding an $S(\beta)$. Let us denote the limits of $\mu_{300}/I_\theta\sqrt{I_\theta}$ as $\theta \to \infty$ and $\theta \to -\infty$ by c_1 and c_2, respectively.

Then we have the following theorem:

THEOREM 4.1.

(1) *When* $c_1 > 0$, *criterion* (3.1) *is satisfied iff* $\beta \leq 1/2$.

(2) *When* $c_1 < 0$, *criterion* (3.1) *is satisfied iff* $\beta \geq 1/2$.

(3) *When* $c_2 > 0$, *criterion* (3.2) *is satisfied iff* $\beta \geq 1/2$.

(4) *When* $c_2 < 0$, *criterion* (3.2) *is satisfied iff* $\beta \leq 1/2$.

PROOF. Since each proposition is proved in the same manner, it suffices to show only proposition (1). As shown in Section 3, without loss of generality, we can take a parametrization θ with $I_\theta = 1$ on $(-\infty, \infty)$. For a sufficiently large θ, we have $\frac{\mu_{300}}{I_\theta\sqrt{I_\theta}}\sqrt{I_\theta} > c_1 - \epsilon$ for sufficiently small $\epsilon > 0$. For $\beta < 1/2$, it follows that

$$\left(\beta - \frac{1}{2}\right)\frac{\mu_{300}}{I_\theta\sqrt{I_\theta}}\sqrt{I_\theta} < \left(\beta - \frac{1}{2}\right)(c_1 - \epsilon) = c_\epsilon < 0.$$

Now, criterion (3.1) is reduced to

$$\begin{aligned}
&\int_{\theta_0}^{\infty} I_\theta \exp\left\{-\int_{\theta_0}^{\theta}\left(-\frac{\mu_{110}}{I_\theta} + (\beta - 1)\frac{\mu_{300}}{I_\theta}\right)du\right\}d\theta \\
&= \int_{\theta_0}^{\infty} \sqrt{I_\theta}\exp\left\{-\int_{\theta_0}^{\theta}\left(\left(\beta - \frac{1}{2}\right)\frac{\mu_{300}}{I_\theta}\right)du\right\}d\theta \\
&\geq \int_{\theta_0}^{\infty} \exp\{-(\theta - \theta_0)c_\epsilon\}d\theta = \infty.
\end{aligned}$$

Thus, for $\beta < 1/2$, (3.1) is satisfied. Next, when $\beta = 1/2$, it follows from Theorem 3.4 that (3.1) is satisfied. Finally, when $\beta > 1/2$, we have

$$\begin{aligned}
&\int_{\theta_0}^{\infty} I_\theta \exp\left\{-\int_{\theta_0}^{\theta}\left(-\frac{\mu_{110}}{I} + (\beta - 1)\frac{\mu_{300}}{I_\theta}\right)du\right\}d\theta \\
&\leq \int_{\theta_0}^{\infty} \exp\{-(\theta - \theta_0)c_\epsilon\}d\theta < \infty.
\end{aligned}$$

These lead to proposition (1) of Theorem 4.1. □

By Theorem 4.1, we can obtain an easy method to get the region $S(\beta)$ with parametrization invariance.

THEOREM 4.2. (1) *When c_1 and c_2 have the same sign, $\beta = 1/2$ is a necessary and sufficient condition for the estimator $\tilde{\theta}_{ML}$ to be parametrization invariance with respect to second order admissibility.*

(2) *When c_1 and c_2 have the different sign, a necessary and sufficient condition is $\beta \leq 1/2$ if $c_1 > 0$ and $\beta \geq 1/2$ if $c_1 < 0$.*

We note that Theorem 4.2 is applicable to such a statistical model that its parameter space is the whole real line. When the parameter space is not the whole real line, we need to develop a theorem in a different form.

Suppose that the parameter space for a parametrization θ is an interval with end points, $\underline{\theta}$ and $\bar{\theta}$. Assume that the parameter space for a monotone increasing reparametrization η is the whole real line, and the regularity conditions are satisfied under η. Then, as noted at the end of Section 2, we have

$$\frac{\mu_{300}}{I_\theta\sqrt{I_\theta}} = \frac{\nu_{300}}{I_\eta\sqrt{I_\eta}}.$$

Therefore, it holds that

$$\lim_{\theta \to \underline{\theta}} \frac{\mu_{300}}{I_\theta \sqrt{I_\theta}} = \lim_{\eta \to -\infty} \frac{\nu_{300}}{I_\eta \sqrt{I_\eta}}$$

and

$$\lim_{\theta \to \bar{\theta}} \frac{\mu_{300}}{I_\theta \sqrt{I_\theta}} = \lim_{\eta \to \infty} \frac{\nu_{300}}{I_\eta \sqrt{I_\eta}}.$$

This shows that the criterion of Theorem 4.2 holds regardless of parametrization. Then we have the following corollary:

COROLLARY 4.3. *Suppose that the parameter space for a parametrization θ is an interval with end points, $\underline{\theta}$ and $\bar{\theta}$. Defining*

$$\lim_{\theta \to \bar{\theta}} \frac{\mu_{300}}{I_\theta \sqrt{I_\theta}} = d_1, \qquad \lim_{\theta \to \underline{\theta}} \frac{\mu_{300}}{I_\theta \sqrt{I_\theta}} = d_2,$$

we have the following propositions.

(1) *When d_1 and d_2 have the same sign, $\beta = 1/2$ is a necessary and sufficient condition for the estimator $\tilde{\theta}_{ML}$ to be parametrization-invariant with respect to second order admissibility.*

(2) *When d_1 and d_2 have a different sign, a necessary and sufficient condition is $\beta \leq 1/2$ if $d_1 > 0$ and $\beta \geq 1/2$ if $d_1 < 0$.*

Now, we should notice that our results in this section are derived from the consequence of Lemma 3.3. Therefore, in actual examples, it suffices to check the equalities (3.8) instead of the regularity condition (3.7).

5. Examples

In this section, we will actually constitute the class of estimators with parametrization invariance with respect to second order admissibility. By virtue of Theorem 4.2 and Corollary 4.3, it is easy to obtain the region $S(\beta)$ for several cases. Now, it is easy to see that all the examples below hold the equalities (3.8).

Example 5.1. We consider the location model with the density $f(x, \theta) = f(x - \theta)$. Then we note that the quantities μ_{300} and I_θ are constants. Therefore, for $\mu_{300} = 0$, we have only one solution (4.1). When $\mu_{300} \neq 0$, we get $S(\beta) = \{1/2\}$ by Theorem 4.2, since $c_1 = c_2 \neq 0$.

Example 5.2. We deal with the scale model with the density $f(x, \theta) = f(x/\theta)$, $\theta > 0$. Then, it is easily shown that $\mu_{300}/I_\theta \sqrt{I_\theta}$ is a constant. Thus, we have $S(\beta) = \{1/2\}$ in view of Corollary 4.3.

Example 5.3. We treat the two-dimensional normal model $N(\eta_\theta, I_2)$ with $\eta_\theta^t = (\theta, \theta^2/2)$ (see Efron (1975)). Since $\mu_{300} = 0$, we have only one solution (4.1). Here, $\mu_{110} = \theta$ and $I_\theta = 1 + \theta^2$.

Example 5.4. We deal with the two-dimensional normal model with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix Σ where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}.$$

Let us now consider the estimation problem of the correlation coefficient θ. Now, by straightforward calculation, we have

$$I_\theta = \frac{1+\theta^2}{(1-\theta^2)^2}, \qquad \mu_{110} = 0, \quad \mu_{300} = -\frac{2\theta(\theta^2+3)}{(1-\theta^2)^3}.$$

Accordingly, it holds that

$$\frac{\mu_{300}}{I_\theta\sqrt{I_\theta}} = \frac{2\theta(\theta^2+3)}{(1+\theta^2)\sqrt{1+\theta^2}}.$$

Therefore, it follows that

$$d_1 = -2\sqrt{2}, \qquad d_2 = 2\sqrt{2}.$$

Thus, by Corollary 4.3, we obtain $S(\beta) = \{\beta \geq 1/2\}$.

Example 5.5. We consider a model with the density

$$f(x,\theta) = \frac{1}{\sqrt{2\pi a(\theta^2+b)}} \exp\left\{-\frac{(x-\theta)^2}{2a(\theta^2+b)}\right\},$$

where $a, b > 0$. Then, we have

$$I_\theta = \frac{1}{(\theta^2+b)^2}\left(2\theta^2 + \frac{\theta^2+b}{a}\right), \qquad \mu_{110} = \frac{2\theta}{(\theta^2+b)^2}\left(1+\frac{1}{a}\right),$$

$$\mu_{300} = \frac{8\theta^3}{(\theta^2+b)^3} + \frac{6\theta}{a(\theta^2+b)^2}.$$

Therefore, it holds that

$$c_1 = \frac{a\sqrt{a}}{(2a+1)\sqrt{2a+1}}\left(8+\frac{6}{a}\right) > 0$$

and

$$c_2 = -\frac{a\sqrt{a}}{(2a+1)\sqrt{2a+1}}\left(8+\frac{6}{a}\right) < 0.$$

Thus, we have $S(\beta) = \{\beta \leq 1/2\}$.

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