Ann. Inst. Statist. Math.
Vol. 51, No. 1, 17–29 (1999)

SOONER AND LATER WAITING TIME PROBLEMS FOR RUNS IN MARKOV DEPENDENT BIVARIATE TRIALS*

SIGEO AKI[1] AND KATUOMI HIRANO[2]

[1] *Department of Informatics and Mathematical Science, Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama-cho, Toyonaka 560-8531, Japan*
[2] *The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan*

(Received July 7, 1997; revised March 2, 1998)

Abstract. In this paper we study exact distributions of sooner and later waiting times for runs in Markov dependent bivariate trials. We give systems of linear equations with respect to conditional probability generating functions of the waiting times. By considering bivariate trials, we can treat very general and practical waiting time problems for runs of two events which are not necessarily mutually exclusive. Numerical examples are also given in order to illustrate the feasibility of our results.

Key words and phrases: Discrete distributions, probability generating function, run, waiting time problem.

1. Introduction

The sooner and later waiting time problems between a success run of length k and a failure run of length r in independent trials were investigated and the probability generating functions (p.g.f.'s) of the exact distributions were derived by Ebneshahrashoob and Sobel (1990). Ling (1992) and Sobel and Ebneshahrashoob (1992) developed the sooner and later problems in the case of a frequency quota. Balasubramanian *et al.* (1993) and Aki and Hirano (1993) solved the problems in the sequence of $\{0, 1\}$-valued Markov dependent trials. Uchida and Aki (1995) studied more general sooner and later waiting time problems between a specified number of "1"-runs of length k and another specified number of "0"-runs of length r in $\{0, 1\}$-valued Markov dependent trials. Aki *et al.* (1996) studied sooner and later waiting time problems in a sequence of $\{0, 1\}$-valued higher order Markov dependent trials.

* This research was partially supported by the ISM Cooperative Research Program (96-ISM-CRP-A11).

In this paper, we investigate sooner and later waiting time problems in a sequence of bivariate $\{0,1\}^2$-valued Markov dependent trials. We wait for "1"-run of length k in the sequence of the first components and for "1"-run of length r in the sequence of the second components of the bivariate Markov dependent trials. The results in this paper can be appplied to the statistical analyses based on $\{0,1\}^2$-valued bivariate data. For example we can mention a start-up demonstration test in which a vender repeats start-ups of two equipments simultaneously until specified numbers of consecutive successful start-ups for each of the equipments in order to demonstrate to a customer the reliability of the equipments with regard to their starting. For start-up demonstration tests we refer the reader Hahn and Gage (1983), Viveros and Balakrishnan (1993), Balakrishnan *et al.* (1995) and Balakrishnan *et al.* (1997).

In addition to the above applications, we have to emphasize that our results can be applied to very general sooner and later waiting time problems in a sequence of random variables. Let $(S, \mathcal{S})$ be a sample space. Suppose we are given a sequence of S-valued i.i.d. random variables $Z_1, Z_2, \dots$ and suppose we are interested in consecutive occurrences of two events A and $B \in \mathcal{S}$, where A and B are not necessarily mutually exclusive. We say that A-run of length k occurs at the n-th trial if $Z_i \in A$ for $i = n-k+1, \dots, n$. When we consider the sooner and later waiting time problems between an A-run of length k and a B-run of length r, it is sufficient to observe the sequence of $\{0,1\}^2$-valued bivariate random vectors $\binom{I_A(Z_i)}{I_B(Z_i)}$, $i = 1, 2, \dots$, where $I_A(\cdot)$ means the indicator function of A. Therefore, our situation is suitable for studying the waiting time problems. Of course, if A and B are mutually exclusive, it is sufficient to observe the sequence of $\{0,1\}$-valued random variables $I_A(Z_1), I_A(Z_2), \dots$, which is the case having been studied by many authors.

2. Main results

Let $\binom{X_0}{Y_0}, \binom{X_1}{Y_1}, \binom{X_2}{Y_2}, \dots$ be a sequence of $\{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\}$-valued Markov chain with the following probabilities: for $x, y, u, v = 0, 1$ and $i = 1, 2, \dots$, $p_{xy} = P(\binom{X_0}{Y_0} = \binom{x}{y})$ and

$$p(x,y,u,v) = P\left(\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \;\middle|\; \begin{pmatrix} X_{i-1} \\ Y_{i-1} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}\right),$$

where for every $x, y = 0, 1$, $p(x,y,0,0) + p(x,y,0,1) + p(x,y,1,0) + p(x,y,1,1) = 1$ and for every $u, v = 0, 1$, $0 < p(u,v,0,0) + p(u,v,0,1) < 1$ and $0 < p(u,v,0,0) + p(u,v,1,0) < 1$. Let τ_x (resp. τ_y) be the waiting time for the first "1"-run of length k (resp. of length r) in $X_0, X_1, X_2, \dots$ (resp. in $Y_0, Y_1, Y_2, \dots$). We set $\tau_S = \min\{\tau_x, \tau_y\}$ and $\tau_L = \max\{\tau_x, \tau_y\}$. The random variables τ_S and τ_L are called the sooner and the later waiting times, respectively.

First, we study the distribution of the sooner waiting time τ_S. Let t_0 be any positive integer. Suppose we have observed until the (t_0-1)-th trial and the sooner run has not yet occurred. For every $i = 0, 1, \dots, k-1$, $j = 0, 1, \dots, r-1$, $u = 0, 1$ and $v = 0, 1$, we consider the following conditions; at the (t_0-1)-th trial we have

$X_{t_0-1} = u$ and $Y_{t_0-1} = v$ and observe currently "1"-run of length i in X's at X_{t_0-1} and observe currently "1"-run of length j in Y's at Y_{t_0-1}. Let $\phi(i,j,u,v;t)$ be the p.g.f. of the conditional distribution of the sooner waiting time from the (t_0-1)-th trial given each of the above conditions. Note that $\phi(i,j,u,v;t)$ does not depend on (t_0-1).

PROPOSITION 2.1. *The p.g.f.'s of the conditional distributions of the waiting time satisfy the system of linear equations: for every* $i = 0,1,\ldots,k-2$, $j = 0,1,\ldots,r-2$, $u = 0,1$ *and* $v = 0,1$,

$$\begin{aligned}\phi(i,j,u,v;t) &= p(u,v,0,0)t\phi(0,0,0,0;t)\\ &\quad + p(u,v,0,1)t\phi(0,j+1,0,1;t)\\ &\quad + p(u,v,1,0)t\phi(i+1,0,1,0;t)\\ &\quad + p(u,v,1,1)t\phi(i+1,j+1,1,1;t),\end{aligned} \tag{2.1}$$

for every $j = 0,1,\ldots,r-2$, $u = 0,1$ *and* $v = 0,1$,

$$\begin{aligned}\phi(k-1,j,u,v;t) &= p(u,v,0,0)t\phi(0,0,0,0;t)\\ &\quad + p(u,v,0,1)t\phi(0,j+1,0,1;t)\\ &\quad + p(u,v,1,0)t\cdot 1\\ &\quad + p(u,v,1,1)t\cdot 1,\end{aligned} \tag{2.2}$$

for every $i = 0,1,\ldots,k-2$, $u = 0,1$ *and* $v = 0,1$,

$$\begin{aligned}\phi(i,r-1,u,v;t) &= p(u,v,0,0)t\phi(0,0,0,0;t)\\ &\quad + p(u,v,0,1)t\cdot 1\\ &\quad + p(u,v,1,0)t\phi(i+1,0,1,0;t)\\ &\quad + p(u,v,1,1)t\cdot 1,\end{aligned} \tag{2.3}$$

and for every $u = 0,1$ *and* $v = 0,1$,

$$\begin{aligned}\phi(k-1,r-1,u,v;t) &= p(u,v,0,0)t\phi(0,0,0,0;t)\\ &\quad + p(u,v,0,1)t\cdot 1\\ &\quad + p(u,v,1,0)t\cdot 1\\ &\quad + p(u,v,1,1)t\cdot 1.\end{aligned} \tag{2.4}$$

PROOF. Suppose we have observed until the (t_0-1)-th trial and we have $X_{t_0-1} = u$ and $Y_{t_0-1} = v$. Further, suppose that we currently observe "1"-run of length i in X's and "1"-run of length j in Y's at the (t_0-1)-th trial. Then, the p.g.f. of the conditional distribution of the sooner waiting time from the (t_0-1)-th trial is $\phi(i,j,u,v;t)$. Given the condition we observe the t_0-th trial. For every $x,y = 0,1$, the conditional probability that we observe $\binom{X_{t_0}}{Y_{t_0}} = \binom{x}{y}$ is $p(u,v,x,y)$. If $i = 0,1,\ldots,$ or $k-2$ and $j = 0,1,\ldots,$ or $r-2$, then we never observe the sooner event at the t_0-th trial. If we have $\binom{X_{t_0}}{Y_{t_0}} = \binom{0}{0}$, then the p.g.f. of the conditional distribution of the sooner waiting time from the t_0-th trial becomes $\phi(0,0,0,0;t)$. If we have $\binom{X_{t_0}}{Y_{t_0}} = \binom{0}{1}$, then the p.g.f. of the conditional distribution of the sooner waiting time from the t_0-th trial becomes $\phi(0,j+1,0,1;t)$. Similarly, we can see the other cases. Since the events $\{\binom{X_{t_0}}{Y_{t_0}} = \binom{0}{0}\}$, $\{\binom{X_{t_0}}{Y_{t_0}} = \binom{0}{1}\}$, $\{\binom{X_{t_0}}{Y_{t_0}} = \binom{1}{0}\}$, $\{\binom{X_{t_0}}{Y_{t_0}} = \binom{1}{1}\}$ are mutually exclusive, we have (2.1). If $i = k-1$ or $j = r-1$, then we possibly observe the sooner event at the t_0-th trial, then the p.g.f. of the conditional distribution of the sooner waiting time from the t_0-th trial becomes 1, hence we have (2.2), (2.3) and (2.4). This completes the proof.

The system of the equations (2.1), (2.2), (2.3) and (2.4) in the conditional p.g.f.'s can be solved by virtue of its linearlity. By using the solutions, we can write the p.g.f. of the sooner waiting time as

$$\begin{aligned}\phi(t) &= p_{00}\phi(0,0,0,0;t) + p_{01}\phi(0,1,0,1;t)\\ &\quad + p_{10}\phi(1,0,1,0;t) + p_{11}\phi(1,1,1,1;t).\end{aligned}$$

For given k and r, the number of equations in Proposition 2.1 is $4kr$. Therefore, for practical values of k and r we can easily solve the system of the equations by usual uses of computer algebra systems. This will be illustrated in Section 3.

As a special case, we consider the case that the sequence $\binom{X_0}{Y_0}, \binom{X_1}{Y_1}, \binom{X_2}{Y_2}, \dots,$ is a $\{\binom{0}{1}, \binom{1}{0}\}$-valued Markov chain. This case is equivalent to observing a $\{0,1\}$-valued univariate (dependent) trials as we explained in Section 1.

COROLLARY 2.1. *Assume that we observe only* $\binom{0}{1}$ *or* $\binom{1}{0}$ *at each trial. Then, denoting* $p_0 = p_{10}$, $q_0 (= 1 - p_0) = p_{01}$, $p_1 = p(1,0,1,0)$, $q_1 (= 1 - p_1) = p(1,0,0,1)$, $p_2 = p(0,1,1,0)$ *and* $q_2 (= 1 - p_2) = p(0,1,0,1)$, *we can write the p.g.f. of the sooner waiting time as*

$$\phi(t) = p_0 \frac{A(q_2t)^{r-1} + (p_1t)^{k-1}}{1 - AB} + q_0 \frac{B(p_1t)^{k-1} + (q_2t)^{r-1}}{1 - AB},$$

where

$$A = q_1 t \frac{1 - (p_1t)^{k-1}}{1 - p_1 t} \quad \text{and} \quad B = p_2 t \frac{1 - (q_2t)^{r-1}}{1 - q_2 t}.$$

PROOF. From Proposition 2.1, we have the system of linear equations in the p.g.f.'s of the conditional distributions of the sooner waiting time,

$$\begin{cases} \phi(1,0,1,0;t) = q_1 t\phi(0,1,0,1;t) + p_1 t\phi(2,0,1,0;t) \\ \phi(2,0,1,0;t) = q_1 t\phi(0,1,0,1;t) + p_1 t\phi(3,0,1,0;t) \\ \quad\vdots \\ \phi(k-2,0,1,0;t) = q_1 t\phi(0,1,0,1;t) + p_1 t\phi(k-1,0,1,0;t) \\ \phi(k-1,0,1,0;t) = q_1 t\phi(0,1,0,1;t) + p_1 t, \end{cases}$$

$$\begin{cases} \phi(0,1,0,1;t) = q_2t\phi(0,2,0,1;t) + p_2t\phi(1,0,1,0;t) \\ \phi(0,2,0,1;t) = q_2t\phi(0,3,0,1;t) + p_2t\phi(1,0,1,0;t) \\ \quad\vdots \\ \phi(0,r-2,0,1;t) = q_2t\phi(0,r-1,0,1;t) + p_2t\phi(1,0,1,0;t) \\ \phi(0,r-1,0,1;t) = q_2t + p_2t\phi(1,0,1,0;t). \end{cases}$$

By solving the system of linear equations, we have easily

$$\phi(1,0,1,0;t) = \frac{A(q_2t)^{r-1} + (p_1t)^{k-1}}{1-AB}$$

and

$$\phi(0,1,0,1;t) = \frac{B(p_1t)^{k-1} + (q_2t)^{r-1}}{1-AB}.$$

This completes the proof.

Corollary 2.1 agrees with Corollary 1 of Balasubramanian *et al.* (1993).

Remark 2.1. By using the result of Proposition 2.1, the reliability of the linear connected-(r,s)-out-of-$(r+1,n)$:F lattice system (cf. Yamamoto and Miyakawa (1995)) can be given. The system is a rectangular grid of size $(r+1)\times n$ containing $(r+1)n$ components and it fails if and only if there is at least one rectangular block of failed components of size $r\times s$. Define for $i=1,2,\ldots,n$,

$$X_i = \begin{cases} 1 & \text{if all of components } (1,i),(2,i),\ldots,(r,i) \text{ fail} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_i = \begin{cases} 1 & \text{if all of components } (2,i),(3,i),\ldots,(r+1,i) \text{ fail} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the system does not fail if and only if the sooner waiting time between "1"-run of length s in X's and "1"-run of length s in Y's is greater than n. If we assume that all components are independently and identically distributed, we can find the reliability of the system by using Proposition 2.1.

Before we study the distribution of the later waiting time τ_L, we shall derive the p.g.f.'s of τ_x and τ_y. Let t_0 be any positive integer. Suppose we have observed until the (t_0-1)-th trial. Suppose that we have not yet observed the first "1"-run of length r in Y's and we have $X_{t_0-1}=u$ and $Y_{t_0-1}=v$. Further, for every $j=0,1,\ldots,r-1$, suppose we currently observe "1"-run of length j in Y's at the (t_0-1)-th trial. Let $\xi(j,u,v;t)$ be the p.g.f. of the conditional distribution of τ_y from the (t_0-1)-th trial given each of the above conditions.

Similarly, suppose we have observed until the (t_0-1)-th trial. Suppose that we have not yet observed the first "1"-run of length k in X's and we have $X_{t_0-1}=u$

and $Y_{t_0-1} = v$. Further, for every $i = 0, 1, \dots, k-1$, suppose we currently observe "1"-run of length i in X's at the $(t_0 - 1)$-th trial. Let $\eta(i, u, v; t)$ be the p.g.f. of the conditional distribution of τ_x from the $(t_0 - 1)$-th trial given each of the above conditions. Note that $\xi(j, u, v; t)$ and $\eta(i, u, v; t)$ do not depend on $(t_0 - 1)$.

Similarly as the proof of Proposition 2.1, by considering the condition of one-step ahead from every condition, we have the following lemmas.

LEMMA 2.1. *The p.g.f.'s of the conditional distributions of the waiting time τ_y satisfy the system of linear equations: for every $j = 0, 1, \dots, r-2$, $u = 0, 1$ and $v = 0, 1$,*

$$\begin{aligned}\xi(j, u, v; t) &= p(u, v, 0, 0)t\xi(0, 0, 0; t) \\ &\quad + p(u, v, 0, 1)t\xi(j+1, 0, 1; t) \\ &\quad + p(u, v, 1, 0)t\xi(0, 1, 0; t) \\ &\quad + p(u, v, 1, 1)t\xi(j+1, 1, 1; t),\end{aligned} \tag{2.5}$$

for every $u = 0, 1$ and $v = 0, 1$,

$$\begin{aligned}\xi(r-1, u, v; t) &= p(u, v, 0, 0)t\xi(0, 0, 0; t) \\ &\quad + p(u, v, 0, 1)t \cdot 1 \\ &\quad + p(u, v, 1, 0)t\xi(0, 1, 0; t) \\ &\quad + p(u, v, 1, 1)t \cdot 1.\end{aligned} \tag{2.6}$$

LEMMA 2.2. *The p.g.f.'s of the conditional distributions of the waiting time τ_x satisfy the system of linear equations: for every $i = 0, 1, \dots, k-2$, $u = 0, 1$ and $v = 0, 1$,*

$$\begin{aligned}\eta(i, u, v; t) &= p(u, v, 0, 0)t\eta(0, 0, 0; t) \\ &\quad + p(u, v, 0, 1)t\eta(0, 0, 1; t) \\ &\quad + p(u, v, 1, 0)t\eta(i+1, 1, 0; t) \\ &\quad + p(u, v, 1, 1)t\eta(i+1, 1, 1; t),\end{aligned} \tag{2.7}$$

for every $u = 0, 1$ and $v = 0, 1$,

$$\begin{aligned}\eta(k-1, u, v; t) &= p(u, v, 0, 0)t\eta(0, 0, 0; t) \\ &\quad + p(u, v, 0, 1)t\eta(0, 0, 1; t) \\ &\quad + p(u, v, 1, 0)t \cdot 1 \\ &\quad + p(u, v, 1, 1)t \cdot 1.\end{aligned} \tag{2.8}$$

Next, we consider the later waiting time problem. Let t_0 be any positive integer. Suppose we have observed until the $(t_0 - 1)$-th trial and we have not yet observed the sooner run. For every $i = 0, 1, \dots, k-1$, $j = 0, 1, \dots, r-1$, $u = 0, 1$

and $v = 0, 1$, we consider the following conditions; at the $(t_0 - 1)$-th trial we have $X_{t_0-1} = u$ and $Y_{t_0-1} = v$ and observe currently "1"-run of length i in X's at X_{t_0-1} and observe currently "1"-run of length j in Y's at Y_{t_0-1}. Let $\psi(i, j, u, v; t)$ be the p.g.f. of the conditional distribution of the later waiting time from the $(t_0 - 1)$-th trial given each of the above conditions. Note that $\psi(i, j, u, v; t)$ does not depend on $(t_0 - 1)$.

PROPOSITION 2.2. *The p.g.f.'s of the conditional distributions of the later waiting time satisfy the system of linear equations: for every* $i = 0, 1, \ldots, k-2$, $j = 0, 1, \ldots, r-2$, $u = 0, 1$ *and* $v = 0, 1$,

$$\begin{aligned}\psi(i, j, u, v; t) = {} & p(u, v, 0, 0)t\psi(0, 0, 0, 0; t) \\ & + p(u, v, 0, 1)t\psi(0, j+1, 0, 1; t) \\ & + p(u, v, 1, 0)t\psi(i+1, 0, 1, 0; t) \\ & + p(u, v, 1, 1)t\psi(i+1, j+1, 1, 1; t),\end{aligned} \tag{2.9}$$

for every $j = 0, 1, \ldots, r-2$, $u = 0, 1$ *and* $v = 0, 1$,

$$\begin{aligned}\psi(k-1, j, u, v; t) = {} & p(u, v, 0, 0)t\psi(0, 0, 0, 0; t) \\ & + p(u, v, 0, 1)t\psi(0, j+1, 0, 1; t) \\ & + p(u, v, 1, 0)t\xi(0, 1, 0; t) \\ & + p(u, v, 1, 1)t\xi(j+1, 1, 1; t),\end{aligned} \tag{2.10}$$

for every $i = 0, 1, \ldots, k-2$, $u = 0, 1$ *and* $v = 0, 1$,

$$\begin{aligned}\psi(i, r-1, u, v; t) = {} & p(u, v, 0, 0)t\psi(0, 0, 0, 0; t) \\ & + p(u, v, 0, 1)t\eta(0, 0, 1; t) \\ & + p(u, v, 1, 0)t\psi(i+1, 0, 1, 0; t) \\ & + p(u, v, 1, 1)t\eta(i+1, 1, 1; t),\end{aligned} \tag{2.11}$$

and for every $u = 0, 1$ *and* $v = 0, 1$,

$$\begin{aligned}\psi(k-1, r-1, u, v; t) = {} & p(u, v, 0, 0)t\psi(0, 0, 0, 0; t) \\ & + p(u, v, 0, 1)t\eta(0, 0, 1; t) \\ & + p(u, v, 1, 0)t\xi(0, 1, 0; t) \\ & + p(u, v, 1, 1)t \cdot 1.\end{aligned} \tag{2.12}$$

PROOF. Similarly as the previous results, consider the condition of one-step ahead from every condition. Since we assumed that the sooner run does not occur until the $(t_0 - 1)$-th trial, (2.9) has the same form as (2.1). When $i = k-1$, $0 \le j \le r-2$, $u = 0, 1$ and $v = 0, 1$, if we observe $\binom{X_{t_0}}{Y_{t_0}} = \binom{1}{0}$ or $\binom{1}{1}$, the sooner run occurs at the t_0-th trial and the later run becomes "1"-run of length r in Y's automatically. Since in the case we never observe also the later run at the

t_0-th trial simultaneously, from the t_0-th trial we wait for only "1"-run of length r in Y's. Hence, we have (2.10). Similarly, we have (2.11). When $i = k - 1$ and $j = r - 1$, if we observe $\binom{X_{t_0}}{Y_{t_0}} = \binom{1}{1}$, then we observe the sooner and later run at the t_0-th trial simultaneously. Taking it into consideration, we have (2.12). This completes the proof.

The differences between the systems of linear equations given in Propositions 2.1 and 2.2 are only that some "1"'s in (2.2), (2.3) and (2.4) are replaced by ξ's or η's in (2.10), (2.11) and (2.12). The ξ's and η's can easily be obtained by solving the systems of linear equations given in Lemmas 2.1 and 2.2, respectively. Hence, by solving the system of equations given in Proposition 2.2, we can write the p.g.f. of the later waiting time as

$$\begin{aligned}\psi(t) &= p_{00}\psi(0,0,0,0;t) + p_{01}\psi(0,1,0,1;t)\\ &\quad + p_{10}\psi(1,0,1,0;t) + p_{11}\psi(1,1,1,1;t).\end{aligned}$$

3. Numerical examples

In this section we illustrate how to obtain the distributions of sooner and later waiting times by using Propostions 2.1 and 2.2 and computer algebra systems.

Example 1. The possible outcomes of a throw of a die is $\{1,2,3,4,5,6\}$. We are interested in two events $A = \{1,2,3\}$ and $B = \{2,4,6\}$. When we observe a sequence $Z_1, Z_2, \ldots$ of the independent throwings of the die, which is assumed to be unbiased, we consider the sooner and later waiting time problems between an A-run of length k and a B-run of length r. By setting $X_i = I_A(Z_i)$ and $Y_i = I_B(Z_i)$ for $i = 1, 2, \ldots$, we have the sequence of $\{0,1\}^2$-valued bivariate random vectors $\binom{X_i}{Y_i}$, $i = 1, 2, \ldots$. For $x, y = 0, 1$, we denote $p(x,y) = P(X_i = x, Y_i = y)$. Then, in this example, we have $p(1,1) = \frac{1}{6}$, $p(1,0) = \frac{1}{3}$, $p(0,1) = \frac{1}{3}$ and $p(0,0) = \frac{1}{6}$. All we have to do is only to solve the system of linear equations in Proposition 2.1 or 2.2 when k and r are given. However, technically, it may be a problem to input the system of linear equations when k or r is not so small. Most computer algebra systems can carry out solving the system of linear equations when a list of linear equations and a list of variables are given. Hence, if we write a procedure to make lists of the equations and the variables (the conditional p.g.f.'s in this case) for arbitrarily given k and r, we need not input the necessary system of equations with our hands one by one. Since algorithms for that are given essentially in Propositions 2.1 and 2.2, we can easily write the procedure.

For example, for $k = r = 3$, we can obtain the p.g.f.'s of the sooner and later waiting times as

$$\begin{aligned}\phi_S(t) = &\,(t^3(-5t^3 + 6t^2 - 90t + 2862))/\\ &\,(t^6 + 18t^4 - 810t^3 - 2268t^2 - 5832t + 11664)\end{aligned}$$

and

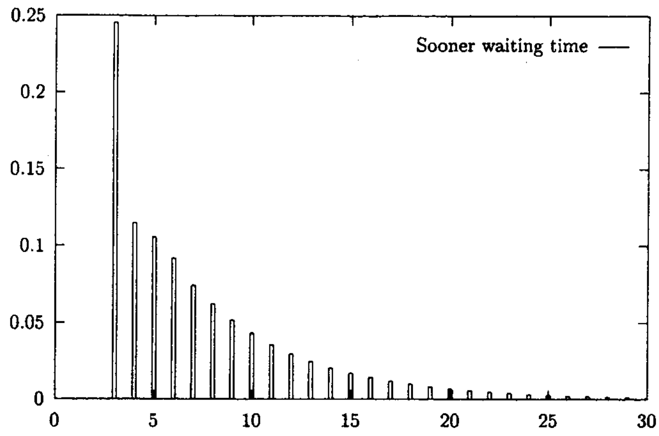

Fig. 1. Probability function of the sooner waiting time of Example 1.

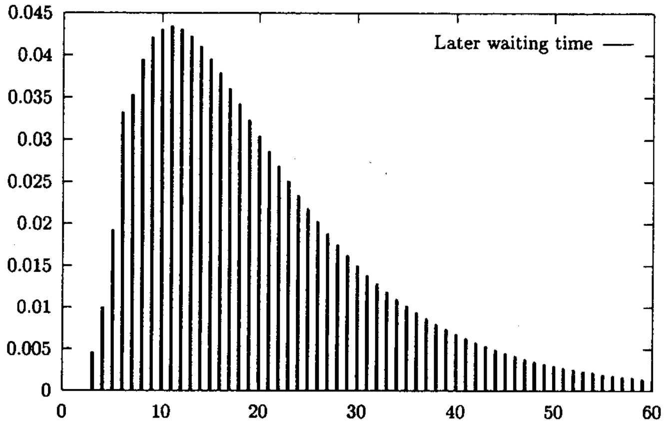

Fig. 2. Probability function of the later waiting time of Example 1.

$$\begin{aligned}\phi_L(t) = {} & (t^3(3t^6 + 4t^5 + 62t^4 - 1126t^3 - 780t^2 - 504t - 432))/\\ & (t^9 + 2t^8 + 22t^7 - 782t^6 - 3816t^5 - 13752t^4\\ & \quad - 2592t^3 + 18144t^2 + 93312t - 93312),\end{aligned}$$

respectively. By expanding them in the power series in t, we get the probability distributions of the waiting times. The values of probabilities are given in Table 1. Figures 1 and 2 are the graphs of the probability functions of the sooner and later waiting times of the example with $k = r = 3$, respectively.

Example 2. Example 1 treated independent bivariate trials. In order to show that purely dependent trials can be treated, we consider the following bivariate Markov dependent trials with the initial probabilities $p_{xy} = 1/4$ for every $x, y =$

Table 1. Values of probabilities in Examples 1 and 2. Probab. I and Probab. II are the probabilities of sooner and later waiting times in Example 1, respectively. Probab. III and Probab. IV are the probabilities of sooner and later waiting times in Example 2, respectively.

x	Probab. I	Probab. II	Probab. III	Probab. IV
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0.24537	0.00462963	0.4275	0.0625
4	0.114969	0.0100309	0.10575	0.04125
5	0.10571	0.0192901	0.09595	0.05105
6	0.091821	0.033179	0.083945	0.063055
7	0,0740705	0.0353045	0.0632822	0.0621087
8	0.0620529	0.0395096	0.0491978	0.0610669
9	0.0516213	0.0421287	0.0384764	0.0592551
10	0.0428687	0.0430688	0.0299762	0.0562593
11	0.0356577	0.0434439	0.0233604	0.052747
12	0.0296456	0.0431083	0.0182095	0.0489839
13	0.0246472	0.0422473	0.0141933	0.0451246
14	0.0204928	0.0410306	0.0110628	0.0413019
15	0.0170382	0.0395414	0.00862289	0.0376042
16	0.014166	0.0378665	0.00672106	0.0340877
17	0.011778	0.0360736	0.0052387	0.0307869
18	0.00979254	0.0342138	0.00408327	0.0277197
19	0.00814178	0.0323283	0.00318269	0.0248927
20	0.00676929	0.0304488	0.00248073	0.0223039
21	0.00562817	0.0285992	0.00193359	0.019946
22	0.00467941	0.0267976	0.00150713	0.017808
23	0.00389059	0.025057	0.00117472	0.0158764
24	0.00323474	0.0233867	0.00091563	0.014137
25	0.00268945	0.0217928	0.000713683	0.0125746
26	0.00223608	0.0202789	0.000556277	0.0111745
27	0.00185914	0.0188466	0.000433587	0.00992219
28	0.00154573	0.0174961	0.000337957	0.00880402
29	0.00128517	0.0162266	0.000263419	0.00780702
30	0.00106852	0.015036	0.00020532	0.00691918

$0, 1$ and transition probabilities

$$
\begin{array}{llll}
p(1,1,1,1) = 0.5 & p(1,1,1,0) = 0.2 & p(1,1,0,1) = 0.2 & p(1,1,0,0) = 0.1 \\
p(1,0,1,1) = 0.2 & p(1,0,1,0) = 0.5 & p(1,0,0,1) = 0.1 & p(1,0,0,0) = 0.2 \\
p(0,1,1,1) = 0.2 & p(0,1,1,0) = 0.1 & p(0,1,0,1) = 0.5 & p(0,1,0,0) = 0.2 \\
p(0,0,1,1) = 0.1 & p(0,0,1,0) = 0.2 & p(0,0,0,1) = 0.2 & p(0,0,0,0) = 0.5.
\end{array}
$$

Table 1. (continued).

x	Probab. I	Probab. II	Probab. III	Probab. IV
31	0.000888397	0.013922	0.000160036	0.0061294
32	0.000738637	0.0128817	0.000124739	0.00542751
33	0.000614122	0.0119117	0.0000972272	0.00480424
34	0.000510598	0.0110087	0.0000757832	0.00425118
35	0.000424525	0.0101691	0.0000590688	0.00376073
36	0.000352961	0.00938942	0.0000460409	0.00332604
37	0.000293461	0.00866606	0.0000358863	0.00294095
38	0.000243992	0.00799557	0.0000279714	0.00259995
39	0.000202861	0.0073746	0.0000218022	0.0022981
40	0.000168664	0.0067999	0.0000169936	0.00203099
41	0.000140232	0.00626836	0.0000132455	0.00179469
42	0.000116593	0.00577702	0.0000103242	0.0015857
43	0.0000969381	0.00532309	0.00000804712	0.00140091
44	0.000080597	0.00490389	0.00000627229	0.00123754
45	0.0000670105	0.00451694	0.0000048889	0.00109314
46	0.0000557143	0.00415989	0.00000381063	0.000965514
47	0.0000463224	0.00383053	0.00000297017	0.000852739
48	0.0000385136	0.00352681	0.00000231508	0.000753096
49	0.0000320213	0.0032468	0.00000180448	0.000665065
50	0.0000266233	0.00298873	0.00000140649	0.000587299
51	0.0000221354	0.00275091	0.00000109628	0.000518606
52	0.0000184039	0.00253181	0.000000854492	0.000457934
53	0.0000153015	0.00232998	0.000000666029	0.000404347
54	0.0000127221	0.0021441	0.000000519133	0.000357022
55	0.0000105775	0.00197293	0.000000404635	0.000315229
56	0.0000087944	0.00181533	0.000000315391	0.000278323
57	0.0000073119	0.00167023	0.00000024583	0.000245733
58	0.00000607931	0.00153666	0.000000191611	0.000216956
59	0.0000050545	0.00141372	0.00000014935	0.000191546
60	0.00000420245	0.00130056	0.00000011641	0.000169111

By using the algorithms given by Propositions 2.1 and 2.2, we can derive the p.g.f's of the sooner and later waiting times. For example, for $k = r = 3$, we obtain the p.g.f.'s of the sooner and later waiting times as

$$\begin{aligned}\phi_S(t) = (t^3(28t^7 - 276t^6 - 4015t^5 - 39275t^4 + 966000t^3\\ - 32500t^2 + 10800000t - 42750000))/\\ (2(4t^9 - 8t^8 - 665t^7 - 9350t^6 + 70000t^5 + 310000t^4\\ + 4100000t^3 + 5000000t^2 + 25000000t - 50000000))\end{aligned}$$

and

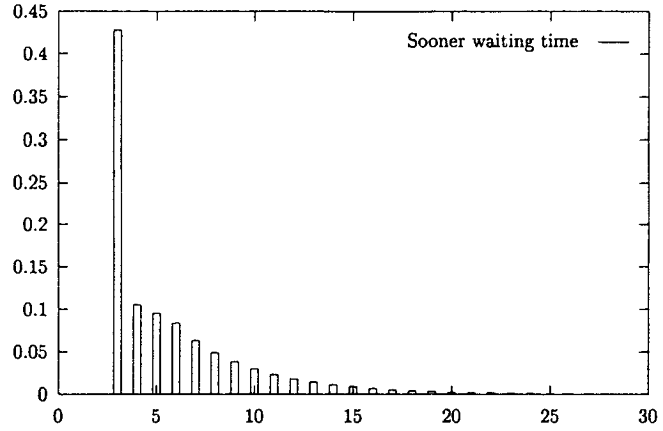

Fig. 3. Probability function of the sooner waiting time of Example 2.

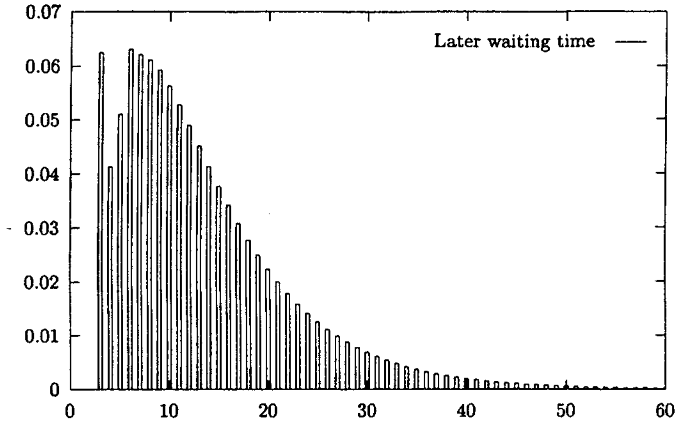

Fig. 4. Probability function of the later waiting time of Example 2.

$$\begin{aligned}\phi_L(t) = {} & (t^3(-196t^{10} + 7812t^9 + 5345t^8 + 43375t^7 - 18185750t^6 \\ & - 8495000t^5 - 89550000t^4 + 652000000t^3 + 1155000000t^2 \\ & - 3375000000t + 6250000000))/ \\ & (2(252t^{12} - 144t^{11} - 39815t^{10} - 658500t^9 + 3111000t^8 \\ & + 19950000t^7 + 344550000t^6 + 831000000t^5 + 4585000000t^4 \\ & + 1500000000t^3 + 8000000000t^2 - 60000000000t + 50000000000)),\end{aligned}$$

respectively. By expanding them, we get the probability distributions of the waiting times. The values of probabilities are given in Table 1. Figures 3 and 4 are the graphs of the probability functions of the sooner and later waiting times of this example with $k = r = 3$, respectively.

REFERENCES

Aki, S., Balakrishnan, N. and Mohanty, S. G. (1996). Sooner and later waiting time problems and failure runs in higher order Markov dependent trials, *Ann. Inst. Statist. Math.*, **48**, 773–787.

Aki, S. and Hirano, K. (1993). Discrete distributions related to succession events in a two-state Markov chain, *Statistical Science & Data Analysis* (eds. K. Matusita, M. L. Puri and T. Hayakawa), 467–474, VSP Publishers, Amsterdam.

Balakrishnan, N., Balasubramanian, K. and Viveros, R. (1995). Start-up demonstration tests under correlation and corrective action, *Naval Res. Logist.*, **42**, 1271–1276.

Balakrishnan, N., Mohanty, S. and Aki, S. (1997). Start-up demonstration tests under Markov dependence model with corrective actions, *Ann. Inst. Statist. Math.*, **49**, 155–169.

Balasubramanian, K., Viveros, R. and Balakrishnan, N. (1993). Sooner and later waiting time problems for Markovian Bernoulli trials, *Statist. Probab. Lett.*, **18**, 153–161.

Ebneshahrashoob, M. and Sobel, M. (1990). Sooner and later problems for Bernoulli trials: frequency and run quotas, *Statist. Probab. Lett.*, **9**, 5–11.

Hahn, G. J. and Gage, J. B. (1983). Evaluation of a start-up demonstration test, *Journal of Quality Technology*, **15**, 103–106.

Ling, K. D. (1992). A generalization of the sooner and later waiting time problems for Bernoulli trials: frequency quota, *Statist. Probab. Lett.*, **14**, 401–405.

Sobel, M. and Ebneshahrashoob, M. (1992). Quota sampling for multinomial via Dirichlet, *J. Statist. Plann. Inference*, **33**, 157–164.

Uchida, M. and Aki, S. (1995). Sooner and later waiting time problems in two-state Markov chain, *Ann. Inst. Statist. Math.*, **47**, 415–433.

Viveros, R. and Balakrishnan, N. (1993). Statistical inference from start-up demonstration test data, *Journal of Quality Technology*, **25**, 119–130.

Yamamoto, H. and Miyakawa, M. (1995). Reliability of a linear connected-(r,s)-out-of-(m,n):F lattice system, *IEEE Transactions on Reliability*, **44**, 333–336.